第2章 热物理问题的数学描述与偏微分方程的分类

胡 茂 彬

http://staff.ustc.edu.cn/~humaobin/
humaobin@ustc.edu.cn

内容提要

- 热物理问题的控制方程
- 方程的物理分类
- 方程的数学分类

2.1 热物理问题的控制方程

- 热物理过程：流动、传热、传质、燃烧

- 三个基本的守恒定律：

  质量守恒

  动量守恒

  能量守恒

关于场论的一点复习

梯度
散度
旋度

2.1.1 连续方程

单位时间系统的质量增量 = 流入系统的净质量 + 源项

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \boldsymbol{U}) = 0$$

2.1.2 动量方程 (牛顿流体)

单位时间内系统动量增量=外界流入净动量流率 + 体系外力之和

$$\frac{\partial \rho u}{\partial t} + \mathrm{div}(\rho u \boldsymbol{U}) = \mathrm{div}(\mu\,\mathrm{grad}\,u) - \frac{\partial p}{\partial x} + S_x + B_x$$

$$\frac{\partial \rho v}{\partial t} + \mathrm{div}(\rho v \boldsymbol{U}) = \mathrm{div}(\mu\,\mathrm{grad}\,v) - \frac{\partial p}{\partial y} + S_y + B_y$$

$$\frac{\partial \rho w}{\partial t} + \mathrm{div}(\rho w \boldsymbol{U}) = \mathrm{div}(\mu\,\mathrm{grad}\,w) - \frac{\partial p}{\partial z} + S_z + B_z$$

分子粘性之外的粘性力广义源项

$$S_x = \frac{\partial}{\partial x}(\mu\frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(\mu\frac{\partial v}{\partial x}) + \frac{\partial}{\partial z}(\mu\frac{\partial w}{\partial x}) - \frac{2}{3}\frac{\partial}{\partial x}\mathrm{div}(\mu\boldsymbol{U})$$

$$S_y = \frac{\partial}{\partial x}(\mu\frac{\partial u}{\partial y}) + \frac{\partial}{\partial y}(\mu\frac{\partial v}{\partial y}) + \frac{\partial}{\partial z}(\mu\frac{\partial w}{\partial y}) - \frac{2}{3}\frac{\partial}{\partial y}\mathrm{div}(\mu\boldsymbol{U})$$

$$S_z = \frac{\partial}{\partial x}(\mu\frac{\partial u}{\partial z}) + \frac{\partial}{\partial y}(\mu\frac{\partial v}{\partial z}) + \frac{\partial}{\partial z}(\mu\frac{\partial w}{\partial z}) - \frac{2}{3}\frac{\partial}{\partial z}\mathrm{div}(\mu\boldsymbol{U})$$

2.1.3 能量方程（用内能e表示）

单位时间内体系总能量增量 = 净能量流入率 + 表面力做功 + 体积力做功

$$\frac{\partial \rho e}{\partial t}+\mathrm{div}(\rho \boldsymbol{U} e)+p\,\mathrm{div}(\boldsymbol{U})=\mathrm{div}(\lambda\,\mathrm{grad}\,T)+\Phi+S_{in}$$

其中粘性耗散：

$$\Phi=2\mu[(\frac{\partial u}{\partial x})^2+(\frac{\partial v}{\partial y})^2+(\frac{\partial w}{\partial z})^2+\frac{1}{2}(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x})^2+\frac{1}{2}(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y})^2+\frac{1}{2}(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z})^2]-\frac{2}{3}\mu(\mathrm{div}\,\boldsymbol{U})^2$$

用焓h表示

$$\frac{\partial \rho e}{\partial t} + \mathrm{div}(\rho \boldsymbol{U} e) + p\,\mathrm{div}(\boldsymbol{U}) = \mathrm{div}(\lambda\,\mathrm{grad}\,T) + \Phi + S_{in}$$

$$h = e + \frac{p}{\rho}$$

$$\frac{\partial \rho h}{\partial t} + \mathrm{div}(\rho \boldsymbol{U} h) = \frac{Dp}{Dt} + \mathrm{div}(\lambda\,\mathrm{grad}\,T) + \Phi + S_h$$

用温度T表示

$$\frac{\partial \rho T}{\partial t} + \mathrm{div}(\rho \boldsymbol{U} T) = \frac{1}{c_p}\frac{Dp}{Dt} + \mathrm{div}(\frac{\lambda}{c_p}\,\mathrm{grad}\,T) + S_T$$

对于固体和不可压流体： $\mathrm{div}(\boldsymbol{U}) = 0$

$$\frac{\partial T}{\partial t} + \mathrm{div}(\boldsymbol{U} T) = \mathrm{div}(\frac{\lambda}{\rho c}\,\mathrm{grad}\,T) + \frac{S_T}{\rho}$$

2.1.4 化学组分方程

单位时间内某化学组分增量=对流、扩散净流率 + 化学反应产生率

$$\frac{\partial \rho m_l}{\partial t} + \mathrm{div}(\rho m_l \boldsymbol{U}) = \mathrm{div}(\Gamma_l \,\mathrm{grad}\, m_l) + R_l$$

约束条件：

$$\sum_l m_l = 1$$

$$\sum_l \Gamma_l \,\mathrm{grad}(m_l) = 0$$

$$\sum_l R_l = 0$$

2.1.5 控制方程的通用形式

$$\frac{\partial \rho \phi}{\partial t} + \operatorname{div}(\rho \boldsymbol{U} \phi) = \operatorname{div}(\Gamma_{\phi} \operatorname{grad} \phi) + S$$

非稳项　　对流项　　扩散项　　源项