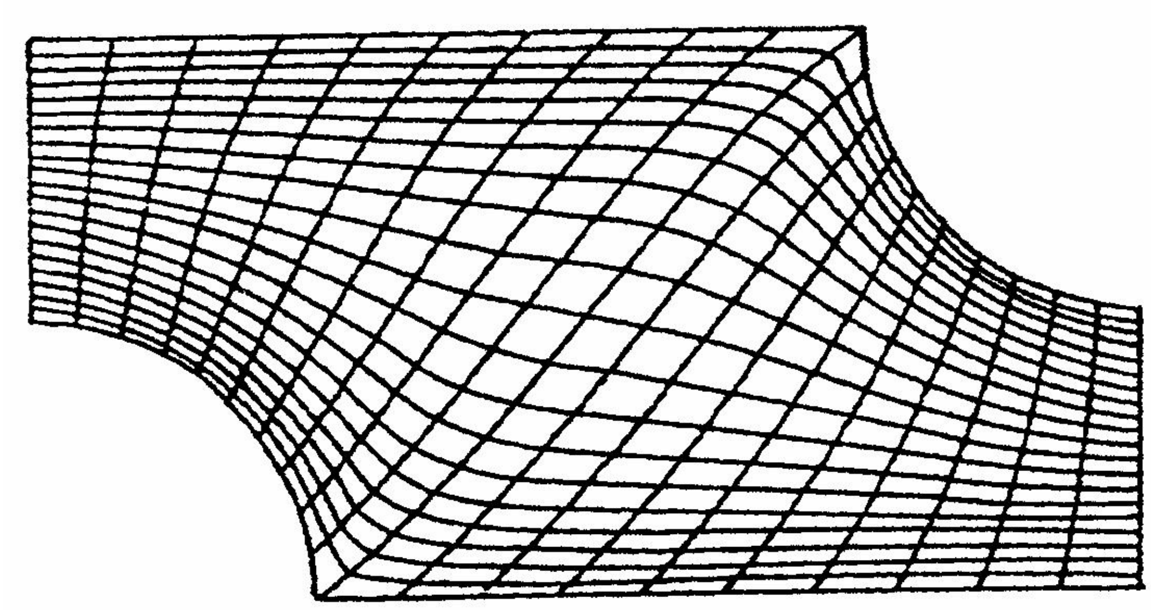
9.2 贴体坐标和贴体坐标转换



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1 贴体坐标

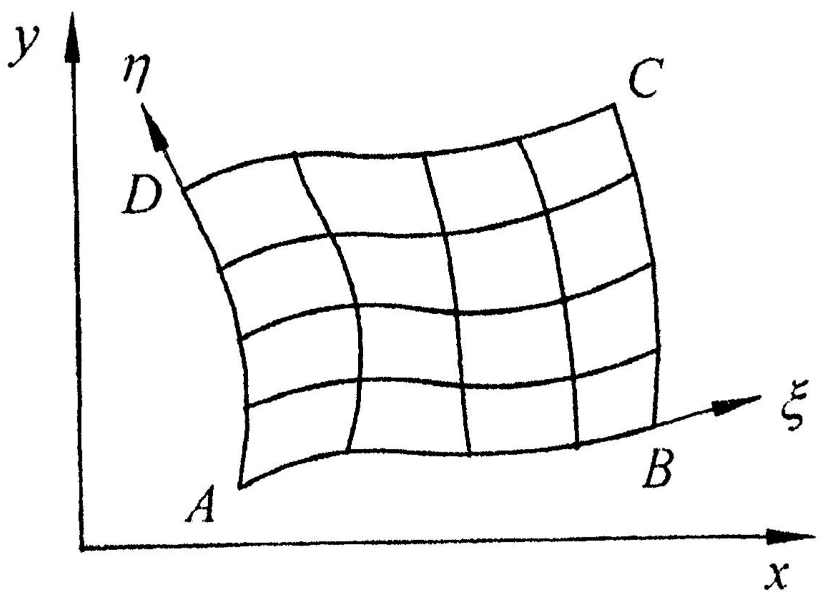
- 理想的坐标系：求解区域边界与坐标系的坐标轴一一相符合。

- 直角坐标系 - 矩形区域
- 极坐标 - 圆形区域
- 球坐标 - 球形区域

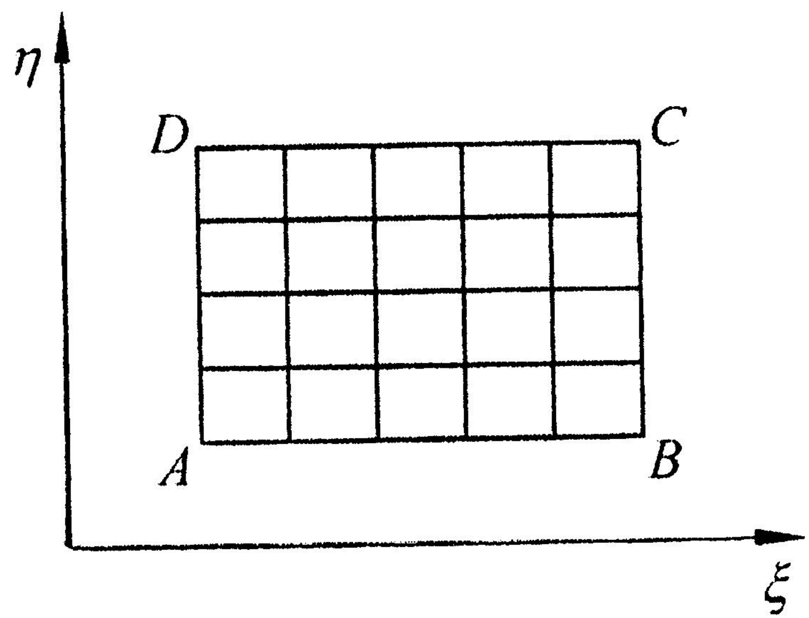
- 称为贴体坐标 (或适体坐标)。在贴体坐标下所建立的计算网格，就叫贴体网格。

