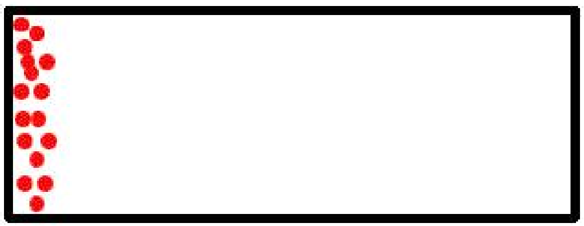
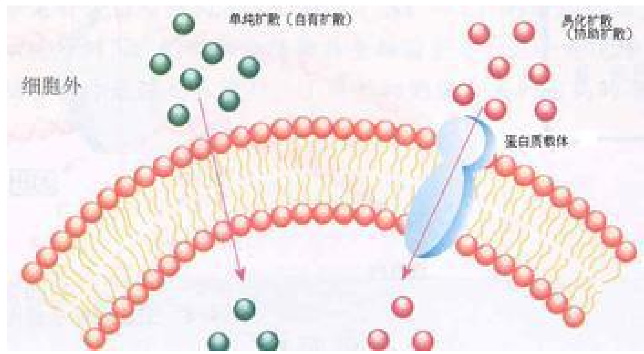
4.2 多维导热问题的数值解法





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主要内容

1 非稳态二维导热方程的全隐式离散

2 非稳态三维导热方程的全隐式离散

3 边界条件处理

4 线化代数方程组的迭代解法

此外没有讲到、但仍然需要的有：

5 源项线化

6 界面当量导热系数的确定

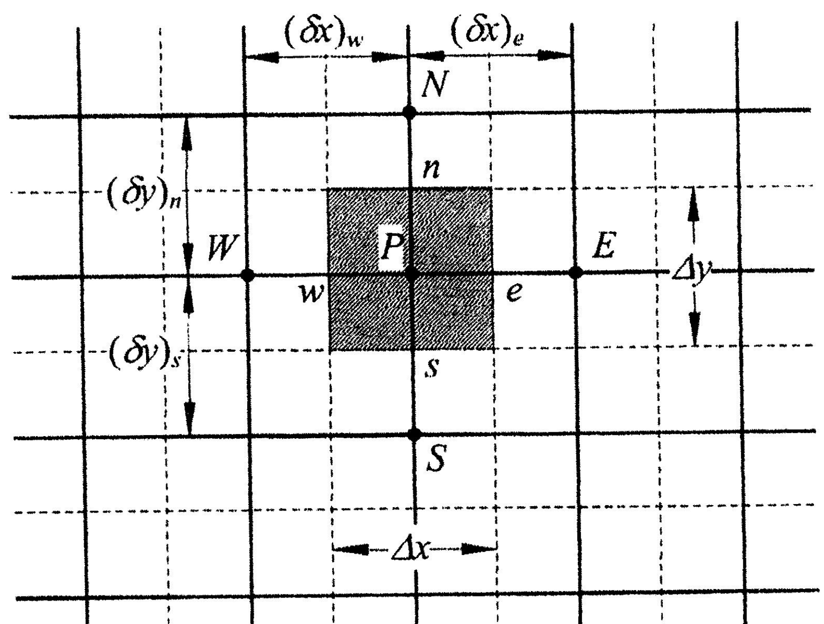
7方程非线性性质的处理

4.2.1 非稳态二维导热方程的全隐式离散 (直角坐标系)

- 二维导热方程

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda \frac{\partial T}{\partial y}\right) + S$$

控制体



第1步积分

须对x,y方向分布
取假设

须对y，t方向
取假设

$$\int_s^n \int_w^e \int_t^{t+\Delta t} \rho c \frac{\partial T}{\partial t} dt dx dy = \int_t^{t+\Delta t} \int_s^n \int_w^e \frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right) dx dy dt$$

$$+ \int_t^{t+\Delta t} \int_w^e \int_s^n \frac{\partial}{\partial y}\left(\lambda \frac{\partial T}{\partial y}\right) dy dx dt + \int_s^n \int_w^e \int_t^{t+\Delta t} S dx dy dt$$

须对x，t方向
取假设

须对x，y，t方向
取假设
源项需要线化

第2步积分

全隐式：积分时 阶梯分布取积分上限的值

$$\rho c(T_P - T_P^0)\ \Delta x \Delta y = \left[\lambda_e \frac{T_E - T_P}{(\delta x)_e} - \lambda_w \frac{T_P - T_W}{(\delta x)_w}\right]\Delta y \Delta t$$

$$+\left[\lambda_n \frac{T_N - T_P}{(\delta y)_n} - \lambda_s \frac{T_P - T_S}{(\delta y)_s}\right]\Delta x \Delta t + \underline{(S_C + S_P T_P)}\Delta x \Delta y \Delta t$$

线化源项

界面当量导热系数

- 整理成通用形式：

$$a_P T_P = a_E T_E + a_W T_W + a_N T_N + a_S T_S + b$$

系数

导热面积

$$a_E = \frac{\lambda_e \Delta y}{(\delta x)_e},\quad a_W = \frac{\lambda_w \Delta y}{(\delta x)_w},\quad a_N = \frac{\lambda_n \Delta x}{(\delta y)_n},\quad a_S = \frac{\lambda_s \Delta x}{(\delta y)_s}$$

距离

$$a_P = a_E + a_W + a_N + a_S + a_P^0 - S_P \Delta x \Delta y,$$

$$a_P^0 = \frac{\rho c \Delta x \Delta y}{\Delta t},\qquad b = S_C \Delta x \Delta y + a_P^0 T_P^0$$

