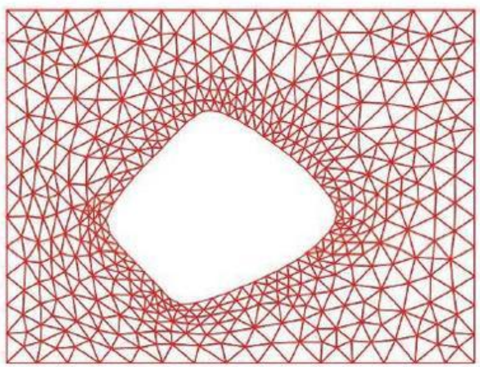
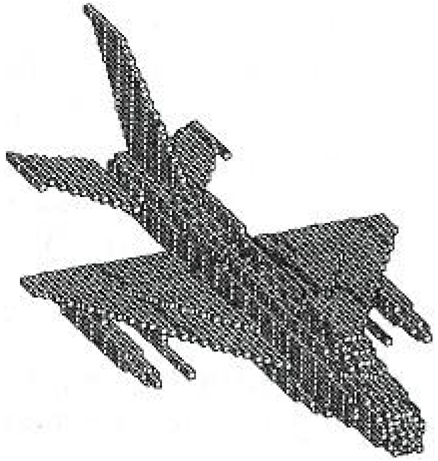
3.2.4 其它构成导数有限差分离散的方法





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1.多项式拟合求解函数构造函数导数的差分格式

用关于求解函数的

拟合多项式来逼近函数

A) 线性拟合

$$u(x,t_n) = a + bx$$

- 令 j 点坐标为 0

$$u_{j-1}^n = a - b\Delta x \qquad u_j^n = a \qquad u_{j+1}^n = a + b\Delta x$$

$$\left(\frac{\partial u}{\partial x}\right)_j^n = b = \frac{u_j^n - u_{j-1}^n}{\Delta x} \qquad \left(\frac{\partial u}{\partial x}\right)_j^n = b = \frac{u_{j+1}^n - u_j^n}{\Delta x}$$

B) 二次拟合

$$u(x,t_n) \cong a + bx + cx^2$$

- 令 j 点坐标为 0

$$u_{j-1}^n = a - b\Delta x + c\Delta x^2 \qquad u_j^n = a \qquad u_{j+1}^n = a + b\Delta x + c\Delta x^2$$

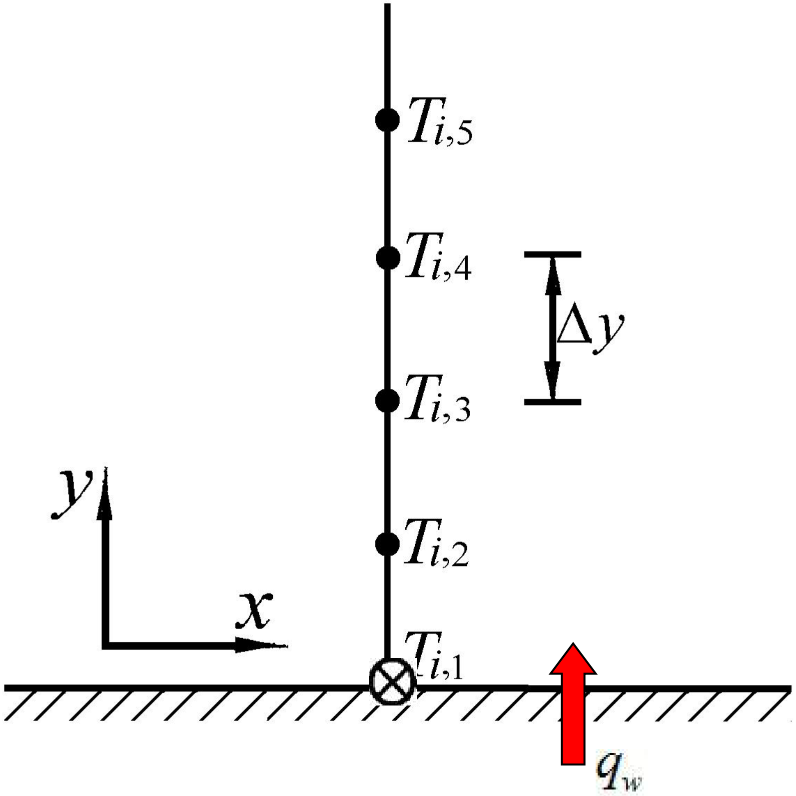
$$\left(\frac{\partial u}{\partial x}\right)_j^n = b = \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} \qquad \left(\frac{\partial^2 u}{\partial x^2}\right)_j^n = 2c = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\cancel{2}\Delta x^2}$$

