形式化方法导引

第 3 章 经典数理逻辑-问题求解基础

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$\longrightarrow$ 教学课程 $\longrightarrow$ 形式化方法导引

本章内容

回顾:

- Define verification
  - $\mathcal{M} \models \phi$
- A method of define $\mathcal{M}$ and ϕ: Logics
  - Propositional logic
  - Predicate logic
  - Higher-order logic

本章内容:

- A method of verification
  - *Rules* of Natural Deduction

Verifier —— Logic (回顾)

回顾: 定义: Verification in Logics

Most logics used in the design, specification and verification of computer systems fundamentally deal with a *satisfaction relation*:

$$\mathcal{M} \models \phi$$

- $\mathcal{M}$ is some sort of situation or *model* of a system
- ϕ is a *specification*, a formula of that logic, expressing what should be true in situation $\mathcal{M}$.
- At the *heart* of this set-up is that one can often *specify and implement algorithms* for computing $\models$.

下一个问题:

- 问: 如何统一化定义 $\mathcal{M}$ 和 ϕ? 答: *Logics* （已介绍）
- 问: 如何支持 $\models$ 和 algorithms? 答: *Rules*

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问题: 可计算性与复杂度

(回顾) 命题逻辑-真值表-复杂度?

- 若 $\mathcal{M}$ 原子命题的个数为 n, 判定所需时间为 $O(2^n)$.

一阶逻辑-复杂度?

- 至少不比命题逻辑简单

一阶逻辑-可计算性?

定理 (*Undecidability* in First-order logic)

The decision problem of validity in predicate logic is *undecidable*: no program exists which, given any ϕ, decides whether $\models \phi$.

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支持 $\vDash$ 和 algorithms

回顾, 定义: Semantic entailment relation

If, for all valuations in which all $\phi_1, \phi_2, \ldots, \phi_n$ evaluate to **T**, ψ evaluates to **T** as well, we say that

$$\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$$

holds and call $\vDash$ the *semantic* entailment relation.

问题: $\vDash$ 求解复杂度过高

解决方法: New rules: a collection of *proof rules* in *natural deduction*.

- 不使用 Truth Tables 进行求解
- 定义并使用 proof rules
- 使用 proof rules 产生结论 (即 $\vdash$), 取代 $\vDash$, 即

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支持 $\vDash$ 和 algorithms

定义: rules for conjunction: $\wedge i$, $\wedge e_1$, $\wedge e_2$

$$\frac{\phi \quad \psi}{\phi \wedge \psi} \wedge i \qquad \frac{\phi \wedge \psi}{\phi} \wedge e_1 \qquad \frac{\phi \wedge \psi}{\psi} \wedge e_2$$

例: Prove that $p \wedge q,\ r \vdash q \wedge r$ is valid

1	$p \wedge q$	premise
2	r	premise
3	q	$\wedge e_2$ 1
4	$q \wedge r$	$\wedge i$ 3, 2

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定义: rules for double negation: $\neg\neg e, \neg\neg i$

$$\frac{\neg\neg\phi}{\phi}\ \neg\neg e \qquad \frac{\phi}{\neg\neg\phi}\ \neg\neg i$$

例: Prove that $p, \neg\neg(q \wedge r) \vdash \neg\neg p \wedge r$ is valid

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例: Prove that $p, p \to q, p \to (q \to r) \vdash r$ is valid

1	$p \to (q \to r)$	premise
2	$p \to q$	premise
3	p	premise
4	$q \to r$	$\to e$ 1, 3
5	q	$\to e$ 2, 3
6	r	e 4, 5

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定义: rules for elminiating implication: *modus tollens*, MT

$$\frac{\phi \to \psi \quad \neg\psi}{\neg\phi} \quad \text{MT}$$

例: If Abraham Lincoln was Ethiopian, then he was African.

- Abraham Lincoln was *not* African; therefore he was *not* Ethiopian.

注意: MT is not a primitive rule.

例: Prove that $p \to (q \to r), p, \neg r \vdash \neg q$ is valid

1	$p \to (q \to r)$	premise
2	p	premise
3	$\neg r$	premise
4	$q \to r$	$\to e$ 1, 2
5	$\neg q$	MT 4, 3

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支持 $\vDash$ 和 algorithms

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定义: rule implies introduction: $\rightarrow i$

$$\frac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \psi\end{array}}}{\phi \rightarrow \psi} \rightarrow i$$

To prove $\phi \rightarrow \psi$, make a *temporary assumption* of ϕ and then prove ψ.

