8.2 拓展的三对角阵算法

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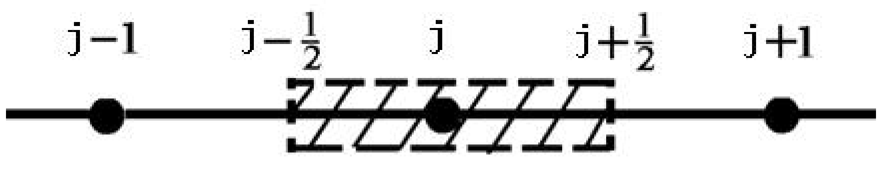
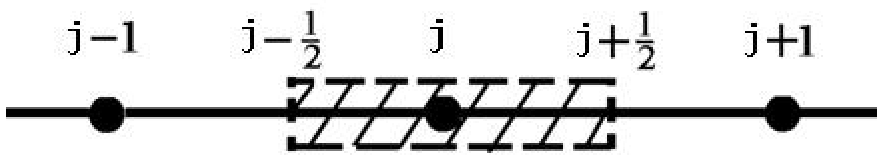
1) 块三角阵算法(BTDMA)
2) 环形三角阵算法(CTDMA)
3) 五对角阵算法(PTDMA)

1、块三对角阵算法(BTDMA)

- 实际问题有多个关联变量
- 一维问题，每个节点都是一个小的方程组



N个变量关联的块三角方程组



$$A_j U_{j-1} + B_j U_j + C_j U_{j+1} = D_j \qquad (j = 1, 2, \cdots, J-1, J)$$

$$A_1 = 0, \quad C_J = 0$$

J：　一维问题划分的节点个数

U：　矢量，具有n个分量，该节点的待求变量

A、B、C：都是 nxn 的小矩阵

消元过程

$A_1 = 0$

- j=1 $B_1^{-1} \times B_1 U_1 + C_1 U_2 = D_1$

- j=2 $A_2^{-1} \times A_2 U_1 + B_2 U_2 + C_2 U_3 = D_2$

- 用A2、B1矩阵求逆消去U_1

$$\left(B_1^{-1} C_1 - A_2^{-1} B_2\right) U_2 - A_2^{-1} C_2 U_3 = B_1^{-1} D_1 - A_2^{-1} D_2$$

简化公式

$$\boldsymbol{E}_2\boldsymbol{U}_2 + \boldsymbol{F}_2\boldsymbol{U}_3 = \boldsymbol{G}_2$$

$$\boldsymbol{E}_2 = \boldsymbol{B}_1^{-1}\boldsymbol{C}_1 - \boldsymbol{A}_2^{-1}\boldsymbol{B}_2$$

$$\boldsymbol{F}_2 = -\boldsymbol{A}_2^{-1}\boldsymbol{C}_2$$

$$\boldsymbol{G}_2 = \boldsymbol{B}_1^{-1}\boldsymbol{D}_1 - \boldsymbol{A}_2^{-1}\boldsymbol{D}_2$$

递推公式

$$\boldsymbol{E}_j\boldsymbol{U}_j + \boldsymbol{F}_j\boldsymbol{U}_{j+1} = \boldsymbol{G}_j$$

$$\boldsymbol{E}_j = \boldsymbol{E}_{j-1}^{-1}\boldsymbol{F}_{j-1} - \boldsymbol{A}_j^{-1}\boldsymbol{B}_j$$

$$\boldsymbol{F}_{\mathrm{j}} = -\boldsymbol{A}_j^{-1}\boldsymbol{C}_j$$

$$\boldsymbol{G}_j = \boldsymbol{E}_{j-1}^{-1}\boldsymbol{G}_{j-1} - \boldsymbol{A}_j^{-1}\boldsymbol{D}_j$$

递推过程

$$\boldsymbol{E}_j \boldsymbol{U}_j + \boldsymbol{F}_j \boldsymbol{U}_{j+1} = \boldsymbol{G}_j$$

- j=1: $$\boldsymbol{B}_1 \boldsymbol{U}_1 + \boldsymbol{C}_1 \boldsymbol{U}_2 = \boldsymbol{D}_1$$

$$\boldsymbol{E}_1 = \boldsymbol{B}_1, \quad \boldsymbol{F}_1 = \boldsymbol{C}_1, \quad \boldsymbol{G}_1 = \boldsymbol{D}_1,$$

