6.8 非原始变量顺序求解的涡—流函数法

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流场求解

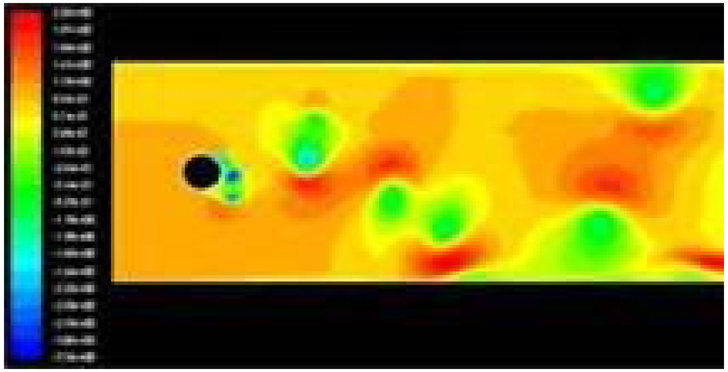
- 原始变量法

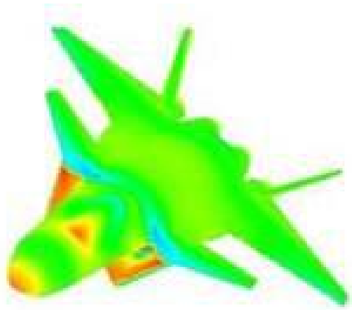
顺序求解 – SIMPLE 算法

耦合求解

- 间接变量解法

涡流函数法





6.8.1 二维方腔内自然对流的控制方程

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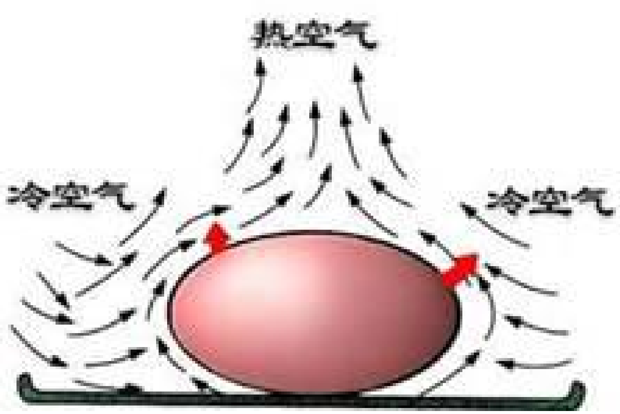
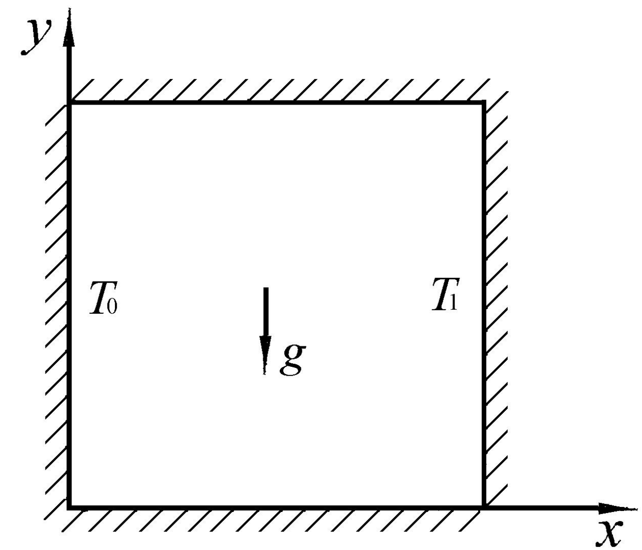
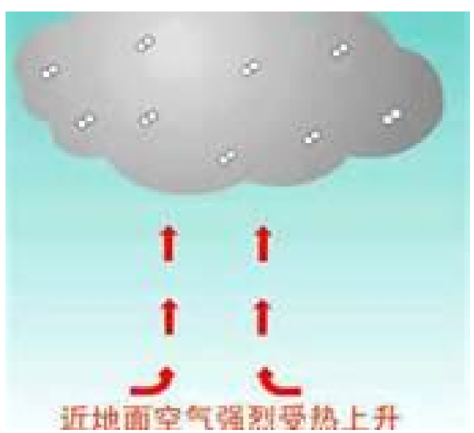