例: Prove that $\neg q \to \neg p \vdash p \to \neg\neg q$ is valid

1	$\neg q \to \neg p$	premise
2	p	assumption
3	$\neg\neg p$	$\neg\neg i$ 2
4	$\neg\neg q$	MT 1, 3
5	$p \to \neg\neg q$	$\to i$ 2 – 4

例: Prove that $p \wedge q \to r \vdash p \to (q \to r)$ is valid

1	$p \wedge q \to r$	premise
2	p	assumption
3	q	assumption
4	$p \wedge q$	$\wedge i\ 2, 3$
5	r	$\to e\ 1, 4$
6	$q \to r$	$\to i\ 3 - 5$
7	$p \to (q \to r)$	$\to i\ 2 - 6$

例: Prove that $p \to (q \to r) \vdash p \wedge q \to r$ is valid

1	$p \to (q \to r)$	premise
2	$p \wedge q$	assumption
3	p	$\wedge e_1$ 2
4	q	$\wedge e_2$ 2
5	$q \to r$	$\wedge e$ 1, 3
6	r	$\wedge e$ 5, 4
7	$p \wedge q \to r$	$\to i$ 2 – 6

注: $\dashv\vdash$

$p \to (q \to r) \dashv\vdash p \wedge q \to r$

例: Prove that $p \to (q \to r) \vdash p \wedge q \to r$ is valid

1	$p \to (q \to r)$	premise
2	$p \wedge q$	assumption
3	p	$\wedge e_1$ 2
4	q	$\wedge e_2$ 2
5	$q \to r$	$\wedge e$ 1, 3
6	r	$\wedge e$ 5, 4
7	$p \wedge q \to r$	$\to i$ 2 – 6

注: $\dashv\vdash$

$p \to (q \to r) \dashv\vdash p \wedge q \to r$

定义: rules for disjunction: $\vee i_1, \vee i_2, \vee e$

$$\frac{\phi}{\phi \vee \psi} \vee i_1 \qquad \frac{\psi}{\phi \vee \psi} \vee i_2 \qquad \frac{\phi \vee \psi \quad \boxed{\begin{array}{c}\phi \\ \vdots \\ \chi\end{array}} \quad \boxed{\begin{array}{c}\psi \\ \vdots \\ \chi\end{array}}}{\chi} \to \vee e$$

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例: Prove that $p \lor q \vdash q \lor p$ is valid

1	$p \lor q$	premise
2	p	assumption
3	$q \lor p$	$\lor i_2$ 2
4	q	assumption
5	$q \lor p$	$\lor i_1$ 4
6	$q \lor p$	$\lor e$ $1, 2-3, 4-5$

定义: Contradictions (矛盾)

Contradictions are expressions of the form $\phi \wedge \neg\phi$ or $\neg\phi \wedge \phi$, where ϕ is any formula.

定理

Any formula can be derived from a contradiction:

$$p \wedge \neg p \vdash q$$

定义: rules for negation: $\bot e, \neg e$

$$\frac{\bot}{\phi}\ \bot e \qquad \frac{\phi \quad \neg\phi}{\bot}\ \neg e$$

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例: Prove that $\neg p \vee q \vdash p \to q$ is valid

1	$\neg p \vee q$	premise
2	$\neg p$	assumption
3	p	assumption
4	$\bot$	$\neg e$ 3, 2
5	q	$\bot e$ 4
6	$p \to q$	$\to i$ 3 – 5
7	$p \to q$	$\vee e$ 1, 2 – 6

q	assumption
p	assumption
q	copy 2
$p \to q$	$\to i$ 3 – 4

定义: rules for negation: $\bot e, \neg e$

$$\frac{\bot}{\phi}\ \bot e \qquad\qquad \frac{\phi \quad \neg\phi}{\bot}\ \neg e$$

例: Prove that $\neg p \vee q \vdash p \to q$ is valid

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7	$p \to q$	$\vee e$ 1, 2 – 6

q	assumption
p	assumption
q	copy 2
$p \to q$	$\to i$ 3 – 4

定义: rule for negation: $\neg i$

$$\frac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \bot\end{array}}}{\neg\phi}\ \neg i$$

