形式化方法导引
第 3 章 经典数理逻辑-问题求解基础

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⟶ 教学课程 ⟶ 形式化方法导引

本章内容

回顾:

- Define verification
    - $\mathcal{M} \vDash \phi$
- A method of define $\mathcal{M}$ and ϕ: Logics
    - Propositional logic
    - Predicate logic
    - Higher-order logic

本章内容:

- A method of verification
    - *Rules* of Natural Deduction

回顾: 定义: Verification in Logics

Most logics used in the design, specification and verification of computer systems fundamentally deal with a *satisfaction relation*:

$$\mathcal{M} \vDash \phi$$

- $\mathcal{M}$ is some sort of situation or *model* of a system
- ϕ is a *specification*, a formula of that logic, expressing what should be true in situation $\mathcal{M}$.
- At the *heart* of this set-up is that one can often *specify and implement algorithms* for computing $\vDash$.

下一个问题:

- 问: 如何统一化定义 $\mathcal{M}$ 和 ϕ? 答: *Logics* （已介绍）
- 问: 如何支持 $\vDash$ 和 algorithms? 答: *Rules*

问题: 可计算性与复杂度

(回顾) 命题逻辑-真值表-复杂度?
- 若 $\mathcal{M}$ 原子命题的个数为 n, 判定所需时间为 $O(2^n)$.

一阶逻辑-复杂度?
- 至少不比命题逻辑简单

一阶逻辑-可计算性?

定理（*Undecidability* in First-order logic）

The decision problem of validity in predicate logic is *undecidable*: no program exists which, given any ϕ, decides whether $\models \phi$.

针对复杂度: 怎么办? **首先**，定义新的 *rules*, 即本章内容
针对可计算性: 怎么办？利用前述定理，推导出后续方法的极限
下面: 先介绍命题逻辑的求解基础

支持 ⊨ 和 algorithms

回顾, 定义: Semantic entailment relation

If, for all valuations in which all $\phi_1, \phi_2, \ldots, \phi_n$ evaluate to **T**, ψ evaluates to **T** as well, we say that

$$\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$$

holds and call $\vDash$ the *semantic* entailment relation.

问题: $\vDash$ **求解复杂度过高**
解决方法: New rules: a collection of *proof rules* in *natural deduction*.

- **不使用** Truth Tables **进行求解**
- **定义并使用** proof rules
- **使用** proof rules **产生结论** (即 $\vdash$), **取代** $\vDash$, **即**

$$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$$

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定义: rules for conjunction: $\wedge i$, $\wedge e_1$, $\wedge e_2$

$$\frac{\phi \quad \psi}{\phi \wedge \psi} \quad \wedge i \qquad\qquad \frac{\phi \wedge \psi}{\phi} \quad \wedge e_1 \qquad\qquad \frac{\phi \wedge \psi}{\psi} \quad \wedge e_2$$

例: Prove that $p \wedge q,\ r \vdash q \wedge r$ is valid

1	$p \wedge q$	premise
2	r	premise
3	q	$\wedge e_2$ 1
4	$q \wedge r$	$\wedge i\ 3, 2$

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定义: rules for double negation: $\neg\neg e, \neg\neg i$

$$\frac{\neg\neg\phi}{\phi} \quad \neg\neg e \qquad\qquad\qquad \frac{\phi}{\neg\neg\phi} \quad \neg\neg i$$

例: Prove that $p, \neg\neg(q \wedge r) \vdash \neg\neg p \wedge r$ is valid

1	p	premise
2	$\neg\neg(q \wedge r)$	premise
3	$\neg\neg p$	$\neg\neg i\ 1$
4	$q \wedge r$	$\neg\neg e\ 2$
5	r	$\wedge e_2\ 4$
6	$\neg\neg p \wedge r$	$\wedge i\ 3, 5$

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定义: rules for eliminating implication: $\rightarrow e$

$$\frac{\phi \quad \phi \rightarrow \psi}{\psi} \quad \rightarrow e$$

例: Prove that $p, p \rightarrow q, p \rightarrow (q \rightarrow r) \vdash r$ is valid

1	$p \rightarrow (q \rightarrow r)$	premise
2	$p \rightarrow q$	premise
3	p	premise
4	$q \rightarrow r$	$\rightarrow e \; 1, 3$
5	q	$\rightarrow e \; 2, 3$
6	r	$e \; 4, 5$

支持 ⊨ 和 algorithms

定义: rules for elminiating implication: *modus tollens*, MT

$$\frac{\phi \to \psi \quad \neg\psi}{\neg\phi} \quad \text{MT}$$

例: If Abraham Lincoln was Ethiopian, then he was African.