封闭方腔内自然对流

1. Boussinesq 假设

- 1）不计流体内粘性耗散；
- 2）流体除密度外的物性为常数；
- 3）控制方程内，除计算浮升力（重力）时考虑密度变化外，其余各项中密度为常数；而浮升力项中的密度与温度关系为

$$\rho = \rho_0 \left[1 - \beta \left(T - T_0 \right) \right]$$

2．原始变量的控制方程

$$\frac{\partial \rho}{\partial t}+\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}=0$$

$$\frac{\partial \rho u}{\partial t}+\frac{\partial \rho u u}{\partial x}+\frac{\partial \rho v u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)$$

$$\frac{\partial \rho v}{\partial t}+\frac{\partial \rho u v}{\partial x}+\frac{\partial \rho v v}{\partial y}=-\frac{\partial p}{\partial y}+\frac{\partial}{\partial x}\left(\mu\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial y}\right)-\rho g$$

$$\frac{\partial(\rho h)}{\partial t}+\frac{\partial(\rho u h)}{\partial x}+\frac{\partial(\rho v h)}{\partial y}=\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(\lambda\frac{\partial T}{\partial y}\right)$$

引入有效压力

$$p_{eff} = p + \rho_0 g y$$

- 压力导数

$$\frac{\partial p}{\partial x} = \frac{\partial p_{eff}}{\partial x}, \qquad \frac{\partial p}{\partial y} = \frac{\partial p_{eff}}{\partial y} - \rho_0 g$$

y方向动量方程中右端浮力项用Bossinesq假设，
其余所有密度和物性为常数，得到：

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$\frac{\partial u}{\partial t}+\frac{\partial uu}{\partial x}+\frac{\partial vu}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t}+\frac{\partial uv}{\partial x}+\frac{\partial vv}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+g\beta(T-T_0)$$

$$\frac{\partial T}{\partial t}+\frac{\partial (uT)}{\partial x}+\frac{\partial (vT)}{\partial y}=a\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)$$

其中$\nu=\mu/\rho$为运动粘性系数， $a=\lambda/(\rho c_p)$ 为导温系数。

3.转换为涡—流函数形式

- 流函数定义：

$$u=\frac{\partial\psi}{\partial y},\quad v=-\frac{\partial\psi}{\partial x}$$

- 涡函数定义：

$$\omega=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$$




流函数方程

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega$$

简化动量方程，消去压力梯度项

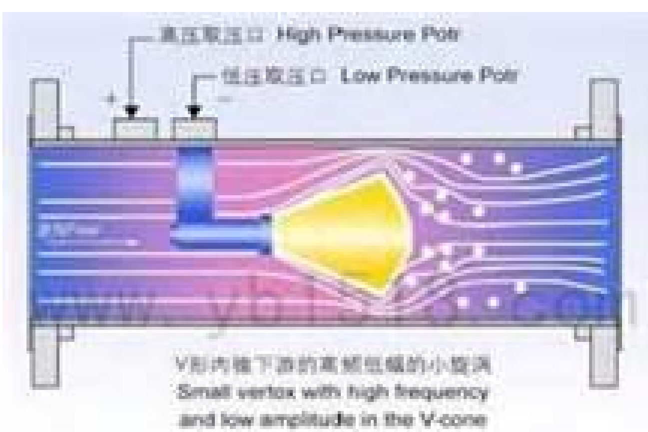
- 两式求导相减，得涡方程 (消去压力梯度)