多项式拟合方法在处理边界条件中的应用

例3.4 二维导热区域固壁边界问题

- 已知：热流 (定义输入壁面热流为正)，内部温度

- 未知：壁面温度



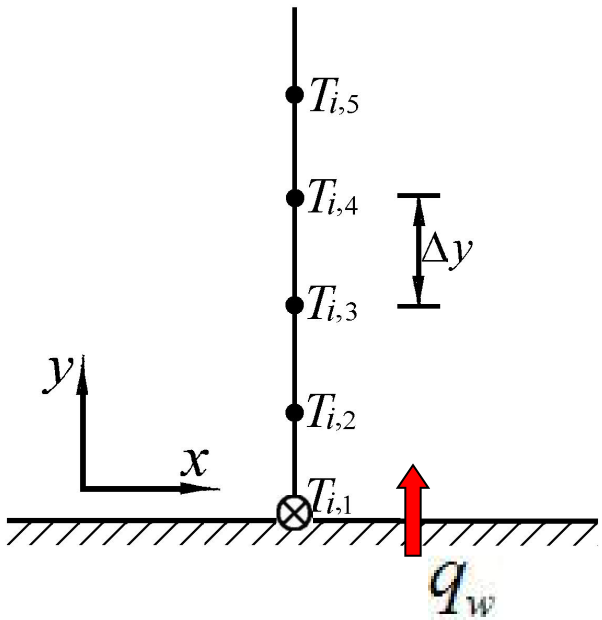
壁面附近温度二次拟合

$$T(x_i, y) \cong a + by + cy^2$$

- 壁面坐标为 0，向上为正方向

$$\begin{cases} T_{i,1} = a \\ T_{i,2} = a + b\Delta y + c\Delta y^2 \\ T_{i,3} = a + 2b\Delta y + 4c\Delta y^2 \end{cases}$$

$$a = T_{i,1}, \quad b = \frac{-3T_{i,1} + 4T_{i,2} - T_{i,3}}{2\Delta y}, \quad c = \frac{T_{i,1} - 2T_{i,2} + T_{i,3}}{2\Delta y^2}$$



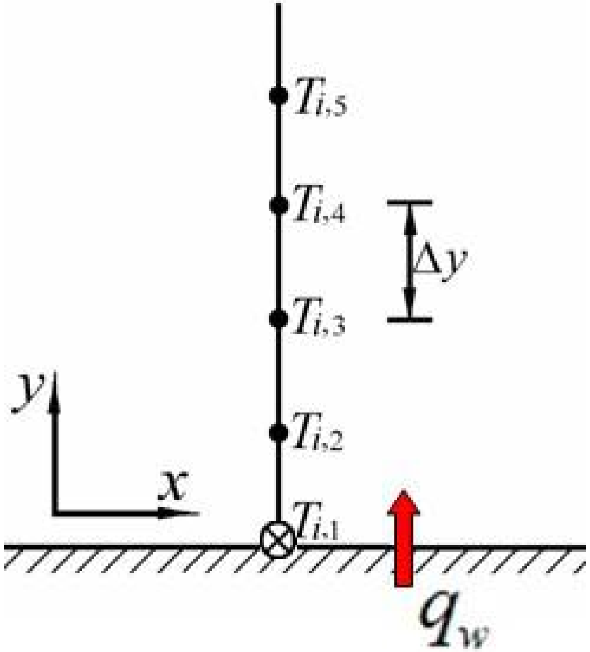
由热流确定壁面温度

$$T(x_i, y) \cong a + by + cy^2$$

$$q_w = -\lambda \frac{\partial T}{\partial y}\bigg|_{y=0} \cong -\lambda b = \frac{\lambda}{2\Delta y}(3T_{i,1} - 4T_{i,2} + T_{i,3}) \qquad O(\Delta y^2)$$

$$T_{i,1} = \frac{1}{3}\left(4T_{i,2} - T_{i,3} - \frac{2\Delta y q_w}{\lambda}\right)$$