控制体体积

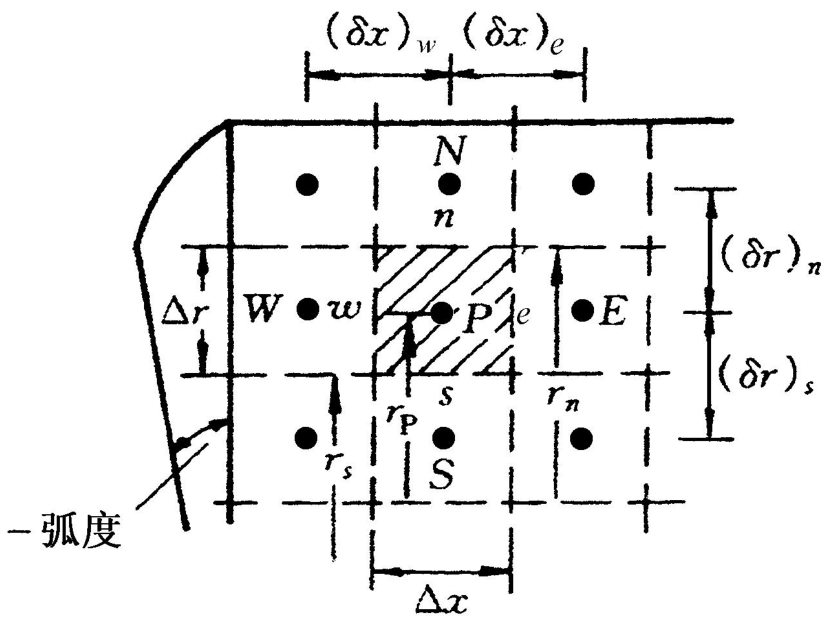
稳态情况是非稳态情况

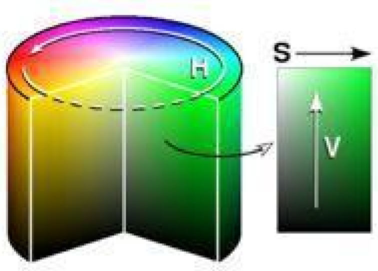
当 $\Delta t \to \infty$ 的特殊情况

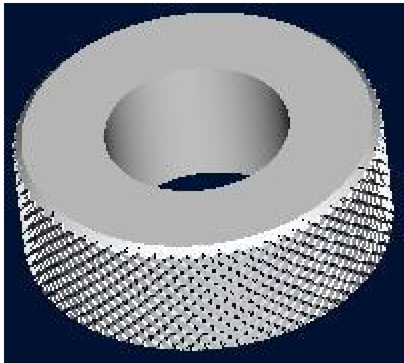
$$\Delta t \to \infty \xrightarrow{\; a_P^0 = \dfrac{\rho c \Delta x \Delta y}{\Delta t} \;} a_P^0 \to 0$$

2.轴对称圆柱坐标系

- 变量：轴向x，径向r







控制体体积

$$\Delta V = \left(r_n + r_s\right)\Delta r \Delta x / 2$$

控制方程

$$\rho c\frac{\partial T}{\partial t}=\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(r\lambda\frac{\partial T}{\partial r}\right)+S$$

Lame因子

离散结果

$$a_P T_P = a_E T_E + a_W T_W + a_N T_N + a_S T_S + b$$

系数形式不同:

多了Lame系数

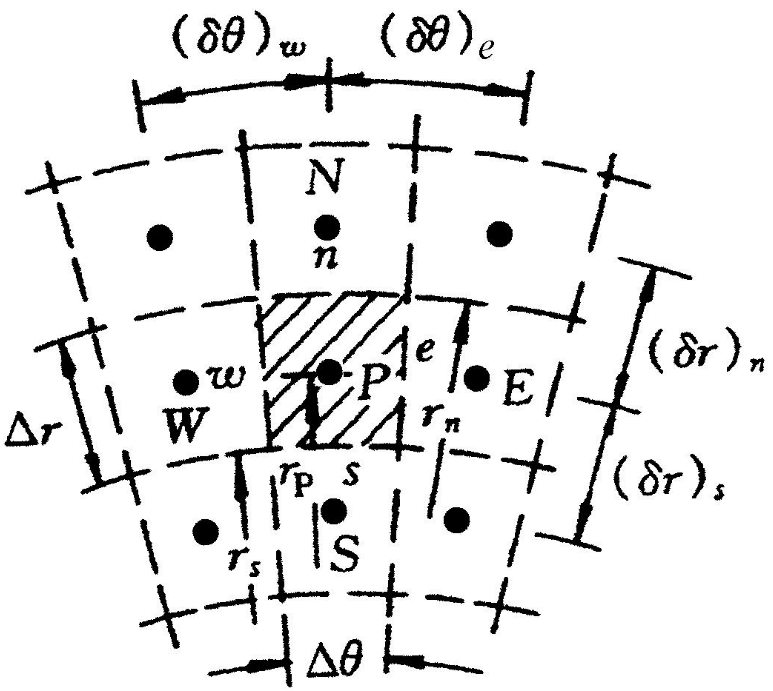
$$a_E = \frac{\lambda_e r_P \Delta r}{(\delta x)_e},\quad a_W = \frac{\lambda_w r_P \Delta r}{(\delta x)_w},\quad a_N = \frac{\lambda_n r_n \Delta x}{(\delta r)_n},\quad a_S = \frac{\lambda_s r_s \Delta x}{(\delta r)_s}$$