- Abraham Lincoln was *not* African; therefore he was *not* Ethiopian.

注意: MT is not a primitive rule.

例: Prove that $p \to (q \to r), p, \neg r \vdash \neg q$ is valid

1	$p \to (q \to r)$	premise
2	p	premise
3	$\neg r$	premise
4	$q \to r$	$\to e\ 1, 2$
5	$\neg q$	MT $4, 3$

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定义: rule implies introduction: $\rightarrow i$

$$
\begin{array}{c}
\boxed{\begin{array}{c} \phi \\ \vdots \\ \psi \end{array}} \\[4pt]
\hline
\phi \rightarrow \psi
\end{array} \qquad \rightarrow i
$$

To prove $\phi \rightarrow \psi$, make a *temporary assumption* of ϕ and then prove ψ.

例: Prove that $\neg q \rightarrow \neg p \vdash p \rightarrow \neg\neg q$ is valid

1	$\neg q \rightarrow \neg p$	premise
2	p	assumption
3	$\neg\neg p$	$\neg\neg i$ 2
4	$\neg\neg q$	MT 1, 3
5	$p \rightarrow \neg\neg q$	$\rightarrow i$ 2 − 4

例: Prove that $p \wedge q \to r \vdash p \to (q \to r)$ is valid

1	$p \wedge q \to r$	premise
2	p	assumption
3	q	assumption
4	$p \wedge q$	$\wedge i\ 2,3$
5	r	$\to e\ 1,4$
6	$q \to r$	$\to i\ 3-5$
7	$p \to (q \to r)$	$\to i\ 2-6$

例: Prove that $p \to (q \to r) \vdash p \land q \to r$ is valid

1	$p \to (q \to r)$	premise
2	$p \land q$	assumption
3	p	$\land e_1\ 2$
4	q	$\land e_2\ 2$
5	$q \to r$	$\land e\ 1,3$
6	r	$\land e\ 5,4$
7	$p \land q \to r$	$\to i\ 2-6$

注: ⊣⊢

$p \to (q \to r) \dashv\vdash p \land q \to r$

定义: rules for disjunction: $\vee i_1, \vee i_2, \vee e$

$$\frac{\phi}{\phi \vee \psi} \quad \vee i_1 \qquad \frac{\psi}{\phi \vee \psi} \quad \vee i_2 \qquad \frac{\phi \vee \psi \quad \begin{array}{c} \boxed{\begin{array}{c} \phi \\ \vdots \\ \chi \end{array}} \end{array} \quad \begin{array}{c} \boxed{\begin{array}{c} \psi \\ \vdots \\ \chi \end{array}} \end{array}}{\chi} \quad \rightarrow \vee e$$

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例: Prove that $p \vee q \vdash q \vee p$ is valid

1	$p \vee q$	premise
2	p	assumption
3	$q \vee p$	$\vee i_2 \; 2$
4	q	assumption
5	$q \vee p$	$\vee i_1 \; 4$
6	$q \vee p$	$\vee e \; 1, 2-3, 4-5$

定义: Contradictions (矛盾)

Contradictions are expressions of the form $\phi \wedge \neg\phi$ or $\neg\phi \wedge \phi$, where ϕ is any formula.

定理

Any formula can be derived from a contradiction:

$$p \wedge \neg p \vdash q$$

定义: rules for negation: $\bot e, \neg e$

$$\frac{\bot}{\phi} \quad \bot e \qquad\qquad \frac{\phi \quad \neg\phi}{\bot} \quad \neg e$$

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定义: rules for negation: $\bot e, \neg e$

$$\frac{\bot}{\phi} \quad \bot e \qquad\qquad \frac{\phi \quad \neg\phi}{\bot} \quad \neg e$$

例: Prove that $\neg p \vee q \vdash p \to q$ is valid

1	$\neg p \vee q$	premise		
2	$\neg p$	assumption	q	assumption
3	p	assumption	p	assumption
4	$\bot$	$\neg e\ 3, 2$	q	copy 2
5	q	$\bot e\ 4$	$p \to q$	$\to i\ 3 - 4$
6	$p \to q$	$\to i\ 3 - 5$		
7	$p \to q$			$\vee e\ 1, 2 - 6$