$$O(\Delta y^3)$$



另一个问题：由壁面温度求热流

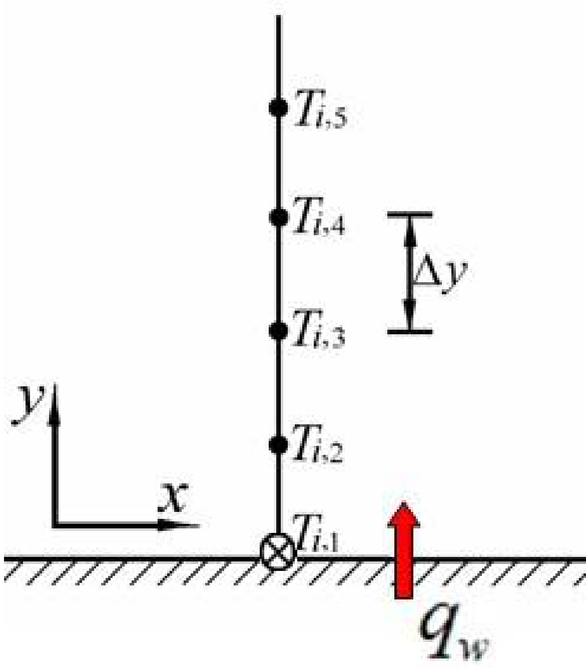
- 三次多项式拟合温度

$$T(x_i, y) \cong a + by + cy^2 + dy^3$$

- 热流表达式

$$q_w = -\lambda \frac{\partial T}{\partial y}\bigg|_{y=0} \cong -\lambda b$$

$$q_w = \frac{\lambda(11T_{i,1} - 18T_{i,2} + 9T_{i,3} - 2T_{i,4})}{6\Delta y}$$



2 预估—校正多步法构成差分格式

第一步称为预估步

其后各步称为校正步

A 交替方向隐式离散格式
(ADI-Alternative Direction Implicit)

- 例：2维无源非稳态导热方程

$$\frac{\partial T}{\partial t}=\sigma\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right) \qquad \sigma>0$$

- Peaceman-Rachford：

  分两步进行，每步推进 $\Delta t/2$

  1) 对 x 方向取隐式，对 y 方向取显式

  2) 对 y 方向取隐式，对 x 方向取显式

1).

$$\frac{T^{*}_{i,j}-T^{n}_{i,j}}{\Delta t/2}=\sigma\left(\frac{\delta_x^2 T^{*}_{i,j}}{\Delta x^2}+\frac{\delta_y^2 T^{n}_{i,j}}{\Delta y^2}\right)$$

$$=\sigma\left(\frac{T^{*}_{i+1,j}-2T^{*}_{i,j}+T^{*}_{i-1,j}}{\Delta x^2}+\frac{T^{n}_{i,j+1}-2T^{n}_{i,j}+T^{n}_{i,j-1}}{\Delta y^2}\right)$$

2).

$$\frac{T^{n+1}_{i,j}-T^{*}_{i,j}}{\Delta t/2}=\sigma\left(\frac{\delta_x^2 T^{*}_{i,j}}{\Delta x^2}+\frac{\delta_y^2 T^{n+1}_{i,j}}{\Delta y^2}\right)$$

$$=\sigma\left(\frac{T^{*}_{i+1,j}-2T^{*}_{i,j}+T^{*}_{i-1,j}}{\Delta x^2}+\frac{T^{n+1}_{i,j+1}-2T^{n+1}_{i,j}+T^{n+1}_{i,j-1}}{\Delta y^2}\right)$$

3维的情况- 须做调整以使计算稳定

- Brain

$$\frac{T^{*}_{i,j,k}-T^{n}_{i,j,k}}{\Delta t/2}=\sigma\left(\frac{\delta_x^2 T^{*}_{i,j,k}}{\Delta x^2}+\frac{\delta_y^2 T^{n}_{i,j,k}}{\Delta y^2}+\frac{\delta_z^2 T^{n}_{i,j,k}}{\Delta z^2}\right)$$

$$\frac{T^{**}_{i,j,k}-T^{n}_{i,j,k}}{\Delta t/2}=\sigma\left(\frac{\delta_x^2 T^{*}_{i,j,k}}{\Delta x^2}+\frac{\delta_y^2 T^{**}_{i,j,k}}{\Delta y^2}+\frac{\delta_z^2 T^{n}_{i,j,k}}{\Delta z^2}\right)$$

$$\frac{T^{n+1}_{i,j,k}-T^{**}_{i,j,k}}{\Delta t/2}=\sigma\left(\frac{\delta_x^2 T^{*}_{i,j,k}}{\Delta x^2}+\frac{\delta_y^2 T^{**}_{i,j,k}}{\Delta y^2}+\frac{\delta_z^2 T^{n+1}_{i,j,k}}{\Delta z^2}\right)$$