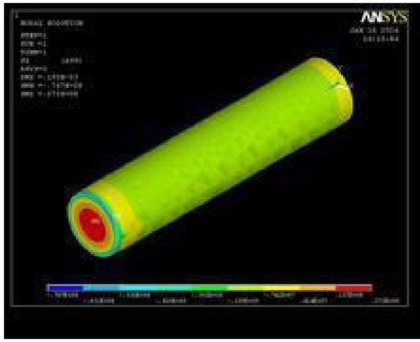
$$\begin{cases} a_P = a_E + a_W + a_N + a_S + a_P^0 - S_P \Delta V \\ a_P^0 = \dfrac{\rho c \Delta V}{\Delta t},\quad b = S_C \Delta V + a_P^0 T_P^0,\quad \Delta V = \dfrac{r_n + r_s}{2} \Delta r \Delta x \end{cases}$$

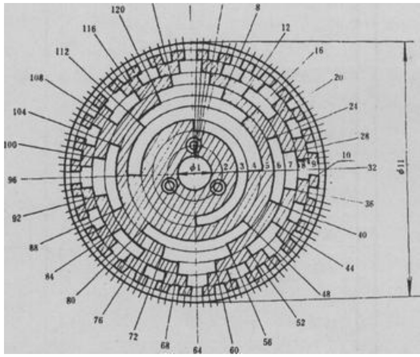
控制体体积

3 平面极坐标系

- 变量：径向r，周向 θ







控制体体积 $$\Delta V = (r_n + r_s)\Delta r \Delta\theta / 2$$

控制方程

$$\rho c\frac{\partial T}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\lambda\frac{\partial T}{\partial r}\right)+\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\lambda}{r}\frac{\partial T}{\partial\theta}\right)+S$$

Lame因子

离散结果

$$a_P T_P = a_E T_E + a_W T_W + a_N T_N + a_S T_S + b$$

系数形式不同：

Lame系数

$$a_E = \frac{\lambda_e \Delta r}{r_e (\delta\theta)_e}, \quad a_W = \frac{\lambda_w \Delta r}{r_w (\delta\theta)_w}, \quad a_N = \frac{\lambda_n r_n \Delta\theta}{(\delta r)_n}, \quad a_S = \frac{\lambda_s r_s \Delta\theta}{(\delta r)_s}$$

$$\begin{cases} a_P = a_E + a_W + a_N + a_S + a_P^0 - S_P \Delta V \\ a_P^0 = \dfrac{\rho c \Delta V}{\Delta t}, \quad b = S_C \Delta V + a_P^0 T_P^0, \quad \Delta V = \dfrac{r_n + r_s}{2} \Delta r \Delta\theta \end{cases}$$

4 三种坐标系下的通用表达式

$$a_P T_P = a_E T_E + a_W T_W + a_N T_N + a_S T_S + b$$

- 三种坐标系下的离散表达式形式都一样，区别只是系数的形式不同

4.2.1 非稳态三维导热方程的全隐式离散 (直角坐标系)

- 三维导热方程

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda \frac{\partial T}{\partial z}\right) + S$$

- 控制容积积分法离散结果

$$a_P T_P = a_E T_E + a_W T_W + a_N T_N + a_S T_S + a_T T_T + a_B T_B + b$$

2 圆柱坐标

- 导热方程

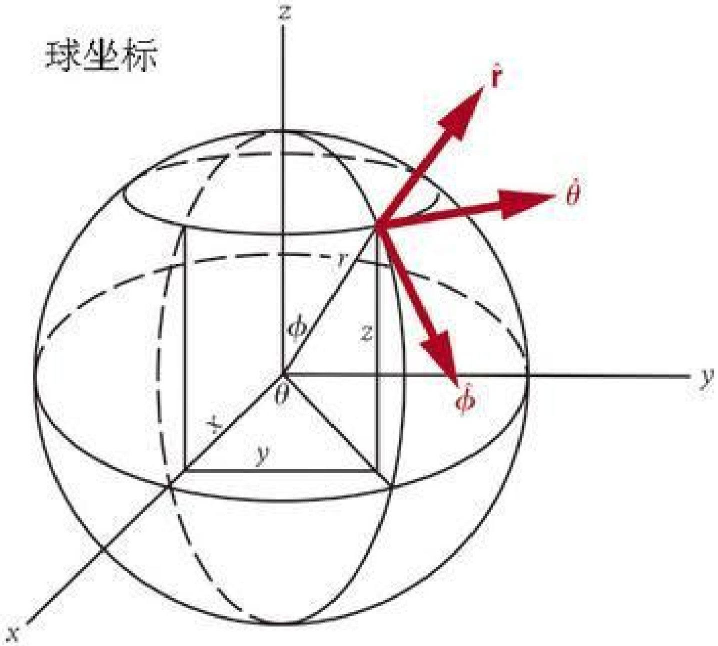
$$\rho c\frac{\partial T}{\partial t}=\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(r\lambda\frac{\partial T}{\partial r}\right)+\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{\lambda}{r}\frac{\partial T}{\partial \theta}\right)+S$$