定义: rule for negation: $\neg i$

$$\frac{\begin{array}{|c|}\hline \phi \\ \vdots \\ \bot \\ \hline\end{array}}{\neg \phi} \; \neg i$$

例: Prove that
$p \to q, p \to \neg q \vdash \neg p$ is valid

1	$p \to q$	premise
2	$p \to \neg q$	premise
3	p	assumption
4	q	$\to e \; 1, 3$
5	$\neg q$	$\to e \; 2, 3$
6	$\bot$	$\neg e \; 4, 5$
7	$\neg p$	$\neg i \; 3 - 6$

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Derived rule: MT

$$\frac{\phi \to \psi \quad \neg\psi}{\neg\phi} \quad \text{MT}$$

例: Prove the derived rule

1	$\phi \to \psi$	premise
2	$\neg\psi$	premise
3	ϕ	assumption
4	ψ	$\to e\ 1,3$
5	$\bot$	$\neg e\ 4,2$
6	$\neg\phi$	$\neg i\ 3-5$

Derived rule: ¬¬i

$$\frac{\phi}{\neg\neg\phi} \quad \neg\neg i$$

例: Prove the derived rule

1	ϕ	premise
2	$\neg\phi$	assumption
3	$\bot$	$\neg e\ 1, 2$
4	$\neg\neg\phi$	$\neg i\ 2 - 3$

Derived rule: PBC

$$\cfrac{\begin{array}{c} \neg\phi \\ \vdots \\ \bot \end{array}}{\phi} \quad \text{PBC}$$

例: Prove the derived rule

1	$\neg\phi \to \bot$	given
2	$\neg\phi$	assumption
3	$\bot$	$\to e\ 1, 2$
4	$\neg\neg\phi$	$\neg i\ 2 - 3$
5	ϕ	$\neg\neg e\ 4$

例: Prove the derived rule

1	$\neg(\phi \vee \neg\phi)$	assumption
2	ϕ	assumption
3	$\phi \vee \neg\phi$	$\vee i_1\ 2$
4	$\bot$	$\neg e\ 3, 1$
5	$\neg\phi$	$\neg i\ 2 - 4$
6	$\phi \vee \neg\phi$	$\vee i_2\ 5$
7	$\bot$	$\neg e\ 6, 1$
8	$\neg\neg(\phi \vee \neg\phi)$	$\neg i\ 1 - 7$
9	$\phi \vee \neg\phi$	$\neg\neg e\ 8$

Derived rule: LEM

$$\frac{}{\phi \vee \neg\phi}\ \text{LEM}$$

The basic rules of natural deduction:

	introduction	*elimination*
$\wedge$	$\dfrac{\phi \quad \psi}{\phi \wedge \psi} \ \wedge\mathrm{i}$	$\dfrac{\phi \wedge \psi}{\phi} \ \wedge\mathrm{e}_1 \qquad \dfrac{\phi \wedge \psi}{\psi} \ \wedge\mathrm{e}_2$
$\vee$	$\dfrac{\phi}{\phi \vee \psi} \ \vee\mathrm{i}_1 \qquad \dfrac{\psi}{\phi \vee \psi} \ \vee\mathrm{i}_2$	$\dfrac{\phi \vee \psi \quad \begin{array}{c} \phi \\ \vdots \\ \chi \end{array} \quad \begin{array}{c} \psi \\ \vdots \\ \chi \end{array}}{\chi} \ \vee\mathrm{e}$

$\rightarrow$

$$\frac{\begin{array}{|c|}\hline \phi \\ \vdots \\ \psi \\ \hline\end{array}}{\phi \rightarrow \psi} \rightarrow\!\mathrm{i} \qquad\qquad \frac{\phi \quad \phi \rightarrow \psi}{\psi} \rightarrow\!\mathrm{e}$$

$\neg$

$$\frac{\begin{array}{|c|}\hline \phi \\ \vdots \\ \bot \\ \hline\end{array}}{\neg\phi} \neg\mathrm{i} \qquad\qquad \frac{\phi \quad \neg\phi}{\bot} \neg\mathrm{e}$$