2 贴体坐标转换

- 对于一般的复杂求解区域，需要采用坐标转换，构造与之相适应的贴体坐标系

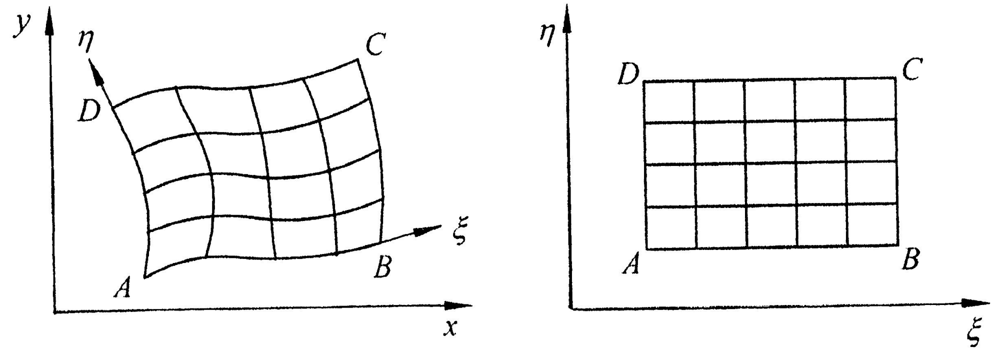


(a) x-y 物理平面



(b) ξ-η 计算平面

坐标转换



(a) x-y 物理平面　(b) ξ-η 计算平面

正变换：

$$\xi = \xi(x, y)$$

$$\eta = \eta(x, y)$$

反变换：

$$x = x(\xi, \eta)$$

$$y = y(\xi, \eta)$$

贴体网格生成：找到计算平面网格节点坐标 (ξ, η)

对应的物理平面网格节点坐标 (x, y)