Douglas 提出的 3D 恒稳格式

$$\frac{T^{*}_{i,j,k}-T^{n}_{i,j,k}}{\Delta t}=\sigma\left(\frac{\delta_x^2(T^{*}_{i,j,k}+T^{n}_{i,j,k})}{2\Delta x^2}+\frac{\delta_y^2 T^{n}_{i,j,k}}{\Delta y^2}+\frac{\delta_z^2 T^{n}_{i,j,k}}{\Delta z^2}\right)$$

$$\frac{T^{**}_{i,j,k}-T^{n}_{i,j,k}}{\Delta t}=\sigma\left(\frac{\delta_x^2(T^{*}_{i,j,k}+T^{n}_{i,j,k})}{2\Delta x^2}+\frac{\delta_y^2(T^{**}_{i,j,k}+T^{n}_{i,j,k})}{2\Delta y^2}+\frac{\delta_z^2 T^{n}_{i,j,k}}{\Delta z^2}\right)$$

$$\frac{T^{n+1}_{i,j,k}-T_{i,j,k}}{\Delta t}=\sigma\left(\frac{\delta_x^2(T^{*}_{i,j,k}+T^{n}_{i,j,k})}{2\Delta x^2}+\frac{\delta_y^2(T^{**}_{i,j,k}+T^{n}_{i,j,k})}{2\Delta y^2}+\frac{\delta_z^2(T^{n+1}_{i,j,k}+T^{n}_{i,j,k})}{2\Delta z^2}\right)$$

加上原来值，取平均。

B MacCormack 格式

这是一类格式，基于预估-校正思想

- 例：匀流方程 $\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}=0 \qquad (a<0)$

- 预估步：FTFS - 体现迎风思想

$$u_j^{\overline{n+1}}=u_j^n-c(u_{j+1}^n-u_j^n)$$

- 校正步：FTBS，并与原始值取平均

$$u_j^{n+1}=\frac{1}{2}\left[u_j^n+u_j^{\overline{n+1}}-c\left(u_j^{\overline{n+1}}-u_{j-1}^{\overline{n+1}}\right)\right]$$

原始值

评述

- 该方法构成格式在预估步必须考虑迎风性
- 尽管每一步的精度都是一阶，两步合起来的最终精度为全二阶
- 该格式广泛应用于流动和传热问题的对流项离散

3. 分裂差分离散方法

与预估校正方法的区别是：

预估校正方法基于原方程离散

而分裂差分方法改造原方程再进行离散

例：一维线性对流扩散方程

$$\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}=\sigma\frac{\partial^2 u}{\partial x^2}$$

a，σ 均为大于零的常数

$$\begin{cases}\dfrac{\partial u}{\partial t}=\sigma\dfrac{\partial^2 u}{\partial x^2}\\[2ex]\dfrac{\partial u}{\partial t}+a\dfrac{\partial u}{\partial x}=0\end{cases}$$

$$\begin{cases}\dfrac{1}{2}\dfrac{\partial u}{\partial t}=\sigma\dfrac{\partial^2 u}{\partial x^2}\\[2ex]\dfrac{1}{2}\dfrac{\partial u}{\partial t}+a\dfrac{\partial u}{\partial x}=0\end{cases}$$

$$\begin{cases} \dfrac{\partial u}{\partial t} = \sigma \dfrac{\partial^2 u}{\partial x^2} \\ \dfrac{\partial u}{\partial t} + a\dfrac{\partial u}{\partial x} = 0 \end{cases} \Longrightarrow \begin{cases} \dfrac{u_j^* - u_j^n}{\Delta t} = \sigma\left(\dfrac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}\right) \\ \dfrac{u_j^{n+1} - u_j^*}{\Delta t} = -a\left(\dfrac{u_{j-2}^* - 4u_{j-1}^* + 3u_j^*}{2\Delta x}\right) \end{cases}$$