- j=J: $$\boldsymbol{U}_J = \boldsymbol{E}_J^{-1} \boldsymbol{G}_J$$

$$\boldsymbol{C}_J = 0$$

回代公式

$$\boldsymbol{E}_j\boldsymbol{U}_j+\boldsymbol{F}_j\boldsymbol{U}_{j+1}=\boldsymbol{G}_j$$

$$\boldsymbol{U}_j=\boldsymbol{E}_j^{-1}\boldsymbol{G}_j-\boldsymbol{E}_j^{-1}\boldsymbol{F}_j\boldsymbol{U}_{j+1}$$

- 顺序： $j=J-1,J-2,\cdots,2,1$

计算实施过程

- 计算消元递推系数(矩阵)：

$$\boldsymbol{E}_j,\ \boldsymbol{F}_j,\ \boldsymbol{G}_j\ (j=1,2,\cdots,J)$$

- 算出最后一个元(矢量)：

$$\boldsymbol{U}_J=\boldsymbol{E}_J^{-1}\boldsymbol{G}_J$$

- 回代算出其它元：

$$\boldsymbol{U}_j=\boldsymbol{E}_j^{-1}\boldsymbol{G}_j-\boldsymbol{E}_j^{-1}\boldsymbol{F}_j\boldsymbol{U}_{j+1}$$

2、环形三对角阵算法

- 常见：周期性边界条件问题

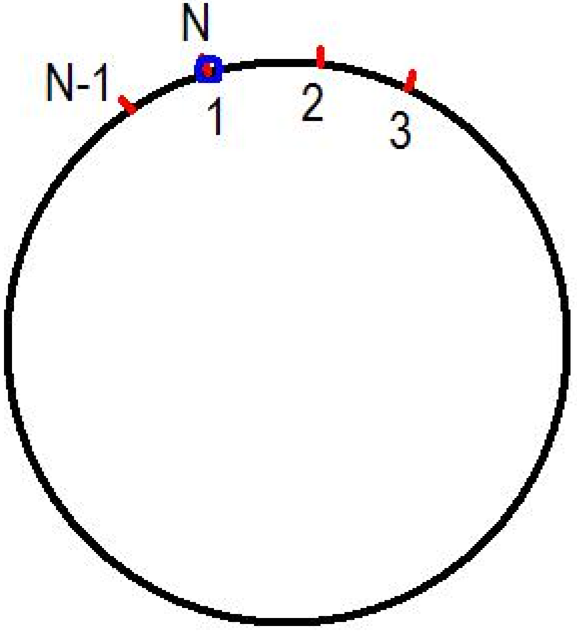
离散方程

$$A_j\phi_j = B_j\phi_{j+1} + C_j\phi_{j-1} + D_j \qquad (j = 1, 2, \cdots, N-1)$$

- 周期性条件：

$$j = 1, \qquad \phi_{j-1} = \phi_{N-1}$$

$$j = N-1, \qquad \phi_{j+1} = \phi_1$$



假定消元递推关系式为

$$\phi_j = E_j \phi_{j+1} + F_j \phi_{N-1} + G_j$$

- 比较：TDMA的递推式

$$T_{i-1} = P_{i-1} T_i + Q_{i-1}$$

第一个节点 j=1

- 原方程： $A_j\phi_j = B_j\phi_{j+1} + C_j\phi_{j-1} + D_j$

$$\phi_1 = (B_1/A_1)\phi_2 + (C_1/A_1)\phi_{N-1} + D_1/A_1$$

$$E_1 = \frac{B_1}{A_1}, \quad F_1 = \frac{C_1}{A_1}, \quad G_1 = \frac{D_1}{A_1}$$