例: Prove that $p \to q, p \to \neg q \vdash \neg p$ is valid

1	$p \to q$	premise
2	$p \to \neg q$	premise
3	p	assumption
4	q	$\to e$ 1, 3
5	$\neg q$	$\to e$ 2, 3
6	$\bot$	$\neg e$ 4, 5
7	$\neg p$	$\neg i$ 3 – 6

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7	$\neg p$	$\neg i$ 3 – 6

Derived rule: MT

$$\frac{\phi \to \psi \quad \neg\psi}{\neg\phi}\ \text{MT}$$

例: Prove the derived rule

1	$\phi \to \psi$	premise
2	$\neg\psi$	premise
3	ϕ	assumption
4	ψ	$\to e\ 1, 3$
5	$\bot$	$\neg e\ 4, 2$
6	$\neg\phi$	$\neg i\ 3-5$

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5	$\bot$	$\neg e\ 4, 2$
6	$\neg\phi$	$\neg i\ 3-5$

Derived rule: $\neg\neg i$

$$\frac{\phi}{\neg\neg\phi}\ \neg\neg i$$

例: Prove the derived rule

1	ϕ	premise
2	$\neg\phi$	assumption
3	$\bot$	$\neg e\ 1, 2$
4	$\neg\neg\phi$	$\neg i\ 2-3$

Derived rule: $\neg\neg i$

$$\frac{\phi}{\neg\neg\phi}\ \neg\neg i$$

例: Prove the derived rule

1	ϕ	premise
2	$\neg\phi$	assumption
3	$\bot$	$\neg e\ 1, 2$
4	$\neg\neg\phi$	$\neg i\ 2-3$

Derived rule: PBC

$$\frac{\boxed{\begin{array}{c}\neg\phi\\ \vdots\\ \bot\end{array}}}{\phi}\ \text{PBC}$$

例: Prove the derived rule

1	$\neg\phi \to \bot$	given
2	$\neg\phi$	assumption
3	$\bot$	$\to e\ 1, 2$
4	$\neg\neg\phi$	$\neg i\ 2-3$
5	ϕ	$\neg\neg e\ 4$

Derived rule: PBC

$$\frac{\boxed{\begin{array}{c}\neg\phi\\ \vdots\\ \bot\end{array}}}{\phi}\ \text{PBC}$$

例: Prove the derived rule

1	$\neg\phi \to \bot$	given
2	$\neg\phi$	assumption
3	$\bot$	$\to e\ 1, 2$
4	$\neg\neg\phi$	$\neg i\ 2-3$
5	ϕ	$\neg\neg e\ 4$

Derived rule: LEM

$$\frac{}{\phi \vee \neg\phi}\ \text{LEM}$$

例: Prove the derived rule

1	$\neg(\phi \vee \neg\phi)$	assumption
2	ϕ	assumption
3	$\phi \vee \neg\phi$	$\vee i_1$ 2
4	$\bot$	$\neg e$ 3, 1
5	$\neg\phi$	$\neg i$ 2 – 4
6	$\phi \vee \neg\phi$	$\vee i_2$ 5
7	$\bot$	$\neg e$ 6, 1
8	$\neg\neg(\phi \vee \neg\phi)$	$\neg i$ 1 – 7
9	$\phi \vee \neg\phi$	$\neg\neg e$ 8

Derived rule: LEM

$$\frac{}{\phi \vee \neg\phi}\ \text{LEM}$$

例: Prove the derived rule

1	$\neg(\phi \vee \neg\phi)$	assumption
2	ϕ	assumption
3	$\phi \vee \neg\phi$	$\vee i_1\ 2$
4	$\bot$	$\neg e\ 3, 1$
5	$\neg\phi$	$\neg i\ 2-4$
6	$\phi \vee \neg\phi$	$\vee i_2\ 5$
7	$\bot$	$\neg e\ 6, 1$
8	$\neg\neg(\phi \vee \neg\phi)$	$\neg i\ 1-7$
9	$\phi \vee \neg\phi$	$\neg\neg e\ 8$

The basic rules of natural deduction:

	introduction	*elimination*
$\wedge$	$\dfrac{\phi \quad \psi}{\phi \wedge \psi}\ \wedge\mathrm{i}$	$\dfrac{\phi \wedge \psi}{\phi}\ \wedge\mathrm{e}_1 \qquad \dfrac{\phi \wedge \psi}{\psi}\ \wedge\mathrm{e}_2$
$\vee$	$\dfrac{\phi}{\phi \vee \psi}\ \vee\mathrm{i}_1 \qquad \dfrac{\psi}{\phi \vee \psi}\ \vee\mathrm{i}_2$	$\dfrac{\phi \vee \psi \quad \boxed{\begin{array}{c}\phi\\ \vdots\\ \chi\end{array}} \quad \boxed{\begin{array}{c}\psi\\ \vdots\\ \chi\end{array}}}{\chi}\ \vee\mathrm{e}$

