ASYMMETRIC EXCLUSION PROCESSES ON LATTICES WITH A JUNCTION: THE EFFECT OF UNEQUAL INJECTION RATES

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Asymmetric exclusion processes (ASEP) on lattices with a junction, in which two or more parallel lattice branches combine into a single one, is important as a model for complex transport phenomena. This paper investigates the effect of unequal injection rates in ASEP with a junction. It is a generalization of the work of Pronina and Kolomeisky [J. Stat. Mech. P07010 (2005)], in which only equal injection rates are considered. It is shown that the unequal rates give rise to new phases and the phase diagram structure is qualitatively changed. The phase diagram and the density profiles are investigated by using Monte Carlo simulations, mean field approximation and domain wall approach. The analytical results are in good agreement with Monte Carlo simulations.

Keywords: ASEP; mean field approximation; junction.

1. Introduction

The analysis of self-driven particle models is a central issue of non-equilibrium statistical mechanics. These models show a variety of generic non-equilibrium effects, in particular, in low dimensions. A very prominent example of such a model is the asymmetric exclusion processes (ASEPs). ASEP was introduced originally as a model to explain the process of creation of the messenger RNA in 1968 and now have become an important tool to investigate various non-equilibrium phenomena in chemistry, physics and biology. They have been applied successfully to understand polymer dynamics in dense media, diffusion through membrane chains, gel electrophoresis, dynamics of motor proteins moving along rigid filaments, the kinetics of synthesis of proteins, and traffic flow analysis.

ASEPs are one-dimensional models where particles hop along discrete lattices. The only interaction between particles is the hard-core exclusion that prevents any
site from being occupied by more than one particle, i.e. each lattice site can be either empty or occupied by a single particle. There are several exact solutions for the stationary state properties of ASEP for periodic\textsuperscript{9} and open boundary conditions\textsuperscript{10} and different update rules.\textsuperscript{11–13}

Recently, to analyze more realistic phenomena, a number of works have been carried out to extend the ASEP to include the possibility of transport on lattices with a more complex geometry than one-channel lattices, e.g. multi-channel lattices,\textsuperscript{14} two-way traffic,\textsuperscript{15} intersected lattices,\textsuperscript{16} lattices with double chains,\textsuperscript{17} lattices with a junction,\textsuperscript{18,19} and lattices with on-ramp and off-ramp.\textsuperscript{20} Here we would like to mention that lattices with a junction, in which two or more parallel lattice branches combine into a single one, is important as a model for complex biological and vehicular transport phenomena. For instance, the junction of protofilaments might lead to motor protein crowding phenomena that are responsible for many human diseases and the road junction might lead to serious traffic congestions.

In this paper we investigate effect of unequal injection rates in ASEP on lattices with a junction. Our work is a generalization of that of Pronina and Kolomeisky,\textsuperscript{18} in which only equal injection rates are considered. It is shown that new phases are observed and the phase diagram can be classified into five regions. The density profile of the stationary phases and phase boundaries are investigated in detail by using Monte Carlo simulation, mean field approximation, and domain wall theory.

The paper is organized as follows. In Sec. 2, we outline our model. Section 3 presents the Monte Carlo simulation results and the theoretical calculation results. Section 4 studies the density profiles on phase boundaries from domain wall approach. We summarize our paper in Sec. 5.

2. Model

Figure 1(a) shows the schematic picture of the model, where particles move along the two parallel lattice chains and then merge into a single lattice chain at the junction. The system consists of chains I, II and III which start from sites 1, $L + 1$, $2L + 1$ and end in sites $L$, $2L$, $3L$, respectively. Random sequential update rules are adopted. In one Monte Carlo step (MCS), the following steps are repeated 3$L$ times. (i) A site $i$ ($1 \leq i \leq 3L$) is chosen randomly; (ii) If $i = 1$ or $i = L + 1$ and the site is empty, a particle is inserted with rate $\alpha_1$ or $\alpha_2$ into chain I or II; (iii) If $i = 3L$ and the site is occupied, the particle is removed with rate $\beta$; (iv) Otherwise, the particle move one step forward if the site is occupied and the target site is empty. Note that in the special case $\alpha_1 = \alpha_2$, the model reduces to that of Pronina and Kolomeisky.\textsuperscript{18}

3. Monte Carlo Simulation and Theoretical Calculation

3.1. Simulation results

In this subsection, the Monte Carlo simulation results are presented. In the simulation, the system size is set to $L = 1000$ unless otherwise mentioned.
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It is well known that there are three stationary phases for ASEP on a single-chain with open boundaries, specified by the processes at the entrance, at the exit and in the bulk of the system. At small values of injection rates $\alpha < 1/2$ and $\alpha < \beta$, the entrance of the particles is the limiting process, the system is in a low-density (LD) phase with $J = \alpha(1 - \alpha)$, $\rho_{\text{bulk}} = \alpha$. At small values of removal rates $\beta < 1/2$ and $\beta < \alpha$, the overall dynamics is governed by the particle exit, the system is in a high-density (HD) phase with $J = \beta(1 - \beta)$, $\rho_{\text{bulk}} = 1 - \beta$. At large values of the injection rate $\alpha > 1/2$ and removal rate $\beta > 1/2$, the dynamics of the system is determined by processes in the bulk, the system is in a maximal-current (MC) phase with $J = 1/4$, $\rho_{\text{bulk}} = 1/2$.