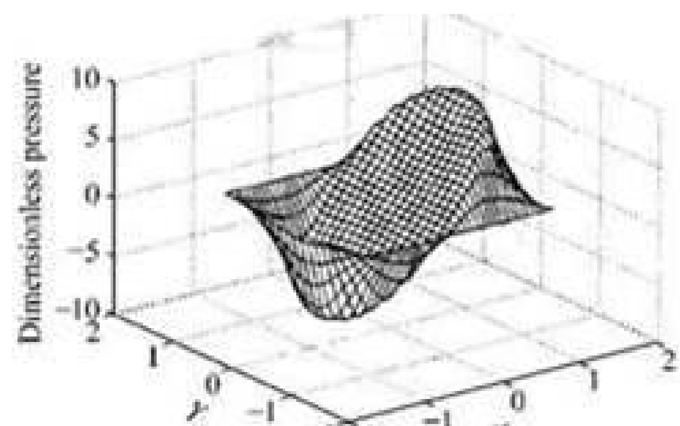
$$
\begin{cases}
\dfrac{\partial u}{\partial t}+\dfrac{\partial uu}{\partial x}+\dfrac{\partial vu}{\partial y}=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}+\nu\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}\right)\\[2ex]
\dfrac{\partial v}{\partial t}+\dfrac{\partial uv}{\partial x}+\dfrac{\partial vv}{\partial y}=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial y}+\nu\left(\dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 v}{\partial y^2}\right)+g\beta(T-T_0)
\end{cases}
$$

$$
\frac{\partial \omega}{\partial t}+\frac{\partial u\omega}{\partial x}+\frac{\partial v\omega}{\partial y}=\nu\left(\frac{\partial^2 \omega}{\partial x^2}+\frac{\partial^2 \omega}{\partial y^2}\right)+g\beta\frac{\partial}{\partial x}\left(T-T_0\right)
$$

无量纲化

$$
\begin{cases}
(X,Y)=(x,y)/H, \quad (U,V)=(u,v)/u^{*}, \\
\Theta=(T-T_0)/(T_1-T_0), \quad \tau=t/t^{*}, \\
\Psi=\psi\big/(Hu^{*}), \quad \Omega=\omega H/u^{*}
\end{cases}
$$





无量纲化之后的控制方程

$$\frac{\partial^2 \Psi}{\partial X^2}+\frac{\partial^2 \Psi}{\partial Y^2}=-\Omega$$

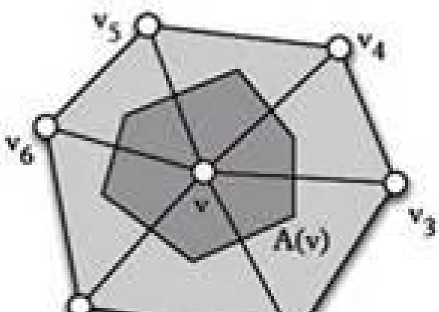
$$\frac{\partial \Omega}{\partial \tau}+\frac{\partial (U\Omega)}{\partial X}+\frac{\partial (V\Omega)}{\partial Y}=A_2\left(\frac{\partial^2 \Omega}{\partial X^2}+\frac{\partial^2 \Omega}{\partial Y^2}\right)+A_1\frac{\partial \theta}{\partial X}$$

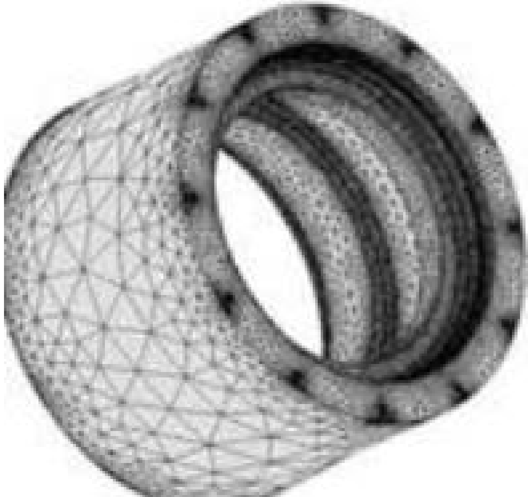
$$\frac{\partial \theta}{\partial \tau}+\frac{\partial (U\theta)}{\partial X}+\frac{\partial (V\theta)}{\partial Y}=A_3\left(\frac{\partial^2 \theta}{\partial X^2}+\frac{\partial^2 \theta}{\partial Y^2}\right)$$