$\bot$ | (no introduction rule for $\bot$) | $\dfrac{\bot}{\phi}\ \bot e$

$\neg\neg$ | | $\dfrac{\neg\neg\phi}{\phi}\ \neg\neg e$

Some useful derived rules:

$$\frac{\phi \to \psi \quad \neg\psi}{\neg\phi} \text{ MT} \qquad\qquad \frac{\phi}{\neg\neg\phi} \text{ } \neg\neg\text{i}$$

$$\frac{\begin{array}{|c|} \hline \neg\phi \\ \vdots \\ \bot \\ \hline \end{array}}{\phi} \text{ PBC} \qquad\qquad \frac{}{\phi \vee \neg\phi} \text{ LEM}$$

Provable equivalence:

$$\neg(p \wedge q) \dashv\vdash \neg q \vee \neg p \qquad \neg(p \vee q) \dashv\vdash \neg q \wedge \neg p$$
$$p \rightarrow q \dashv\vdash \neg q \rightarrow \neg p \qquad p \rightarrow q \dashv\vdash \neg p \vee q$$
$$p \wedge q \rightarrow p \dashv\vdash r \vee \neg r \qquad p \wedge q \rightarrow r \dashv\vdash p \rightarrow (q \rightarrow r).$$

回顾: 问题: ⊨ 求解复杂度过高

解决方法: New rules: a collection of *proof rules* in *natural deduction*.

- **不使用** Truth Tables **进行求解**
- **定义并使用** proof rules
- **使用** proof rules **产生结论** (即 ⊢), **取代** ⊨, **即**

$$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$$

新的问题:

1. rules **太多**: **推演过于复杂，符号也有冗余**
   - **减少冗余的符号，设计自动推演算法** (见第 4 章)
2. **目前所给的是命题逻辑的** rule, **一阶逻辑会有哪些新的** rule?
   - $=, \forall, \exists$ **怎样设计它们的** rules (见下页)
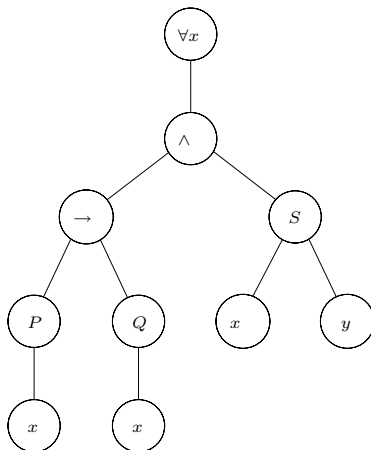
问题 2: $=, \forall, \exists$ 怎样设计它们的 rules?

解决方法:

1. **预定义**
   - **构建** Parse tree
   - **定义** Free and bound variables
   - **定义** Substitution

2. **设计** rules

(1) **构建** Parse tree

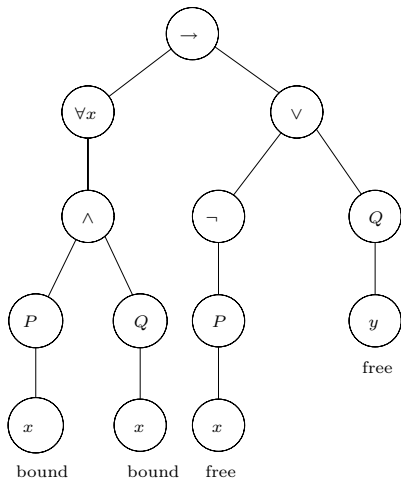- **例**: $\forall x \ ((P(x) \to Q(x)) \land S(x, y))$

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(2) **定义** Free and bound variables

定义: Free and bound variables

Let ϕ be a formula in predicate logic.

- An occurrence of x in ϕ is *free* in ϕ if it is a leaf node in the parse tree of ϕ such that there is no path upwards from that node x to a node $\forall x$ or $\exists x$.
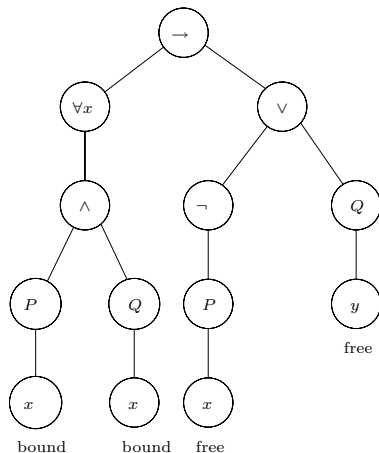- Otherwise, that occurrence of x is called *bound*.