递推公式

$$\left(A_j - C_j E_{j-1}\right)\phi_j = B_j \phi_{j+1} + C_j F_{j-1} \phi_{N-1} + \left(D_j + C_j G_{j-1}\right)$$

$$\phi_j = E_j \phi_{j+1} + F_j \phi_{N-1} + G_j$$

$$E_j = \frac{B_j}{A_j - C_j E_{j-1}}, \quad F_j = \frac{C_j F_{j-1}}{A_j - C_j E_{j-1}}, \quad G_j = \frac{D_j + C_j G_{j-1}}{A_j - C_j E_{j-1}}$$

$$(j = 2, 3, \cdots, N-2)$$

关键：求出 ϕ_{N-1}

- 需再推导出一个方程，与前面方程联立求解
- N-1 节点原方程：

$$A_{N-1}\phi_{N-1} = B_{N-1}\phi_1 + C_{N-1}\phi_{N-2} + D_{N-1}$$

$$p_1 = A_{N-1},\quad q_1 = B_{N-1},\quad r_1 = D_{N-1}$$

$$p_1\phi_{N-1} = q_1\phi_1 + C_{N-1}\phi_{N-2} + r_1$$

$$p_1 \phi_{N-1} = q_1 \phi_1 + C_{N-1} \phi_{N-2} + r_1$$

$$\phi_1 = E_1 \phi_2 + F_1 \phi_{N-1} + G_1$$

$$\left(p_1 - q_1 F_1 \right) \phi_{N-1} = q_1 E_1 \phi_2 + C_{N-1} \phi_{N-2} + \left(r_1 + q_1 G_1 \right)$$

$$\left(p_1 - q_1 F_1\right)\phi_{N-1} = q_1 E_1 \phi_2 + C_{N-1}\phi_{N-2} + \left(r_1 + q_1 G_1\right)$$

$$p_2 = p_1 - q_1 F_1, \quad q_2 = q_1 E_1, \quad r_2 = r_1 + q_1 G_1$$

$$p_2 \phi_{N-1} = q_2 \phi_2 + C_{N-1}\phi_{N-2} + r_2$$

$$p_1 \phi_{N-1} = q_1 \phi_1 + C_{N-1}\phi_{N-2} + r_1$$

$$p_j \phi_{N-1} = q_j \phi_j + C_{N-1} \phi_{N-2} + r_j$$

- 递推关系

$$p_j = p_{j-1} - q_{j-1} F_{j-1}, \quad q_j = q_{j-1} E_{j-1}, \quad r_j = r_{j-1} + q_{j-1} G_{j-1}$$

$$(j = 2, 3, \cdots N-2)$$

- N-1节点：

$$p_{N-2}\phi_{N-1}=\left(q_{N-2}+C_{N-1}\right)\phi_{N-2}+r_{N-2}$$

$$\phi_{N-2}=\left(E_{N-2}+F_{N-2}\right)\phi_{N-1}+G_{N-2}$$

$$\phi_{N-1}=\frac{r_{N-2}+G_{N-2}\left(q_{N-2}+C_{N-1}\right)}{p_{N-2}-\left(q_{N-2}+C_{N-1}\right)\left(E_{N-2}+F_{N-2}\right)}$$

CTDMA具体实施步骤

- 计算系数 $E_j,\ F_j,\ G_j\ (j=1,2,\cdots,N-2)$
- 计算系数 $p_j,\ q_j,\ r_j\ (j=1,2,\cdots,N-2)$
- 求最后一个元素： $\phi_{N-1}=\dfrac{r_{N-2}+G_{N-2}\left(q_{N-2}+C_{N-1}\right)}{p_{N-2}-\left(q_{N-2}+C_{N-1}\right)\left(E_{N-2}+F_{N-2}\right)}$
- 回代求出： $\phi_j=E_j\phi_{j+1}+F_j\phi_{N-1}+G_j$

$$\phi_j\quad (j=N-2,N-3,\cdots 2,1)$$

3、五对角阵算法

- 方程对流项采用高阶迎风型格式离散时
- 一维情况：i-2, i-1, i, i+1, i+2 五节点格式
- 二维情况：9节点格式，每方向上分别求解

某一方向上求解

$$e_i\phi_{i-2} + a_i\phi_{i-1} + b_i\phi_i + c_i\phi_{i+1} + f_i\phi_{i+2} = d_i, \quad i = 1, 2, \cdots N-1, N$$