贴体坐标转换的基本要求

1 转换函数单值，转换节点一一对应

2 物理平面的网格线光滑，提供连续的转换导数

3 物理平面上的网格疏密程度易于控制

4 网格线尽可能避免过于倾斜，以减小截断误差

5 网格线尽可能与边界垂直，以利处理边界条件

3 适体坐标系下的导数变换

正变换：

$$\xi = \xi(x, y)$$
$$\eta = \eta(x, y)$$

$$d\xi = \xi_x dx + \xi_y dy$$
$$d\eta = \eta_x dx + \eta_y dy$$

矩阵形式：

$$\begin{pmatrix} d\xi \\ d\eta \end{pmatrix} = \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}$$

反变换：

$$x = x(\xi, \eta)$$
$$y = y(\xi, \eta)$$

$$dx = x_\xi d\xi + x_\eta d\eta$$
$$dy = y_\xi d\xi + y_\eta d\eta$$

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{pmatrix} \begin{pmatrix} d\xi \\ d\eta \end{pmatrix}$$

坐标转换的Jacobi行列式(Jacobi因子)

$$\begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} = \begin{pmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{pmatrix}^{-1} = \frac{1}{J}\begin{pmatrix} y_\eta & -x_\eta \\ -y_\xi & x_\xi \end{pmatrix}$$

其中

$$J = \begin{vmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{vmatrix} = x_\xi y_\eta - x_\eta y_\xi = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix}^{-1}$$

转换坐标变量之间的微分关系

$$\xi_x = \frac{y_\eta}{J},\quad \xi_y = \frac{-x_\eta}{J},\quad \eta_x = \frac{-y_\xi}{J},\quad \eta_y = \frac{x_\xi}{J}$$

坐标变换式的度量系数

度量系数

- 度量系数：应当用有限差分方法确定，而不应按变换的分析式计算

- 度量系数：不应采用插值方法确定，而应直接按各节点的坐标值用差分式计算，节点的坐标值如果需要则可以采用插值方法确定

导数变换关系

一阶偏导数

$$\frac{\partial}{\partial x}=\frac{y_\eta}{J}\frac{\partial}{\partial \xi}-\frac{y_\xi}{J}\frac{\partial}{\partial \eta},\qquad \frac{\partial}{\partial y}=-\frac{x_\eta}{J}\frac{\partial}{\partial \xi}+\frac{x_\xi}{J}\frac{\partial}{\partial \eta}$$

二阶偏导数

$$\frac{\partial^2}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{y_\eta}{J}\frac{\partial}{\partial \xi}-\frac{y_\xi}{J}\frac{\partial}{\partial \eta}\right)=\frac{y_\eta}{J}\frac{\partial}{\partial \xi}\left(\frac{y_\eta}{J}\frac{\partial}{\partial \xi}\right)-\frac{y_\xi}{J}\frac{\partial}{\partial \eta}\left(\frac{y_\eta}{J}\frac{\partial}{\partial \xi}\right)$$

$$+\frac{y_\eta}{J}\frac{\partial}{\partial \xi}\left(-\frac{y_\xi}{J}\frac{\partial}{\partial \eta}\right)-\frac{y_\xi}{J}\frac{\partial}{\partial \eta}\left(-\frac{y_\xi}{J}\frac{\partial}{\partial \eta}\right)$$

二阶偏导数:

$$\frac{\partial^2}{\partial y^2} = \frac{\partial}{\partial y}\left(-\frac{x_\eta}{J}\frac{\partial}{\partial \xi} + \frac{x_\xi}{J}\frac{\partial}{\partial \eta}\right) = -\frac{x_\eta}{J}\frac{\partial}{\partial \xi}\left(-\frac{x_\eta}{J}\frac{\partial}{\partial \xi}\right)$$

$$+\frac{x_\xi}{J}\frac{\partial}{\partial \eta}\left(-\frac{x_\eta}{J}\frac{\partial}{\partial \xi}\right) - \frac{x_\eta}{J}\frac{\partial}{\partial \xi}\left(\frac{x_\xi}{J}\frac{\partial}{\partial \eta}\right) + \frac{x_\xi}{J}\frac{\partial}{\partial \eta}\left(\frac{x_\xi}{J}\frac{\partial}{\partial \eta}\right)$$