(3) 定义 Substitution

定义 Substitution

Given a variable x, a term t and a formula ϕ, define $\phi[t/x]$ to be the formula obtained by replacing each *free* occurrence of variable x in ϕ with t.
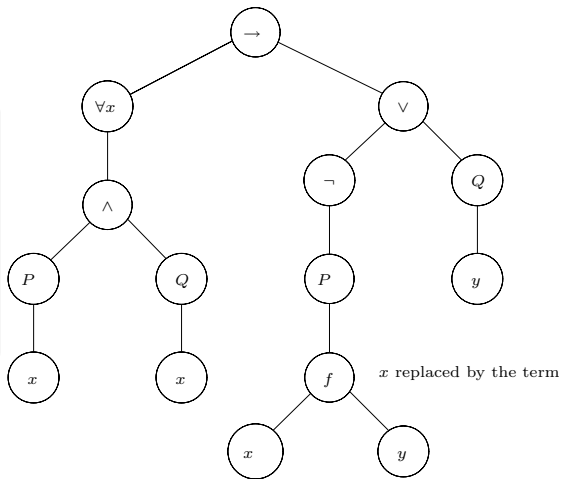
例: x replaced by the term $f(x, y)$

(3) 定义 Substitution

定义 Substitution

Given a variable x, a term t and a formula ϕ, define $\phi[t/x]$ to be the formula obtained by replacing each *free* occurrence of variable x in ϕ with t.

例: x replaced by the term $f(x, y)$

x replaced by the term

定义: rules for equality: $= i$

$$\frac{}{t = t} \quad = i \qquad\qquad \frac{t_1 = t_2 \quad \phi[t_1/x]}{\phi[t_2/x]} \quad = e$$

例: Prove the validity of the sequent:
$x+1 = 1+x, (x+1 > 1) \rightarrow (x+1 > 0) \vdash (1+x) > 1 \rightarrow (1+x) > 0$

1	$(x + 1) = (1 + x)$	premise
2	$(x + 1 > 1) \rightarrow (x + 1 > 0)$	premise
3	$(1 + x > 1) \rightarrow (1 + x > 0)$	$= e\ 1, 2$

支持 ⊨ 和 algorithms
2 First-order Logic | Natural Deduction | Proof rules

定义. rules for equality: $= i$

$$\frac{}{t = t} \; =i \qquad \frac{t_1 = t_2 \quad \phi[t_1/x]}{\phi[t_2/x]} \; =e$$

例. Prove the validity of the sequent:
$x + 1 = 1 + x, (x + 1 > 1) \to (x + 1 > 0) \vdash (1 + x) > 1 \to (1 + x) > 0$

1	$(x + 1) = (1 + x)$	premise
2	$(x + 1 > 1) \to (x + 1 > 0)$	premise
3	$(1 + x > 1) \to (1 + x > 0)$	$= e \; 1, 2$

First, let us state the proof rules for equality. Here equality does not mean syntactic, or intensional, equality, but equality in terms of computation results. In either of these senses, any term t has to be equal to itself. This is expressed by the introduction rule for equality.

This rule is quite evidently sound, but it is not very useful on its own. What we need is a principle that allows us to substitute equals for equals repeatedly. For example, suppose that $y(w + 2)$ equals $yw + y2$; then it certainly must be the case that $z \geq y(w + 2)$ implies $z \geq yw + y2$ and vice versa. We may now express this substitution principle as the rule $= e$

定义: rules for equality: $= i, = e$

$$\frac{}{t = t} \quad = i \qquad\qquad \frac{t_1 = t_2 \quad \phi[t_1/x]}{\phi[t_2/x]} \quad = e$$

例: Prove (symmetric relation 对称性): $t_1 = t_2 \vdash t_2 = t_1$

1	$t_1 = t_2$	premise
2	$t_1 = t_1$	$= i$
3	$t_2 = t_1$	$= e \ 1, 2$

支持 ⊨ 和 algorithms

定义: rules for equality: $= i, = e$

$$\frac{}{t = t} \quad = i \qquad\qquad \frac{t_1 = t_2 \quad \phi[t_1/x]}{\phi[t_2/x]} \quad = e$$

例: Prove (transive relation 传递性): $t_1 = t_2, t_2 = t_3 \vdash t_1 = t_3$

1	$t_2 = t_3$	premise
2	$t_1 = t_2$	premise
3	$t_1 = t_3$	$= e \ 1, 2$

Our discussion of the rules $=i$ and $=e$ has shown that they force equality to be reflexive, symmetric and transitive. These are minimal and necessary requirements for any sane concept of (extensional) equality.