$$U=\frac{\partial \Psi}{\partial Y},\quad V=-\frac{\partial \Psi}{\partial X}$$

- 无量纲量的取法参见课本表6-1

6.8.2 涡流函数形式控制方程的离散化





1. 交替方向隐式格式(ADI)离散 - 以涡方程为例

$$\frac{\partial \Omega}{\partial \tau} + \frac{\partial (U\Omega)}{\partial X} + \frac{\partial (V\Omega)}{\partial Y} = A_2\left(\frac{\partial^2 \Omega}{\partial X^2} + \frac{\partial^2 \Omega}{\partial Y^2}\right) + A_1\frac{\partial \theta}{\partial X}$$

- 离散为:

$$\left(\frac{\partial \Omega}{\partial \tau}\right)_{i,j}^{n+1/2} + \left[\frac{\partial}{\partial X}\left(U\Omega - A_2\frac{\partial \Omega}{\partial X}\right)\right]_{i,j}^{n+1/2} + \left[\frac{\partial}{\partial Y}\left(V\Omega - A_2\frac{\partial \Omega}{\partial Y}\right)\right]_{i,j}^{n} = \left(A_1\frac{\partial \theta}{\partial X}\right)_{i,j}^{n}$$

$$\left(\frac{\partial \Omega}{\partial \tau}\right)_{i,j}^{n+1} + \left[\frac{\partial}{\partial X}\left(U\Omega - A_2\frac{\partial \Omega}{\partial X}\right)\right]_{i,j}^{n+1/2} + \left[\frac{\partial}{\partial Y}\left(V\Omega - A_2\frac{\partial \Omega}{\partial Y}\right)\right]_{i,j}^{n+1} = \left(A_1\frac{\partial \theta}{\partial X}\right)_{i,j}^{n+1/2}$$

整理得到离散方程

$$a_P^{n+1/2}\Omega_P^{n+1/2} = a_E^{n+1/2}\Omega_E^{n+1/2} + a_W^{n+1/2}\Omega_W^{n+1/2} + b^n$$

$$a_P^{n+1}\Omega_P^{n+1} = a_N^{n+1}\Omega_N^{n+1} + a_S^{n+1}\Omega_S^{n+1} + b^{n+1/2}$$

- 系数的表达式见课本方程6-63

能量方程离散
（交替方向隐式ADI）

- 先考虑X向离散，Y向取显式

$$a_P^{n+1/2}\theta_P^{n+1/2} = a_E^{n+1/2}\theta_E^{n+1/2} + a_W^{n+1/2}\theta_W^{n+1/2} + b^n$$

- 再考虑Y向离散，X向取显式

$$a_P^{n+1}\theta_P^{n+1} = a_N^{n+1}\theta_N^{n+1} + a_S^{n+1}\theta_S^{n+1} + b^{n+1/2}$$

2.控制容积积分离散
- 以流函数方程为例

$$\frac{\partial^2 \Psi}{\partial X^2} + \frac{\partial^2 \Psi}{\partial Y^2} = -\Omega$$

- 设函数沿空间为分段线性分布

$$\Psi_P = \frac{1}{2(1+\alpha^2)}\left[\Psi_E + \Psi_W + \alpha^2 \Psi_N + \alpha^2 \Psi_S + \Delta X^2 \Omega_P\right]$$