支持 ⊨ 和 algorithms

$\rightarrow$

$$\frac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \psi\end{array}}}{\phi \rightarrow \psi}\ \rightarrow\mathrm{i} \qquad \frac{\phi \quad \phi \rightarrow \psi}{\psi}\ \rightarrow\mathrm{e}$$

$\neg$

$$\frac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \bot\end{array}}}{\neg\phi}\ \neg\mathrm{i} \qquad \frac{\phi \quad \neg\phi}{\bot}\ \neg\mathrm{e}$$

支持 $\vDash$ 和 algorithms

$\bot$	(no introduction rule for $\bot$)	$\dfrac{\bot}{\phi}\ \bot\mathrm{e}$
$\neg\neg$		$\dfrac{\neg\neg\phi}{\phi}\ \neg\neg\mathrm{e}$

Some useful derived rules:

$$\frac{\phi \to \psi \quad \neg\psi}{\neg\phi}\ \mathrm{MT} \qquad \frac{\phi}{\neg\neg\phi}\ \neg\neg\mathrm{i}$$

$$\frac{\boxed{\begin{array}{c}\neg\phi\\ \vdots\\ \bot\end{array}}}{\phi}\ \mathrm{PBC} \qquad \frac{}{\phi \vee \neg\phi}\ \mathrm{LEM}$$

Provable equivalence:

$$\begin{array}{ll} \neg(p \wedge q) \dashv\vdash \neg q \vee \neg p & \neg(p \vee q) \dashv\vdash \neg q \wedge \neg p \\ p \to q \dashv\vdash \neg q \to \neg p & p \to q \dashv\vdash \neg p \vee q \\ p \wedge q \to p \dashv\vdash r \vee \neg r & p \wedge q \to r \dashv\vdash p \to (q \to r). \end{array}$$

回顾：问题: $\vDash$ 求解复杂度过高
解决方法: New rules: a collection of *proof rules* in *natural deduction*.

- 不使用 Truth Tables 进行求解
- 定义并使用 proof rules
- 使用 proof rules 产生结论 (即 $\vdash$), 取代 $\vDash$, 即

$$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$$

新的问题:

1. rules 太多：推演过于复杂，符号也有冗余
   - 减少冗余的符号，设计自动推演算法 (见第 4 章)
2. 目前所给的是命题逻辑的 rule, 一阶逻辑会有哪些新的 rule?
   - $=, \forall, \exists$ 怎样设计它们的 rules (见下页)

支持 $\vDash$ 和 algorithms

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支持 $\vDash$ 和 algorithms

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支持 $\vDash$ 和 algorithms

问题 2: $=, \forall, \exists$ 怎样设计它们的 rules?

解决方法:

1. 预定义
   - 构建 Parse tree
   - 定义 Free and bound variables
   - 定义 Substitution
2. 设计 rules