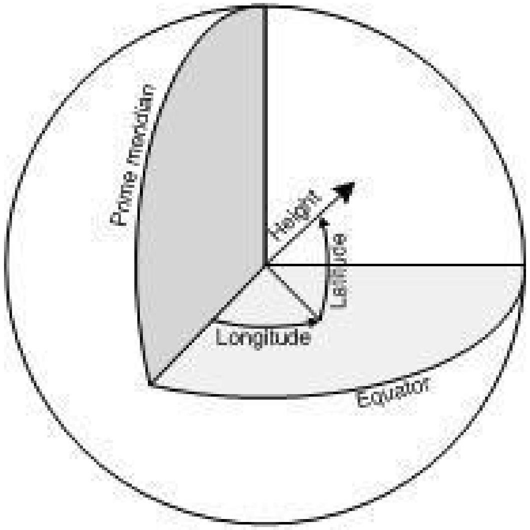
- 控制容积积分法

$$a_P T_P = a_E T_E + a_W T_W + a_N T_N + a_S T_S + a_T T_T + a_B T_B + b$$

区别仅在于系数表达式

导热问题一般不涉及
球坐标系情况





4.2.3 边界条件处理

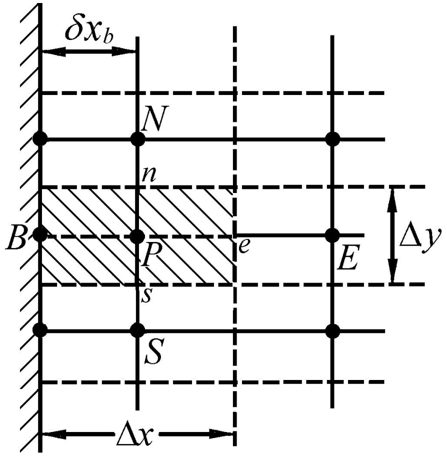
第一类边界条件

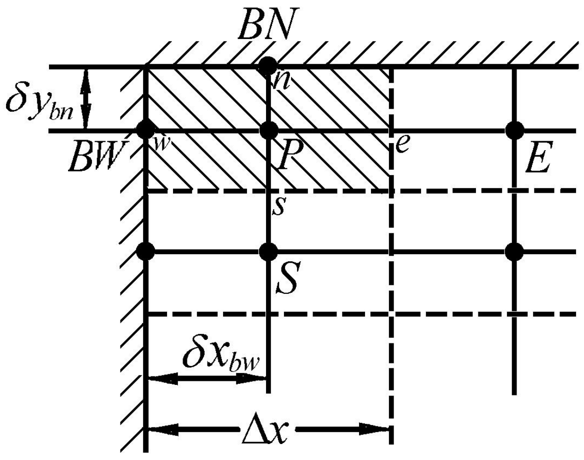
- 点中心：直接转移

- 块中心：

直接外推（一阶精度）

插值（二阶精度）





第2第3类边界条件情况

- 点中心，非角点处
- 点中心，角点处

控制容积平衡法

- 块中心，非角点处
- 块中心，角点处

附加源项法

- 非规则曲线边界条件处理

1 点中心非角点处

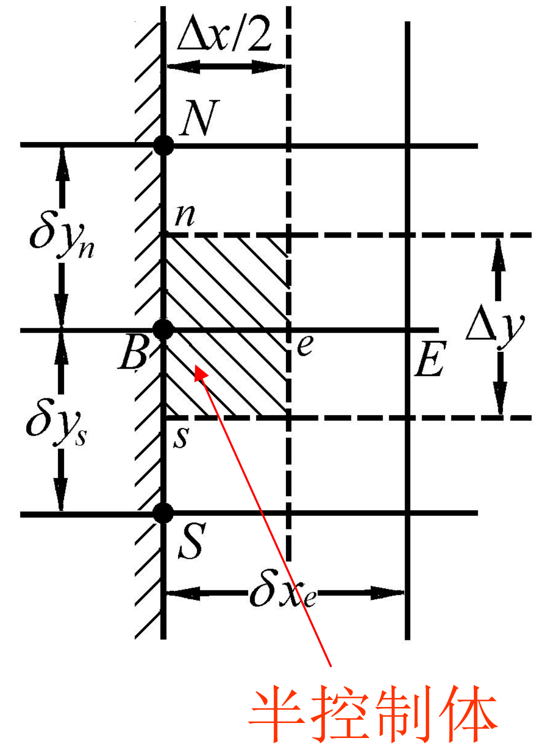
- 能量方程

非稳项　　右边界　　左边界　　导热面积

$$\rho c(T_B - T_B^0)\frac{\Delta x}{2}\Delta y = \left[\lambda_e \frac{T_E - T_B}{(\delta x)_e} + q_B\right]\Delta y \Delta t$$

$$+\left[\lambda_n \frac{T_N - T_B}{(\delta y)_n} - \lambda_s \frac{T_B - T_S}{(\delta y)_s}\right]\frac{\Delta x}{2}\Delta t + \underbrace{(S_C + S_P T_B)}_{\text{源项}}\frac{\Delta x}{2}\Delta y \Delta t$$

北面热流　　南面热流　　导热面积



合并整理结果

$$a_B T_B = a_E T_E + a_N T_N + a_S T_S + b$$

系数

导热面积

$$a_E = \frac{\lambda_e \Delta y}{(\delta x)_e}, \quad a_N = \frac{\lambda_n \Delta x}{2(\delta y)_n}, \quad a_S = \frac{\lambda_s \Delta x}{2(\delta y)_s}, \quad a_B^0 = \frac{\rho c \Delta x \Delta y}{2\Delta t}$$