Based on these, Figs. 2 and 3 present the phase diagram of the system. Due to symmetry, we study $\alpha_1 > \alpha_2$. Denote $n = \alpha_2/\alpha_1$, Fig. 2 shows the diagram in the space $(\alpha_1, \beta)$ with $n$ fixed. It can be seen that five phases are identified, i.e. (LD, LD, LD), (HD, LD, MC), (HD, HD, MC), (HD, LD, HD), and (HD, HD, HD). Here (X, Y, Z) denotes that chains I, II, and III are in the X phase, Y phase and Z phase, respectively. The typical density profiles of the five phases are shown in Fig. 4. Figure 2(b) shows that with the increase/decrease of $n$, (LD, LD, LD) and (HD, HD, MC) shrink/expand. In the special case $n = 1$, (HD, LD, HD) and (HD, LD, MC) disappear. When $n$ is extremely small, (HD, HD, MC) also disappears [Fig. 2(c)].

Figure 3 shows the diagram in the space of $(\alpha_1, \alpha_2)$ with $\beta$ fixed. When $\beta \geq 0.5$, the diagram is shown in Fig. 3(a) and it is independent of $\beta$. Three phases exist,
i.e. (LD, LD, LD), (HD, LD, MC), (HD, HD, MC). When $\beta < 0.5$, (HD, HD, MC) transits into (HD, HD, HD), and (HD, LD, MC) transits into (HD, LD, HD). Moreover, (LD, LD, LD) phase shrinks and the boundary between (HD, LD, HD) and (HD, HD, HD) moves downwards with the decrease of $\beta$ [Fig. 3(b)].

We study the flow rates $J_I$ and $J_{II}$ on chains I and II in (HD, LD, MC) phase and (HD, LD, HD) phase (Fig. 5). It is found that at given $\alpha_1$ and $\beta$, $J_I$ decreases while $J_{II}$ increases with the increase of $\alpha_2$. However, $J_{II}$ is always smaller than $J_I$. On the boundary between (HD, LD, MC) and (HD, HD, MC) [Fig. 5(a)], or between (HD, LD, HD) and (HD, HD, HD) [Fig. 5(b)], $J_{II}$ becomes equal to $J_I$ and both $J_{II}$ and $J_I$ do not change with the further increase of $\alpha_2$ in (HD, HD, MC) phase or (HD, HD, HD) phase.

3.1.1. Mean field analysis

In this subsection, we present the approximate stationary solutions of this model by using the mean field approximation. As shown in Fig. 1(b), we introduce effective...
injection and removal rates for each chain segment. Chain I (II) has the injection rate $\alpha_1$ ($\alpha_2$) and the effective removal rate $\beta_{\text{eff}1}$ ($\beta_{\text{eff}2}$). Chain III has the effective injection rate $\alpha_{\text{eff}}$ and the removal rate $\beta$. Next we discuss the five phases.

The ($\text{HD, HD, MC}$) phase. In HD phase, the overall dynamics is governed by the particle exit. As particles at sites $L$ and $2L$ have the same priority to enter site $2L+1$, it is clear that $\beta_{\text{eff}1} = \beta_{\text{eff}2} = \beta_{\text{eff}}$ and different injection rates $\alpha_1$ and $\alpha_2$ only affect the sites near the entrance of chains I and II. From current conservation, one has

$$2\beta_{\text{eff}}(1 - \beta_{\text{eff}}) = \frac{1}{4}.$$ 

Thus $\beta_{\text{eff}} = (2 - \sqrt{2})/4$.\(^{a}\) Therefore, for the ($\text{HD, HD, MC}$) phase to exist, one has\(^b\)

$$\alpha_2 > \frac{2 - \sqrt{2}}{4}, \quad \beta \geq \frac{1}{2}.$$ 

The ($\text{HD, HD, HD}$) phase. As in ($\text{HD, HD, MC}$) phase, $\beta_{\text{eff}1} = \beta_{\text{eff}2} = \beta_{\text{eff}}$. When chain III is in HD phase, the current is calculated by $J_{111} = \beta(1 - \beta)$. The current conservation thus leads to

$$2\beta_{\text{eff}}(1 - \beta_{\text{eff}}) = \beta(1 - \beta)$$

and the solution is

$$\beta_{\text{eff}} = \frac{1 - \sqrt{1 - 2\beta(1 - \beta)}}{2}.$$ 

\(a\)Note that the other solution of $\beta_{\text{eff}}$ needs to be discarded because $\beta_{\text{eff}}$ should be smaller than $1/2$ in HD phase.