支持 $\vDash$ 和 algorithms

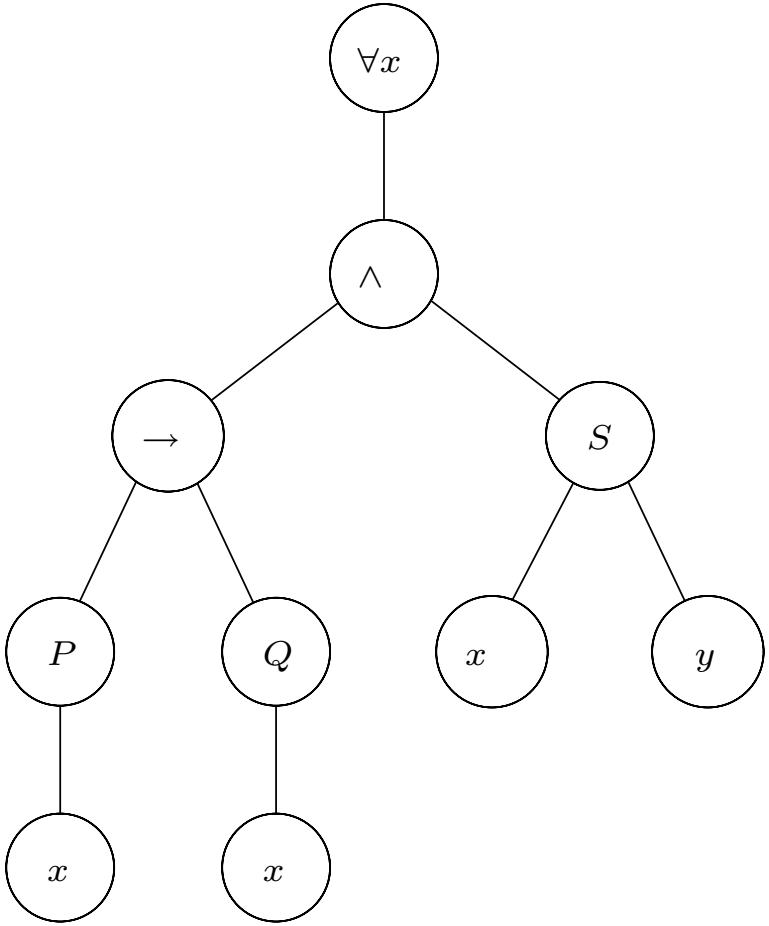
问题 2: $=, \forall, \exists$ 怎样设计它们的 rules?

解决方法:

1. 预定义
   - 构建 Parse tree
   - 定义 Free and bound variables
   - 定义 Substitution
2. 设计 rules

(1) 构建 Parse tree

- 例: $\forall x\ ((P(x) \to Q(x)) \wedge S(x, y))$

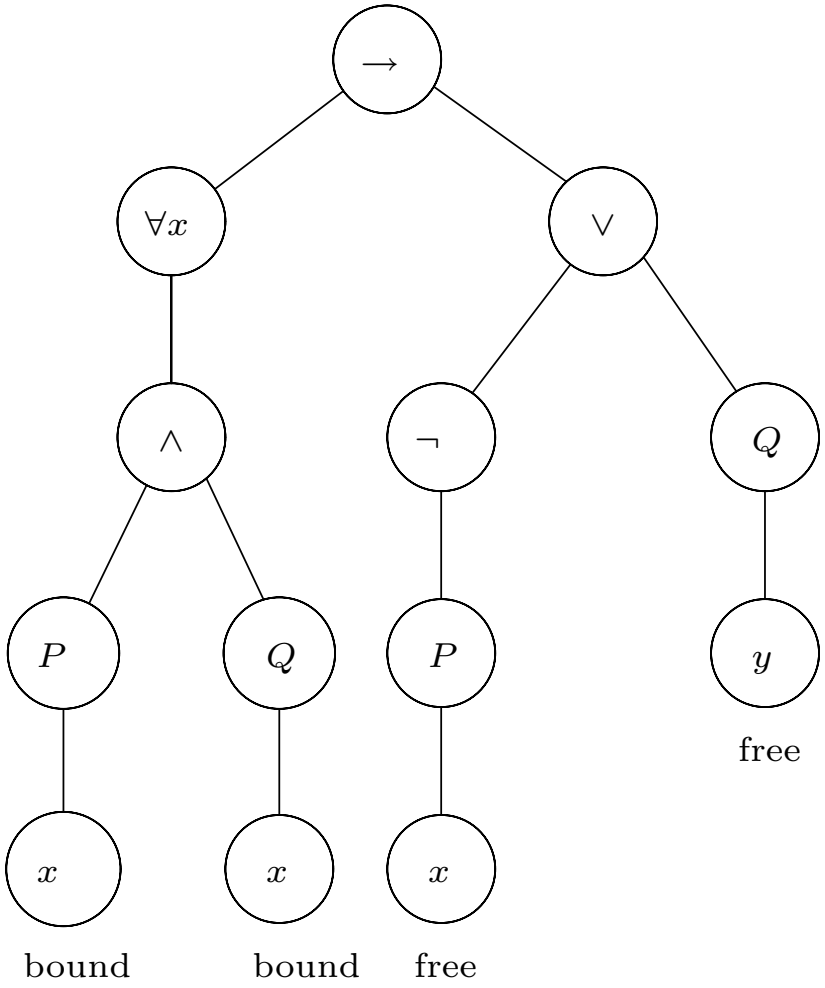


(2) 定义 Free and bound variables

定义: Free and bound variables

Let ϕ be a formula in predicate logic.