- 二者等价

$$\begin{cases} \dfrac{1}{2}\dfrac{\partial u}{\partial t} = \sigma \dfrac{\partial^2 u}{\partial x^2} \\ \dfrac{1}{2}\dfrac{\partial u}{\partial t} + a\dfrac{\partial u}{\partial x} = 0 \end{cases} \Longrightarrow \begin{cases} \dfrac{1}{2}\dfrac{u_j^{n+\frac{1}{2}} - u_j^n}{\Delta t/2} = \sigma\left(\dfrac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}\right) \\ \dfrac{1}{2}\dfrac{u_j^{n+1} - u_j^{n+\frac{1}{2}}}{\Delta t/2} = -a\left(\dfrac{u_{j-2}^{n+\frac{1}{2}} - 4u_{j-1}^{n+\frac{1}{2}} + 3u_j^{n+\frac{1}{2}}}{2\Delta x}\right) \end{cases}$$

例：二维非定常线性热传导方程

$$\frac{\partial T}{\partial t} = \sigma\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \qquad \sigma > 0$$

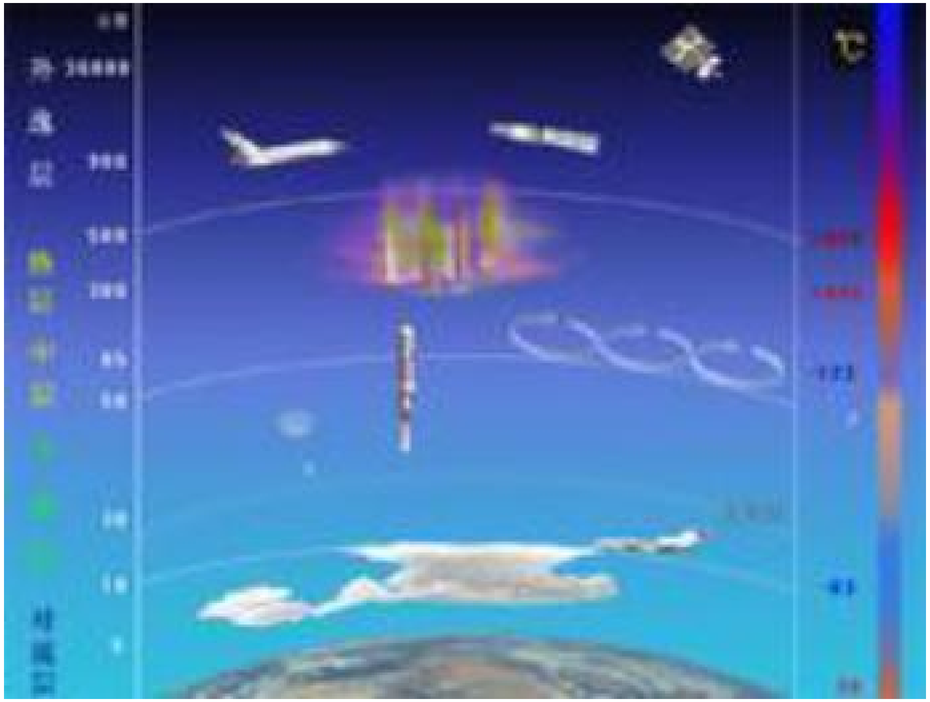
和分裂

$$\begin{cases} T_t = \sigma T_{xx} \\ T_t = \sigma T_{yy} \end{cases}$$

离散化

$$\begin{cases} T_{i,j}^{*} = T_{i,j}^{n} + \alpha_x (T_{i+1,j}^{*} - 2T_{i,j}^{*} + T_{i-1,j}^{*}) \\ T_{i,j}^{n+1} = T_{i,j}^{*} + \alpha_y (T_{i,j+1}^{n+1} - 2T_{i,j}^{n+1} + T_{i,j+1}^{n+1}) \end{cases}$$

4. 半离散化方法



适用于高超音速飞行等对时间导数
离散精度要求非常高的问题

Motivation

- 前差、后差格式精度太低

- 中心差分格式，需要三个甚至更多个时层的离散方程，计算起步困难

- 方法：对时间导数项不离散，而只离散空间导数项，最终解常微分方程

例：一维匀流方程

$$\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}=0 \qquad (a>0)$$

取迎风格式

$$\frac{du_j(t)}{dt}=-\frac{a}{\Delta x_i}\left[u_j(t)-u_{j-1}(t)\right]$$