定义: rules for universal quantification: $\forall x\ e, \forall x\ i$

$$\frac{\forall x\ \phi}{\phi[t/x]} \quad \forall x\ e$$

$$\begin{array}{|c|}
\hline
x_0 \\
\vdots \\
\phi[x_0/x] \\
\hline
\end{array}$$
$$\overline{\forall x\ \phi} \quad \forall x\ i$$

例: Prove: $\forall x\ (P(x) \to Q(x)), \forall x\ P(x) \vdash \forall x\ Q(x)$

1		$\forall x\ (P(x) \to Q(x))$	premise
2		$\forall x\ P(x)$	premise
3	x_0	$P(x_0) \to Q(x_0)$	$\forall x\ e\ 1$
4		$P(x_0)$	$\forall x\ e\ 2$
5		$Q(x_0)$	$\to e\ 3, 4$
6		$\forall x\ Q(x)$	$\forall x\ i\ 3-5$

It says: If $\forall x\ \phi$ is true, then you could replace the x in ϕ by any term t (given, as usual, the side condition that t be free for x in ϕ) and conclude that $\phi[t/x]$ is true as well. The intuitive soundness of this rule is self-evident.

The rule $\forall x\ i$ is a bit more complicated. It employs a proof box similar to those we have already seen in natural deduction for propositional logic, but this time the box is to stipulate the scope of the 'dummy variable' x0 rather than the scope of an assumption. The rule $\forall x\ i$ is written above.

It says: If, starting with a 'fresh' variable x_0, you are able to prove some formula $\phi[x_0/x]$ with x_0 in it, then (because x_0 is fresh) you can derive $\forall x\ \phi$. The important point is that x_0 is a new variable which doesn't occur anywhere outside its box; we think of it as an arbitrary term. Since we assumed nothing about this x_0, anything would work in its place; hence the conclusion $\forall x\ \phi$.

定义: rules for universal quantification: $\forall x\ e, \forall x\ i$

$$\frac{\forall x\ \phi}{\phi[t/x]}\quad \forall x\ e$$

$$\boxed{\begin{array}{c} x_0 \\ \vdots \\ \phi[x_0/x] \end{array}}$$
$$\frac{}{\forall x\ \phi}\quad \forall x\ i$$

例: Prove: $P(t), \forall x\ (P(x) \to \neg Q(x)) \vdash \neg Q(t)$

1	$P(t)$	premise
2	$\forall x\ (P(x) \to \neg Q(x))$	premise
3	$P(t) \to \neg Q(t)$	$\forall x\ e\ 2$
4	$\neg Q(t)$	$\to e\ 3, 1$

定义: rules for existential quantification: $\exists x\ i, \exists e$

$$\frac{\phi[t/x]}{\exists x\ \phi} \quad \exists x\ i$$

$$\exists x\ \phi \qquad \boxed{\begin{array}{c} x_0 \quad \phi[x_0/x] \\ \vdots \\ \chi \end{array}}$$

$$\frac{}{\chi} \quad \exists e$$

例: Prove: $\forall x\ \phi \vdash \exists x\ \phi$

1	$\forall x\ \phi$	premise
2	$\phi[x/x]$	$\forall x\ e\ 1$
3	$\exists x\ \phi$	$\exists x\ i\ 2$

例: Prove: $\forall x\ (P(x) \to Q(x)), \exists x\ P(x) \vdash \exists x\ Q(x)$

1		$\forall x\ (P(x) \to Q(x))$	premise
2		$\exists x\ P(x)$	premise
3	x_0	$P(x_0)$	assumption
4		$P(x_0) \to Q(x_0)$	$\forall x\ e\ 1$
5		$Q(x_0)$	$\to e\ 4,3$
6		$\exists x\ Q(x)$	$\exists x\ i\ 5$
7		$\exists x\ Q(x)$	$\exists x\ e\ 2,3-6$