$$\alpha = \Delta X / \Delta Y$$

超松弛Gauss-Seidel点迭代求解

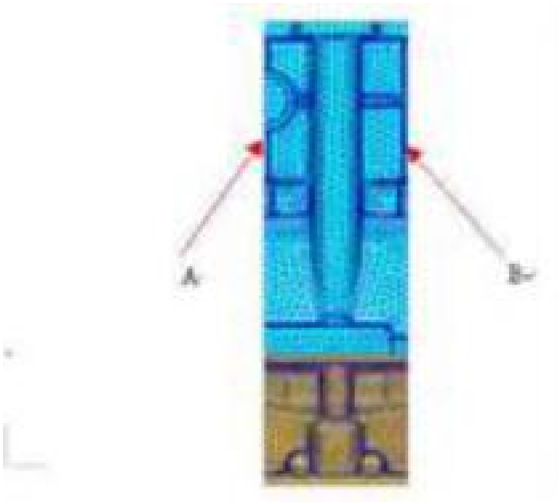
$$\Psi_P^{(k+1)} = \frac{\omega}{2(1+\alpha^2)}\left[\Psi_E^{(k)} + \Psi_W^{(k+1)} + \alpha^2\Psi_N^{(k)} + \alpha^2\Psi_S^{(k+1)} + \Delta X^2\Omega_P\right] + (1-\omega)\Psi_P^{(k)}$$

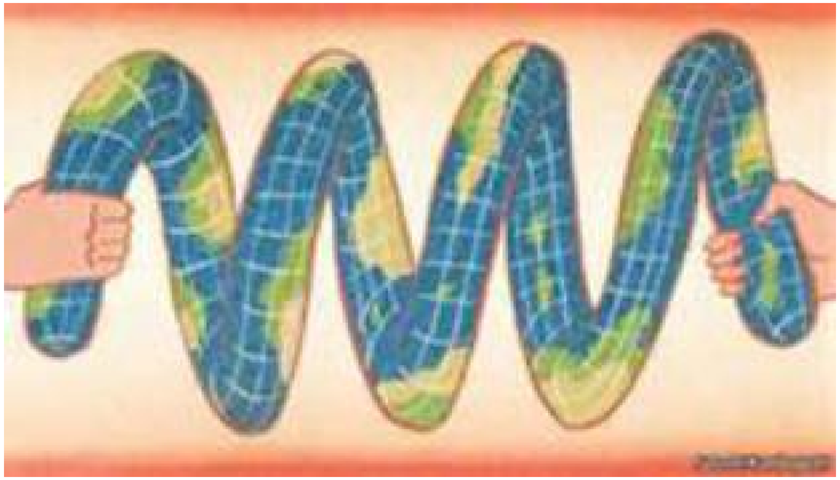
- 最佳松弛因子

$$\omega_0 = 2\left(\frac{1-\sqrt{1-\xi}}{\xi}\right)$$

$$\xi = \left[\frac{\cos\left(\frac{\pi}{M-1}\right) + \alpha^2\cos\left(\frac{\pi}{N-1}\right)}{1+\alpha^2}\right]^2$$

6.8.3 涡—流函数方法中的定解条件处理





初始条件

- 如腔体内起始时处于静止

$$\tau = 0,\ U = V = 0$$

- 温度分布：从高温壁到低温壁作线性分布

边界条件

- 能量方程

$$X=0:\quad \theta=0;\quad X=1:\quad \theta=1$$

$$Y=0:\quad \frac{\partial\theta}{\partial Y}=0;\quad Y=1:\quad \frac{\partial\theta}{\partial Y}=0$$

- 流函数方程

$$X=0,\ 1:\quad \Psi=0,\quad \frac{\partial\Psi}{\partial X}=\frac{\partial\Psi}{\partial Y}=0$$

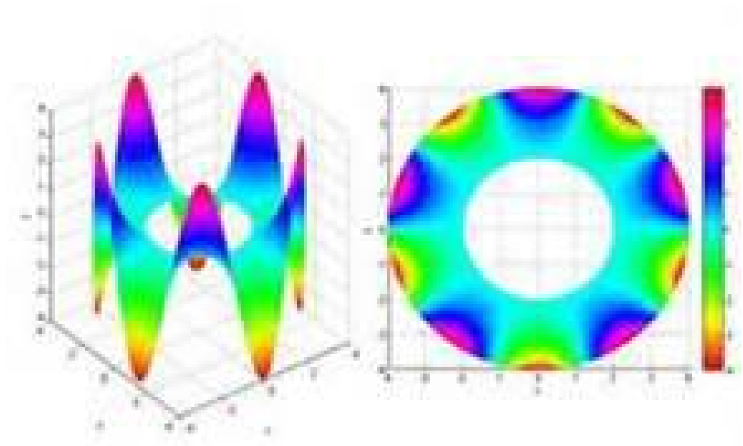
$$Y=0,\ 1:\quad \Psi=0,\quad \frac{\partial\Psi}{\partial Y}=\frac{\partial\Psi}{\partial X}=0$$