距离

$$a_B = a_E + a_N + a_S + a_B^0 - S_P \Delta x \Delta y / 2$$

$$b = \frac{1}{2} S_C \Delta x \Delta y + a_B^0 T_B^0 + q_B \Delta y$$

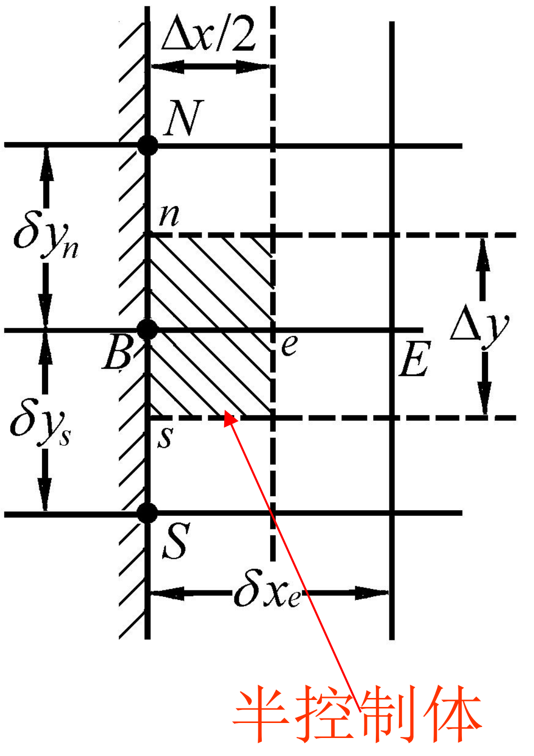
控制体体积

第3类边界条件

- 能量方程

$$q_B = h(T_f - T_B)$$

$$\rho c(T_B - T_B^0)\frac{\Delta x}{2}\Delta y = \left[\lambda_e \frac{T_E - T_B}{(\delta x)_e} + q_B\right]\Delta y \Delta t$$
$$+\left[\lambda_n \frac{T_N - T_B}{(\delta y)_n} - \lambda_s \frac{T_B - T_S}{(\delta y)_s}\right]\frac{\Delta x}{2}\Delta t + (S_C + S_P T_B)\frac{\Delta x}{2}\Delta y \Delta t$$



2 点中心 角点处

- 能量方程

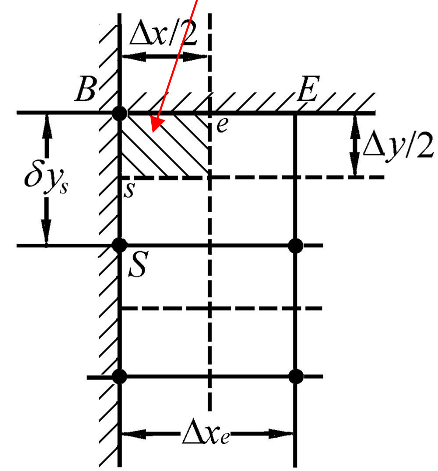
非稳项　　　　右面　　　　左面

$$\rho c(T_B - T_B^0)\frac{\Delta x\Delta y}{4} = \left[\lambda_e \frac{T_E - T_B}{(\delta x)_e} + q_{Bx}\right]\frac{\Delta y}{2}\Delta t$$

$$+\left[q_{By} - \lambda_s \frac{T_B - T_S}{(\delta y)_s}\right]\frac{\Delta x}{2}\Delta t + (S_C + S_P T_B)\frac{\Delta x\Delta y}{4}\Delta t$$

上面　　　　下面

1/4控制体



合并整理结果

$$a_B T_B = a_E T_E + a_S T_S + b$$

- 系数

$$a_E = \frac{\lambda_e \Delta y}{2(\delta x)_e}, \quad a_S = \frac{\lambda_s \Delta x}{2(\delta y)_s}, \quad a_B^0 = \frac{\rho c \Delta x \Delta y}{4\Delta t}$$

$$a_B = a_E + a_S + a_B^0 - \frac{1}{4} S_P \Delta x \Delta y$$

$$b = \frac{1}{4} S_C \Delta x \Delta y + a_B^0 T_B^0 + \frac{1}{2} q_{Bx} \Delta y + \frac{1}{2} q_{By} \Delta x$$

第3类边界条件

- 能量方程

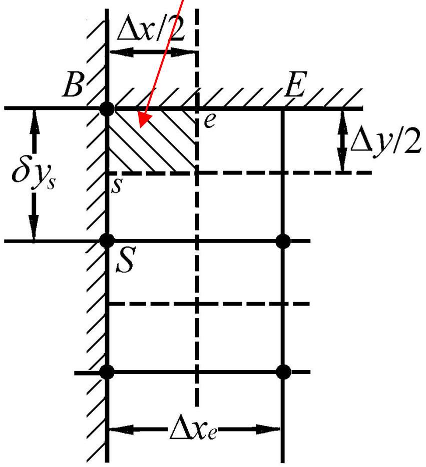
1/4控制体

$$q_{Bx} = h_x(T_{fx} - T_B)$$

$$\rho c(T_B - T_B^0)\frac{\Delta x \Delta y}{4} = \left[\lambda_e \frac{T_E - T_B}{(\delta x)_e} + q_{Bx}\right]\frac{\Delta y}{2}\Delta t$$

$$+\left[q_{By} - \lambda_s \frac{T_B - T_S}{(\delta y)_s}\right]\frac{\Delta x}{2}\Delta t + (S_C + S_P T_B)\frac{\Delta x \Delta y}{4}\Delta t$$