- An occurrence of x in ϕ is *free* in ϕ if it is a leaf node in the parse tree of ϕ such that there is no path upwards from that node x to a node $\forall x$ or $\exists x$.
- Otherwise, that occurrence of x is called *bound*.

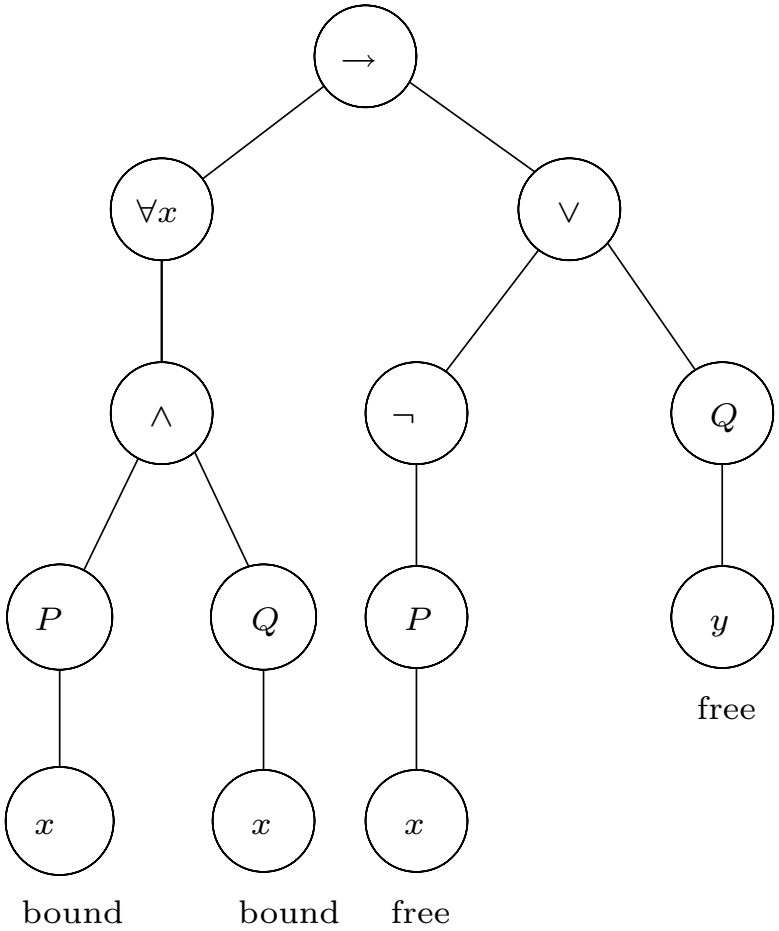


(3) 定义 Substitution

定义 Substitution

Given a variable x, a term t and a formula ϕ, define $\phi[t/x]$ to be the formula obtained by replacing each *free* occurrence of variable x in ϕ with t.