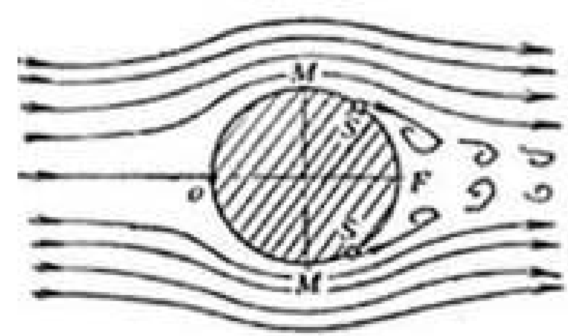
能量方程和流函数方程的边界条件处理
参见第4章

壁涡公式

1.是流场驱动力，不为0

2. 可以由速度表达，也可由流函数表达





- 上下壁面

$$\frac{\partial U}{\partial X} = \frac{\partial V}{\partial X} = 0$$

- 左右壁面

$$\frac{\partial U}{\partial Y} = \frac{\partial V}{\partial Y} = 0$$

- 涡函数

$$\Omega = \frac{\partial V}{\partial X} - \frac{\partial U}{\partial Y}$$

壁涡公式1 – 上下壁面

上下壁面：$\Omega_{wall}=\left(-\frac{\partial U}{\partial Y}\right)_{wall}$ ，即 $\Omega_{i,1}=\left(-\frac{\partial U}{\partial Y}\right)_{i,1}$ ，$\Omega_{i,N}=\left(-\frac{\partial U}{\partial Y}\right)_{i,N}$

$$\left(\frac{\partial U}{\partial Y}\right)_{i,j}=\frac{-3U_{i,j}+4U_{i,j+1}-U_{i,j+2}}{2\Delta Y},\quad \left(\frac{\partial U}{\partial Y}\right)_{i,j}=\frac{3U_{i,j}-4U_{i,j-1}+U_{i,j-2}}{2\Delta Y}$$

下壁： $\Omega_{wall}=\Omega_{i,1}=\left(-4U_{i,2}+U_{i,3}\right)/\left(2\Delta Y\right)$

上壁： $\Omega_{wall}=\Omega_{i,N}=\left(4U_{i,N-1}-U_{i,N-2}\right)/\left(2\Delta Y\right)$

壁涡公式2 – 左右壁面

左右壁面： $\Omega_{wall}=\left(\frac{\partial V}{\partial X}\right)_{wall}$，即 $\Omega_{1,j}=\left(\frac{\partial V}{\partial X}\right)_{1,j}$，$\Omega_{M,j}=\left(\frac{\partial V}{\partial X}\right)_{M,j}$

$$\left(\frac{\partial V}{\partial X}\right)_{i,j}=\frac{-3V_{i,j}+4V_{i+1,j}-V_{i+2,j}}{2\Delta X},\quad \left(\frac{\partial V}{\partial X}\right)_{i,j}=\frac{3V_{i,j}-4V_{i-1,j}+V_{i-2,j}}{2\Delta X}$$

左壁： $\Omega_{wall}=\Omega_{1,j}=\left(4V_{2,j}-V_{3,j}\right)\Big/\left(2\Delta X\right)$

右壁： $\Omega_{wall}=\Omega_{M,j}=\left(-4V_{M-1,j}+V_{M-2,j}\right)\Big/\left(2\Delta X\right)$