$$q_{By} = h_y(T_{fy} - T_B)$$



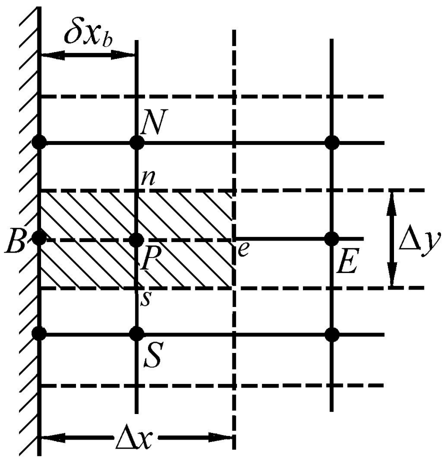
3 块中心非角点处

- 采用附加源项法

通用离散方程

$$a_B = \frac{\lambda_B \Delta y}{(\delta x)_b}$$

$$a_P T_P = a_E T_E + a_B T_B + a_N T_N + a_S T_S + b$$



$$(a_P - a_B) T_P = a_E T_E + a_B (T_B - T_P) + a_N T_N + a_S T_S + b$$

而 $a_B (T_B - T_P) = \dfrac{\lambda_B \Delta y}{(\delta x)_b} (T_B - T_P) = q_B \Delta y$

热流

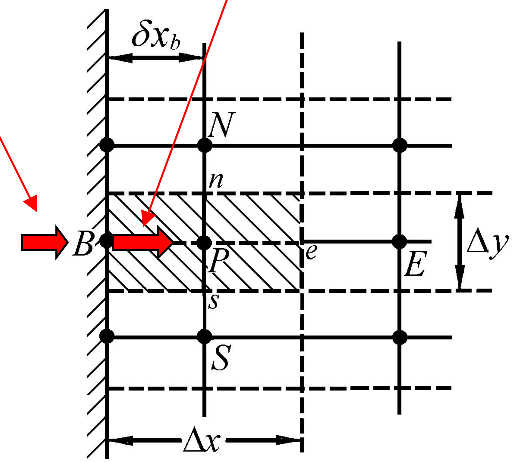
$$\bar{a}_P T_P = a_E T_E + a_N T_N + a_S T_S + \bar{b}$$

$$\bar{a}_P = a_P - a_B = a_E + a_N + a_S + a_P^0 - S_P \Delta x \Delta y, \quad \bar{b} = q_B \Delta y + b$$

第3类边界条件

- 需要消除T_B

$$q_B = h(T_f - T_B) \qquad q_B = \frac{\lambda(T_B - T_P)}{(\delta x)_b}$$



$$q_B = \frac{T_f - T_P}{1/h + (\delta x)_b / \lambda_B}$$

代入得到

$$\bar{\bar{a}}_P T_P = a_E T_E + a_N T_N + a_S T_S + \bar{\bar{b}}$$

$$\bar{\bar{a}}_P = a_E + a_N + a_S + a_P^0 + \frac{\Delta y}{1/h + (\delta x)_b / \lambda_B} - S_P \Delta x \Delta y,$$

$$\bar{\bar{b}} = b + \frac{T_f \Delta y}{1/h + (\delta x)_b / \lambda_B}$$

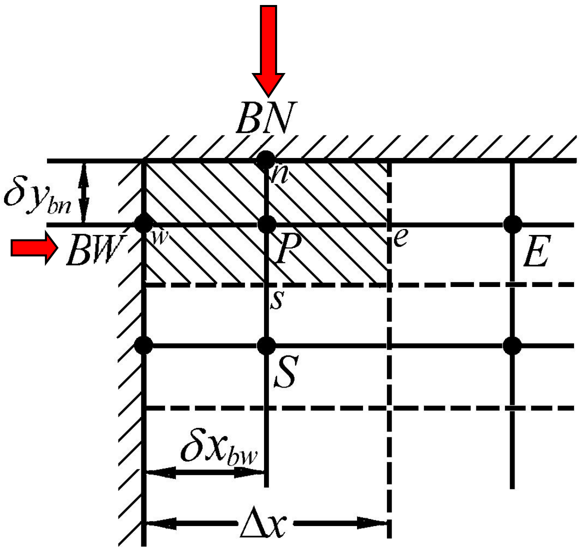
4 块中心 角点处

- 采用附加源项法

$$\overline{a}_P T_P = a_E T_E + a_S T_S + \overline{b}$$

$$\begin{cases} \overline{a}_P = a_P - a_{BW} - a_{BN} = a_E + a_S + a_P^0 - S_P \Delta x \Delta y \\ \overline{b} = q_{Bx} \Delta y + q_{By} \Delta x + b \end{cases}$$

热流



第3类边界条件

- 采用附加源项法

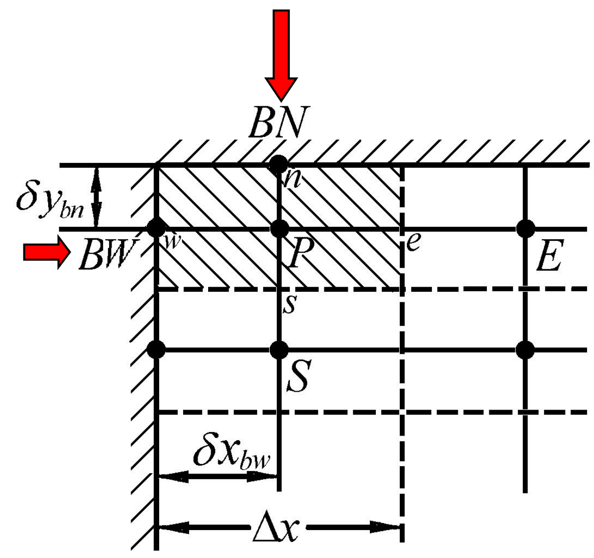
$$\bar{a}_P T_P = a_E T_E + a_S T_S + \bar{b}$$

$$\begin{cases} \bar{a}_P = a_P - a_{BW} - a_{BN} = a_E + a_S + a_P^0 - S_P \Delta x \Delta y \\ \bar{b} = q_{Bx} \Delta y + q_{By} \Delta x + b \end{cases}$$