例: x replaced by the term $f(x, y)$



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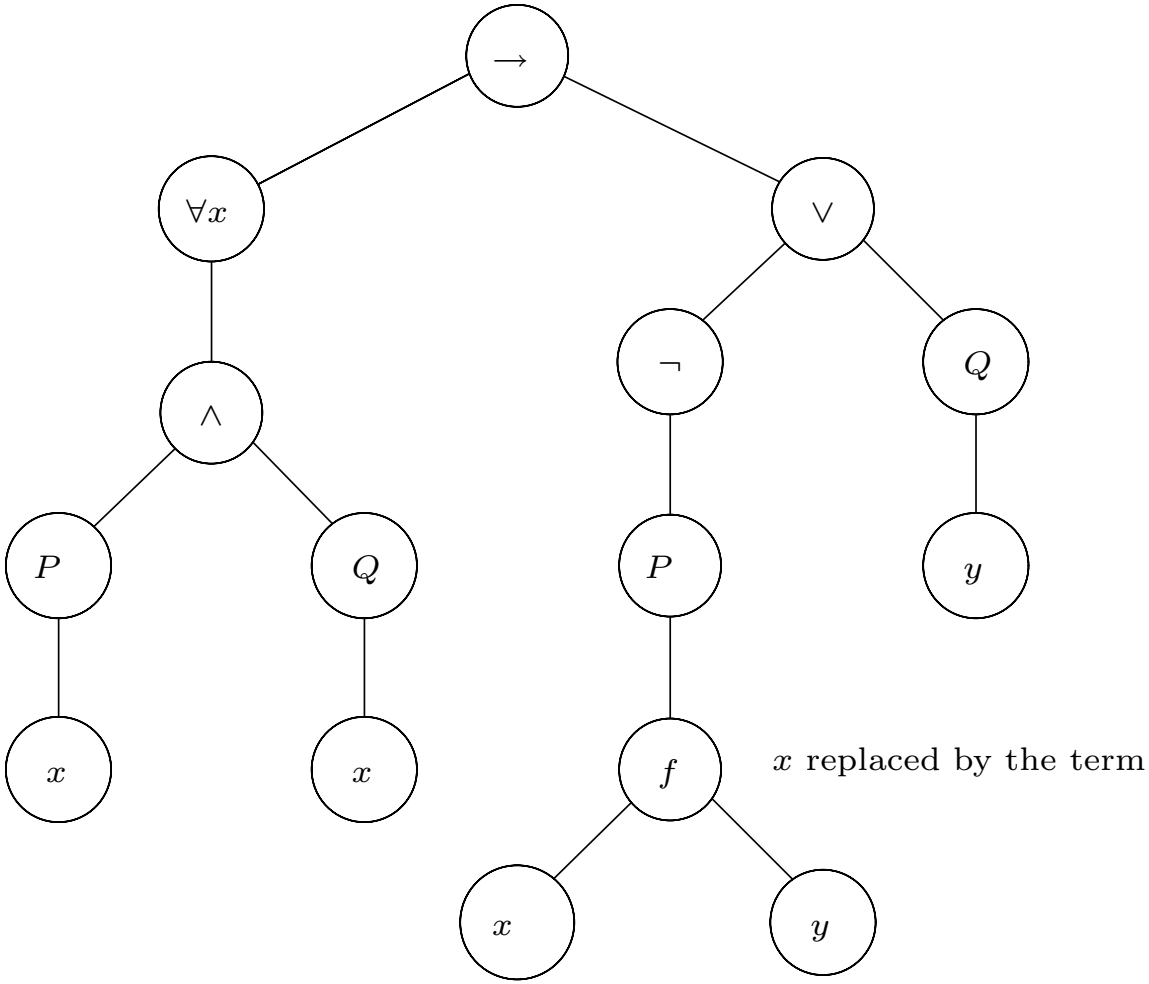

定义: rules for equality: $= i$

$$\frac{}{t = t} \quad = i \qquad\qquad \frac{t_1 = t_2 \quad \phi[t_1/x]}{\phi[t_2/x]} \quad = e$$

例: Prove the validity of the sequent:
$x+1 = 1+x, (x+1 > 1) \to (x+1 > 0) \vdash (1+x) > 1 \to (1+x) > 0$

1	$(x+1) = (1+x)$	premise
2	$(x+1 > 1) \to (x+1 > 0)$	premise
3	$(1+x > 1) \to (1+x > 0)$	$= e$ 1, 2

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例: Prove (symmetric relation 对称性): $t_1 = t_2 \vdash t_2 = t_1$

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支持 $\vDash$ 和 algorithms

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例: Prove (transive relation 传递性): $t_1 = t_2, t_2 = t_3 \vdash t_1 = t_3$

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定义: rules for universal quantification: $\forall x\ e, \forall x\ i$

$$\frac{\forall x\ \phi}{\phi[t/x]}\ \forall x\ e$$

$$\frac{\boxed{\begin{array}{l} x_0 \\ \quad\vdots \\ \quad \phi[x_0/x] \end{array}}}{\forall x\ \phi}\ \forall x\ i$$

例: Prove: $\forall x\ (P(x) \to Q(x)), \forall x\ P(x) \vdash \forall x\ Q(x)$

1		$\forall x\ (P(x) \to Q(x))$	premise
2		$\forall x\ P(x)$	premise
3	x_0	$P(x_0) \to Q(x_0)$	$\forall x\ e\ 1$
4		$P(x_0)$	$\forall x\ e\ 2$
5		$Q(x_0)$	$\to e\ 3, 4$
6		$\forall x\ Q(x)$	$\forall x\ i\ 3-5$

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例: Prove: $P(t), \forall x\ (P(x) \to \neg Q(x)) \vdash \neg Q(t)$

1	$P(t)$	premise
2	$\forall x\ (P(x) \to \neg Q(x))$	premise
3	$P(t) \to \neg Q(t)$	$\forall x\ e\ 2$
4	$\neg Q(t)$	$\to e\ 3, 1$