用流函数表达壁涡

6.8. 4 涡流函数方法离散方程迭代求解步骤

- 三个被求函数 Ψ, Ω, θ
- 离散方程相互关联
- 方程非线性

采用顺序求解，并且需要应用迭代方法

求解步骤

1) 给定一个速度场和温度场，本时步初始起步

2) 求解能量方程得 $\theta^{(1)}$

3) 求解涡量方程，得 $\Omega^{(1)}$

4) 解流函数方程，得 $\Psi^{(1)}$

5) 由流函数定义式，差分求得速度 $U^{(1)},V^{(1)}$

6) 由速度求壁涡 $\Omega_{wall}^{(1)}$

7) 重复迭代 2-6 步，直到本时步迭代收敛

$$\left|\Omega_{i,j}^{(k+1)}-\Omega_{i,j}^{(k)}\right|_{\max}+\left|\theta_{i,j}^{(k+1)}-\theta_{i,j}^{(k)}\right|\max\leq\varepsilon$$

6.8. 5 涡—流函数方法讨论

1．关于压力求解问题

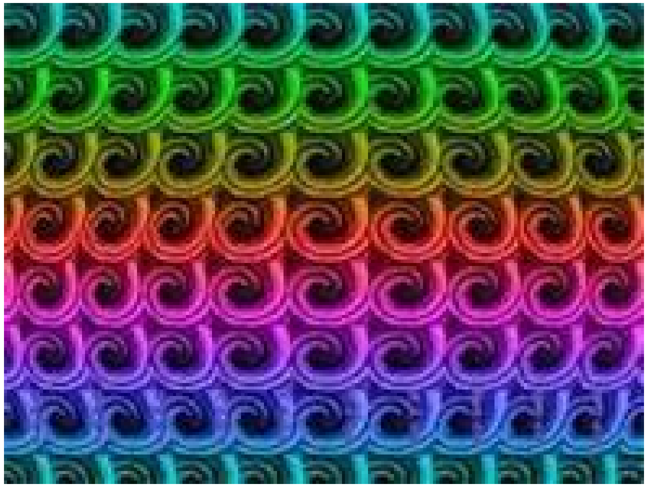
- 自然对流：不一定需要得到压力分布

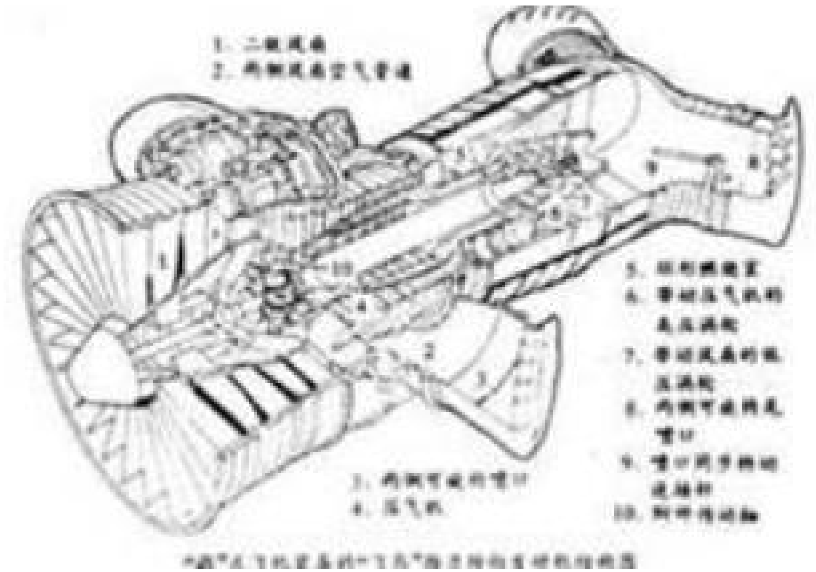
- 强迫对流：需要单独计算压力

$$\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}=2\rho\left[\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial v}{\partial y}\right)-\left(\frac{\partial v}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right)\right]$$

2．涡流函数法一般只适用于二维问题

- 三维问题，须引入三维位势矢量和涡矢量，需解六个方程，太复杂





哪个能用涡流函数法？