热流

$$q_{By} = \frac{T_{fy} - T_P}{1/h_y + (\delta y)_{bn}/\lambda_{BN}}$$

$$q_{Bx} = \frac{T_{fx} - T_P}{1/h_x + (\delta x)_{bw}/\lambda_{BW}}$$



结果

$$\overset{=}{a}_P T_P = a_E T_E + a_S T_S + \overset{=}{b}$$

$$\begin{cases} \overset{=}{a}_P = a_E + a_S + a_P^0 + \dfrac{\Delta y}{1/h_x + (\delta x)_{bw}/\lambda_{BW}} + \dfrac{\Delta x}{1/h_y + (\delta y)_{bn}/\lambda_{BN}} - S_P \Delta x \Delta y \\ \overset{=}{b} = b + \left(\dfrac{\Delta y}{1/h_x + (\delta x)_{bw}/\lambda_{BW}} + \dfrac{\Delta x}{1/h_y + (\delta y)_{bn}/\lambda_{BN}} \right) T_f \end{cases}$$

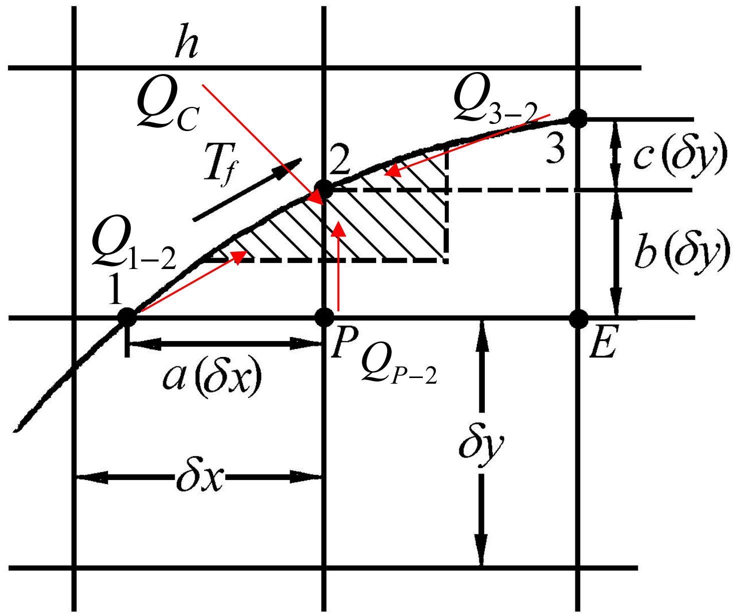
5 非规则 曲线边界

只能采用补充边界节点法

仅考虑稳态无源的情况，以2节点为例

- 热平衡方程：

$$Q_C + Q_{1-2} + Q_{3-2} + Q_{P-2} = 0$$

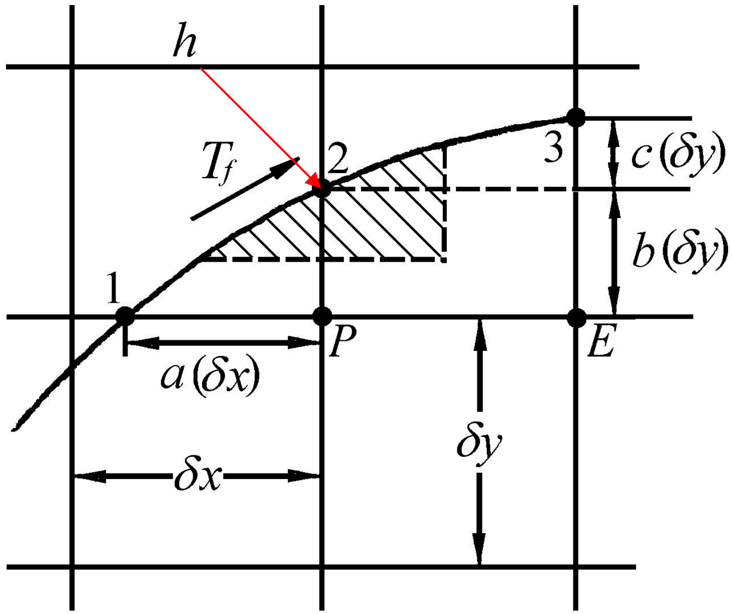


Q_C

导热面积

$$Q_C = \frac{1}{2}\left(\sqrt{(a\delta x)^2 + (\mu b\delta x)^2} + \sqrt{(\delta x)^2 + (\mu c\delta x)^2}\right) h(T_f - T_2)$$