交叉偏导数:

$$\frac{\partial^2}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{y_\eta}{J}\frac{\partial}{\partial \xi} - \frac{y_\xi}{J}\frac{\partial}{\partial \eta}\right) = \frac{-x_\eta}{J}\frac{\partial}{\partial \xi}\left(\frac{y_\eta}{J}\frac{\partial}{\partial \xi}\right) + \frac{x_\xi}{J}\frac{\partial}{\partial \eta}\left(\frac{y_\eta}{J}\frac{\partial}{\partial \xi}\right)$$

$$-\frac{x_\eta}{J}\frac{\partial}{\partial \xi}\left(-\frac{y_\xi}{J}\frac{\partial}{\partial \eta}\right) + \frac{x_\xi}{J}\frac{\partial}{\partial \eta}\left(-\frac{y_\xi}{J}\frac{\partial}{\partial \eta}\right)$$

Laplace 算子转换到计算平面

$$
\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=\frac{1}{J^2}\left\{\left(x_\eta^2+y_\eta^2\right)\frac{\partial^2}{\partial \xi^2}+\left(x_\xi^2+y_\xi^2\right)\frac{\partial^2}{\partial \eta^2}\right.
$$

$$
-2\left(x_\xi x_\eta+y_\xi y_\eta\right)\frac{\partial^2}{\partial \eta \partial \xi}+\left(x_\eta x_{\xi\eta}+y_\eta y_{\xi\eta}-x_\xi x_{\eta\eta}-y_\xi y_{\eta\eta}\right)\frac{\partial}{\partial \xi}
$$

$$
+\left(x_\xi x_{\xi\eta}+y_\xi y_{\xi\eta}-x_\eta x_{\xi\xi}-y_\eta y_{\xi\xi}\right)\frac{\partial}{\partial \eta}-\left(x_\eta^2+y_\eta^2\right)\frac{J_\xi}{J}\frac{\partial}{\partial \xi}
$$

$$
\left.-\left(x_\xi^2+y_\xi^2\right)\frac{J_\eta}{J}\frac{\partial}{\partial \eta}+\left(x_\xi x_\eta+y_\xi y_\eta\right)\frac{J_\eta}{J}\frac{\partial}{\partial \xi}+\left(x_\xi x_\eta+y_\xi y_\eta\right)\frac{J_\xi}{J}\frac{\partial}{\partial \eta}\right\}
$$

Laplace 算子表达式简化形式

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=\frac{1}{J^2}\left[\alpha\frac{\partial^2}{\partial\xi^2}-2\beta\frac{\partial^2}{\partial\xi\partial\eta}+\gamma\frac{\partial^2}{\partial\eta^2}+\tau\frac{\partial}{\partial\xi}+\sigma\frac{\partial}{\partial\eta}\right]$$

其中

$$\alpha=x_\eta^2+y_\eta^2,\qquad \beta=x_\xi x_\eta+y_\xi y_\eta,\qquad \gamma=x_\xi^2+y_\xi^2$$

$$\tau=\left(x_\eta D_y-y_\eta D_x\right)/J,\quad \sigma=\left(y_\xi D_x-x_\xi D_y\right)/J$$

$$D_x=\alpha x_{\xi\xi}-2\beta x_{\xi\eta}+\gamma x_{\eta\eta},\quad D_y=\alpha y_{\xi\xi}-2\beta y_{\xi\eta}+\gamma y_{\eta\eta}$$

4 应用贴体坐标解题过程

1）生成贴体网格：在计算平面上选定与物理平面上复杂求解域相对应的求解区域，给定边界条件，求出两平面内节点的对应关系

2）将控制方程和定解条件在贴体坐标下写出并离散

3）在计算平面求解离散代数方程，并将解传递回物理平面