$$\frac{}{t = t} \quad = i \qquad\qquad \frac{t_1 = t_2 \quad \phi[t_1/x]}{\phi[t_2/x]} \quad = e$$

$$\frac{\forall x \; \phi}{\phi[t/x]} \quad \forall x \; e$$

$$\begin{array}{|c|}\hline x_0 \\ \vdots \\ \phi[x_0/x] \\ \hline\end{array}$$
$$\frac{}{\forall x \; \phi} \quad \forall x \; i$$

$$\frac{\phi[t/x]}{\exists x \; \phi} \quad \exists x \; i \qquad\qquad \exists x \; \phi \quad \begin{array}{|cc|}\hline x_0 & \phi[x_0/x] \\ & \vdots \\ & \chi \\ \hline\end{array}$$
$$\frac{}{\chi} \quad \exists e$$

Quantifier equivalence

- ∀ and ∃
  - $\neg\forall x\ \phi \dashv\vdash \exists x\ \neg\phi$
  - $\neg\exists x\ \phi \dashv\vdash \forall x\ \neg\phi$
- ∧ and ∨
  - $\forall x\ \phi \wedge \forall x\ \psi \dashv\vdash \forall x(\phi \wedge \psi)$
  - $\forall x\ \phi \vee \forall x\ \psi \dashv\vdash \forall x(\phi \vee \psi)$
- double ∀ or ∃
  - $\forall x \forall y\ \phi \dashv\vdash \forall y \forall x\ \phi$
  - $\exists x \exists y\ \phi \dashv\vdash \exists y \exists x\ \phi$

- Assuming that x is not free in ψ
  - $\forall x\ \phi \wedge \psi \dashv\vdash \forall x\ (\phi \wedge \psi)$
  - $\forall x\ \phi \vee \psi \dashv\vdash \forall x\ (\phi \vee \psi)$
  - $\exists x\ \phi \wedge \psi \dashv\vdash \exists x\ (\phi \wedge \psi)$
  - $\exists x\ \phi \vee \psi \dashv\vdash \exists x\ (\phi \vee \psi)$
  - $\forall x(\psi \rightarrow \phi) \dashv\vdash \psi \rightarrow \forall x\ \phi$
  - $\exists x(\psi \rightarrow \phi) \dashv\vdash \psi \rightarrow \exists x\ \phi$
  - $\exists x(\phi \rightarrow \psi) \dashv\vdash \forall x\ \phi \rightarrow \psi$
  - $\forall x(\phi \rightarrow \psi) \dashv\vdash \exists x\ \phi \rightarrow \psi$

问题: ⊢ 演算 (calculus) 方法好用么？

- 问题 1：⊢ 和 ⊨ 的演算结果是否相同？
  - 答：是的，即 Soundness and Completeness
  - 一种表达形式：⊨ ϕ iff ⊢ ϕ
  - 命题逻辑和谓词逻辑均满足
- 问题 2：⊢ 求解的可计算性？
  - 答：同 ⊨，即 The decision problem of validity in predicate logic is undecidable.
- 问题 3：⊢ 演算求解复杂度相对 ⊨ 降低了么？
  - 答：看起来是的
- 问题 3.1：怎样设计算法，以提升效率？
  - 见下章，求解算法的使用与实现

1. Prove the validity of the following sequents:

① $(p \land q) \land r, s \land t \vdash q \land s$

② $q \rightarrow r \vdash (p \rightarrow q) \rightarrow (p \rightarrow r)$

③ $\vdash q \rightarrow (p \rightarrow (p \rightarrow (q \rightarrow p)))$

④ $p \rightarrow q \land r \vdash (p \rightarrow q) \land (p \rightarrow r)$

⑤ $p \land \neg p \vdash \neg(r \rightarrow q) \land (r \rightarrow q)$

2. Prove the validity of the following sequents in predicate logic, where P, and Q have arity 1, and S has arity 0 (a 'propositional atom'):

① $\exists x(S \rightarrow Q(x)) \vdash S \rightarrow \exists x\ Q(x)$

② $\forall x P(x) \rightarrow S \vdash \exists x(P(x) \rightarrow S)$

③ $\forall x(P(x) \land Q(x)) \vdash \forall x P(x) \land \forall x Q(x)$

④ $\neg \forall x \neg P(x) \vdash \exists x P(x)$

⑤ $\forall x \neg P(x) \vdash \neg \exists x P(x)$