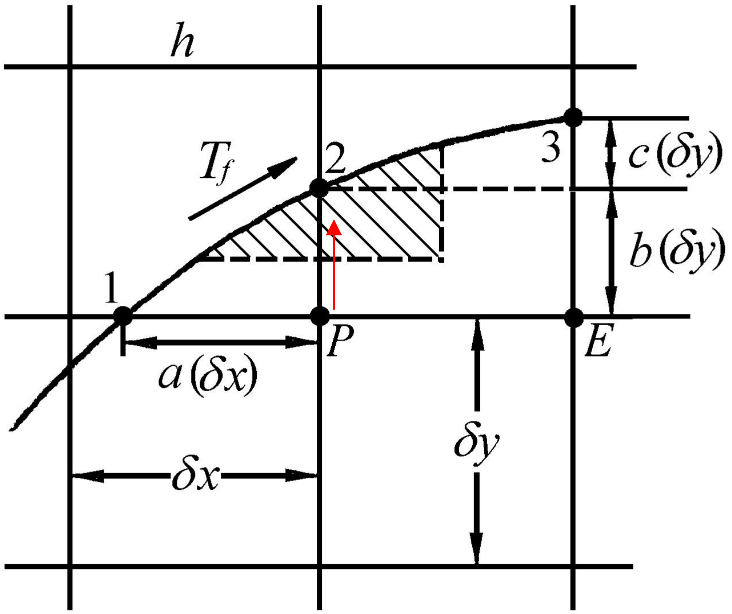
$$= \frac{\delta x}{2}\left(\sqrt{a^2 + \mu^2 b^2} + \sqrt{1 + \mu^2 c^2}\right) h(T_f - T_2)$$



$$Q_{P-2}$$

导热面积

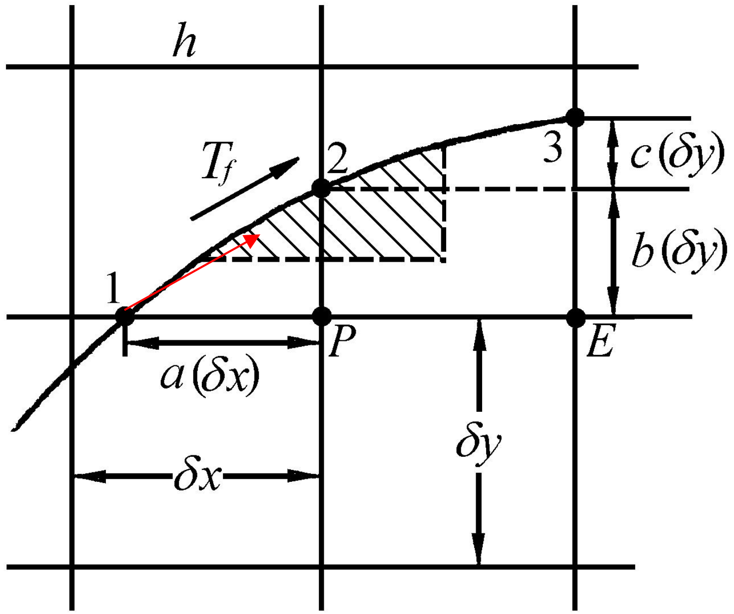
$$Q_{P-2} = \underline{\frac{1}{2}(\delta x + a\delta x)}\lambda \frac{T_P - T_2}{\mu b \delta x} = \frac{1+a}{2\mu b}\lambda (T_P - T_2)$$



$$Q_{1-2}$$

$$Q_{1-2} = \frac{\mu b\delta x}{2}\lambda\frac{T_1 - T_2}{\delta x\sqrt{a^2 + \mu^2 b^2}} = \frac{\mu b\lambda}{2}\frac{T_1 - T_2}{\sqrt{a^2 + \mu^2 b^2}}$$

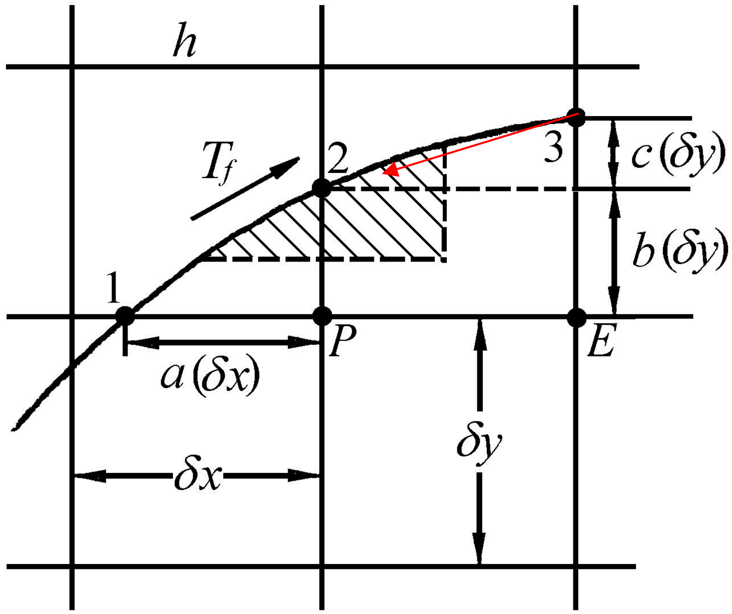
假定的
导热面积



$$Q_{3-2}$$

$$Q_{3-2} = \frac{\mu b \delta x}{2} \lambda \frac{T_3 - T_2}{\delta x \sqrt{1+\mu^2 c^2}} = \frac{\mu b \lambda}{2} \frac{T_3 - T_2}{\sqrt{1+\mu^2 c^2}}$$

假定的
导热面积



合并整理的结果

$$a_2 T_2 = a_1 T_1 + a_3 T_3 + a_P T_P + a_f T_f$$

$$\begin{cases} a_1 = \dfrac{\mu b}{\sqrt{a^2 + \mu^2 b^2}}, \quad a_3 = \dfrac{\mu b}{\sqrt{1 + \mu^2 c^2}}, \quad a_P = \dfrac{1 + a}{\mu b} \\ a_f = \dfrac{h\delta x}{\lambda}\left(\sqrt{a^2 + \mu^2 b^2} + \sqrt{1 + \mu^2 c^2}\right) \\ a_2 = a_1 + a_3 + a_p + a_f \end{cases}$$

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