定义: rules for universal quantification: $\forall x\ e, \forall x\ i$

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3	$P(t) \to \neg Q(t)$	$\forall x\ e\ 2$
4	$\neg Q(t)$	$\to e\ 3, 1$

定义: rules for existential quantification: $\exists x\ i, \exists e$

$$\frac{\phi[t/x]}{\exists x\ \phi}\ \exists x\ i \qquad \frac{\exists x\ \phi \quad \boxed{\begin{array}{c} x_0 \quad \phi[x_0/x] \\ \vdots \\ \chi \end{array}}}{\chi}\ \exists e$$

例: Prove: $\forall x\ \phi \vdash \exists x\ \phi$

1	$\forall x\ \phi$	premise
2	$\phi[x/x]$	$\forall x\ e\ 1$
3	$\exists x\ \phi$	$\exists x\ i\ 2$

定义: rules for existential quantification: $\exists x\ i, \exists e$

$$\frac{\phi[t/x]}{\exists x\ \phi}\ \exists x\ i$$

$$\frac{\exists x\ \phi \quad \boxed{\begin{array}{ll} x_0 & \phi[x_0/x] \\ & \vdots \\ & \chi \end{array}}}{\chi}\ \exists e$$

例: Prove: $\forall x\ \phi \vdash \exists x\ \phi$

1	$\forall x\ \phi$	premise
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例: Prove: $\forall x\ (P(x) \to Q(x)), \exists x\ P(x) \vdash \exists x\ Q(x)$

1		$\forall x\ (P(x) \to Q(x))$	premise
2		$\exists x\ P(x)$	premise
3	x_0	$P(x_0)$	assumption
4		$P(x_0) \to Q(x_0)$	$\forall x\ e\ 1$
5		$Q(x_0)$	$\to e\ 4, 3$
6		$\exists x\ Q(x)$	$\exists x\ i\ 5$
7		$\exists x\ Q(x)$	$\exists x\ e\ 2, 3-6$

支持 $\vDash$ 和 algorithms

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Quantifier equivalence

- $\forall$ and $\exists$
  - $\neg\forall x\ \phi \dashv\vdash \exists x\ \neg\phi$
  - $\neg\exists x\ \phi \dashv\vdash \forall x\ \neg\phi$
- $\wedge$ and $\vee$
  - $\forall x\ \phi \wedge \forall x\ \psi \dashv\vdash \forall x(\phi \wedge \psi)$
  - $\forall x\ \phi \vee \forall x\ \psi \dashv\vdash \forall x(\phi \vee \psi)$
- double $\forall$ or $\exists$
  - $\forall x\forall y\ \phi \dashv\vdash \forall y\forall x\ \phi$
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  - 答: 同 $\vDash$, 即 The decision problem of validity in predicate logic is undecidable.
- 问题 3: $\vdash$ 演算求解复杂度相对 $\vDash$ 降低了么?
  - 答: 看起来是的
- 问题 3.1: 怎样设计算法，以提升效率?
  - 见下章，求解算法的使用与实现

支持 $\vDash$ 和 algorithms

问题: $\vdash$ 演算 (calculus) 方法好用么？

- 问题 1: $\vdash$ 和 $\vDash$ 的演算结果是否相同?
  - 答: 是的，即 Soundness and Completeness
  - 一种表达形式: $\vDash \phi$ iff $\vdash \phi$
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作业

1. Prove the validity of the following sequents:

1. $(p \wedge q) \wedge r, s \wedge t \vdash q \wedge s$
2. $q \to r \vdash (p \to q) \to (p \to r)$
3. $\vdash q \to (p \to (p \to (q \to p)))$
4. $p \to q \wedge r \vdash (p \to q) \wedge (p \to r)$
5. $p \wedge \neg p \vdash \neg(r \to q) \wedge (r \to q)$

2. Prove the validity of the following sequents in predicate logic, where P, and Q have arity 1, and S has arity 0 (a 'propositional atom'):

1. $\exists x(S \to Q(x)) \vdash S \to \exists x\ Q(x)$
2. $\forall x P(x) \to S \vdash \exists x(P(x) \to S)$
3. $\forall x(P(x) \wedge Q(x)) \vdash \forall x P(x) \wedge \forall x Q(x)$
4. $\neg\forall x \neg P(x) \vdash \exists x P(x)$
5. $\forall x \neg P(x) \vdash \neg\exists x P(x)$