- 矩阵形式：

$$\begin{bmatrix} b_1 & c_1 & f_1 & & & & & & \\ a_2 & b_2 & c_2 & f_2 & & & & & \\ e_3 & a_3 & b_3 & c_3 & f_3 & & & & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & & & & e_{N-1} & a_{N-1} & b_{N-1} & c_{N-1} \\ & & & & & & e_N & a_N & b_N \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \cdot \\ \cdot \\ \phi_{N-1} \\ \phi_N \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \cdot \\ \cdot \\ d_{N-1} \\ d_N \end{bmatrix}$$

进行两次消元

- 第一步消元：e
- 第二步消元：

$$\begin{bmatrix} b_1 & c_1 & f_1 & & & & & & \\ a_2 & b_2 & c_2 & f_2 & & & & & \\ e_3 & a_3 & b_3 & c_3 & f_3 & & & & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & & & & e_{N-1} & a_{N-1} & b_{N-1} & c_{N-1} \\ & & & & & & e_N & a_N & b_N \end{bmatrix}$$

第一步消元：消去 e 元素

$$\begin{bmatrix} b_1' & c_1' & f_1' & & & & & \\ a_2' & b_2' & c_2' & f_2' & & & & \\ & a_3' & b_3' & c_3' & f_3' & & & \\ & & \cdot & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \cdot & \\ & & & & & a'_{N-1} & b'_{N-1} & c'_{N-1} \\ & & & & & & a'_N & b'_N \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \cdot \\ \cdot \\ \phi_{N-1} \\ \phi_N \end{bmatrix} = \begin{bmatrix} d_1' \\ d_2' \\ d_3' \\ \cdot \\ \cdot \\ d'_{N-1} \\ d'_N \end{bmatrix}$$

递推关系式

$$
\begin{cases}
a_i' = a_i - e_i b_{i-1}' / a_{i-1}', \qquad b_i' = b_i - e_i c_{i-1}' / a_{i-1}' \\
c_i' = c_i - e_i f_{i-1}' / a_{i-1}', \qquad d_i' = d_i - e_i d_{i-1}' / a_{i-1}' \\
f_i' = f_i, \;\; e_1 = e_2 = a_1 = c_N = f_N = f_{N-1} = 0, \;\; a_0' = a_1' = 1 \\
b_0' = c_0' = f_0' = d_0' = 0, \qquad i = 1, 2, 3, \cdots, N
\end{cases}
$$

第二步消元：消去 a’ 元素

- 消去a’元素，并将对角元变为1

$$
\begin{bmatrix}
1 & c_1'' & f_1'' & & & & \\
 & 1 & c_2'' & f_2'' & & & \\
 & & 1 & c_3'' & f_3'' & & \\
 & & & \cdot & \cdot & \cdot & \\
 & & & & \cdot & \cdot & \cdot \\
 & & & & & 1 & c''_{N-1} \\
 & & & & & & 1
\end{bmatrix}
\begin{bmatrix}
\phi_1 \\ \phi_2 \\ \phi_3 \\ \cdot \\ \cdot \\ \phi_{N-1} \\ \phi_N
\end{bmatrix}
=
\begin{bmatrix}
d_1'' \\ d_2'' \\ d_3'' \\ \cdot \\ \cdot \\ d''_{N-1} \\ d''_N
\end{bmatrix}
$$

系数递推关系式

$$\begin{cases} c_i'' = \dfrac{c_i' - a_i' f_{i-1}''}{b_i' - a_i' c_{i-1}''}, \quad f_i'' = \dfrac{f_i'}{b_i' - a_i' c_{i-1}''}, \quad d_i'' = \dfrac{d_i' - a_i' d_{i-1}''}{b_i' - a_i' c_{i-1}''} \\ c_0'' = f_0'' = d_0'' = 0\ , \qquad i = 1, 2, 3, \cdots, N \end{cases}$$

回代过程

- 回代过程不需要进行两步

$$\begin{cases} \phi_i = d_i'' - c_i''\phi_{i+1} - f_i''\phi_{i+2} \\ \phi_{N+1} = \phi_{N+2} = 0\,, \quad i = N, N-1, \cdots, 2, 1 \end{cases}$$