\(b\)Note that $\alpha_1 > \alpha_2$. 

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**Fig. 3.** Phase diagram of the system. (a) $\beta = 0.7$; (b) $\beta = 0.4, 0.1$ (dashed blue line). Symbols are from Monte Carlo simulations while lines are from mean field calculations.
Fig. 4. Density profiles for (a) (HD, HD, MC) phase with parameters $\alpha_1 = 0.5, \alpha_2 = 0.3, \beta = 0.7$; (b) (HD, LD, MC) phase with parameters $\alpha_1 = 0.1, \alpha_2 = 0.5, \beta = 0.7$; (c) (HD, HD, HD) phase with parameters $\alpha_1 = 0.5, \alpha_2 = 0.3, \beta = 0.4$; (d) (HD, LD, HD) phase with parameters $\alpha_1 = 0.5, \alpha_2 = 0.1, \beta = 0.4$; (e) (LD, LD, LD) phase with parameters $\alpha_1 = 0.1, \alpha_2 = 0.07, \beta = 0.4$. Symbols are obtained from the Monte Carlo simulations while lines are our theoretical predictions.
Thus, the system is in (HD, HD, HD) phase when
\[ \alpha_2 > \frac{1 - \sqrt{1 - 2\beta(1 - \beta)}}{2}, \quad \beta < \frac{1}{2}. \]

The (HD, LD, MC) phase. The current conservation requires that
\[ \beta_{\text{eff1}}(1 - \beta_{\text{eff1}}) + \alpha_2(1 - \alpha_2) = \frac{1}{4}. \]
The solution is
\[ \beta_{\text{eff1}} = \frac{1}{2} - \sqrt{\alpha_2(1 - \alpha_2)}. \]
Moreover, it has been found in previous subsection that \( J_{II} \) is always smaller than \( J_I \) in this phase. In other words, \( J_{II} \) should be smaller than \( J_{III}/2 = 1/8 \). Therefore, one has
\[ \alpha_2(1 - \alpha_2) < \frac{1}{8}. \]
The solution is \( \alpha_2 < (2 - \sqrt{2})/4 \). Thus, the conditions for the existence of the (HD, LD, MC) phase are
\[ \alpha_1 > \frac{1}{2} - \sqrt{\alpha_2(1 - \alpha_2)}, \quad \alpha_2 < \frac{2 - \sqrt{2}}{4}, \quad \beta \geq \frac{1}{2}. \]

The (HD, LD, HD) phase. From current conservation, one has
\[ \beta_{\text{eff1}}(1 - \beta_{\text{eff1}}) + \alpha_2(1 - \alpha_2) = \beta(1 - \beta). \]
The solution is
\[ \beta_{\text{eff1}} = \frac{1 - \sqrt{1 - 4[\beta(1 - \beta) - \alpha_2(1 - \alpha_2)]}}{2}. \]
Similarly, $J_{II}$ should be smaller than $J_{III}/2$, thus

$$\alpha_2(1 - \alpha_2) < \frac{\beta(1 - \beta)}{2}. $$

As a result, the (HD, LD, HD) phase can exist when

$$\alpha_1 > \frac{1 - \sqrt{1 - 2\beta(1 - \beta) - \alpha_2(1 - \alpha_2)}}{2}, \quad \alpha_2 < \frac{1 - \sqrt{1 - 2\beta(1 - \beta)}}{2}, \quad \beta < \frac{1}{2}.$$

The (LD, LD, LD) phase. The current conservation leads to

$$\alpha_{\text{eff}}(1 - \alpha_{\text{eff}}) = \alpha_1(1 - \alpha_1) + \alpha_2(1 - \alpha_2).$$

The solution is\(^{c}\)

$$\alpha_{\text{eff}} = \frac{1 - \sqrt{1 - 4[\alpha_1(1 - \alpha_1) + \alpha_2(1 - \alpha_2)]}}{2}.$$ 

Thus, the conditions for the existence of the (LD, LD, LD) phase are

$$\frac{1 - \sqrt{1 - 4[\alpha_1(1 - \alpha_1) + \alpha_2(1 - \alpha_2)]}}{2} < \min \left( \beta, \frac{1}{2} \right).$$

In the space of $(\alpha_1, \beta)$, taking into account $\alpha_2 = n\alpha_1$, we can obtain the following results.

The (HD, HD, MC) phase:

$$\alpha_1 > \frac{2 - \sqrt{2}}{4n}, \quad \beta \geq \frac{1}{2}.$$ 

The (HD, HD, HD) phase:

$$\alpha_1 > \frac{1 - \sqrt{1 - 2\beta(1 - \beta)}}{2n}, \quad \beta < \frac{1}{2}.$$ 

The (HD, LD, MC) phase:

$$\frac{n + 1 - \sqrt{2n}}{2(1 + n^2)} < \alpha_1 < \frac{2 - \sqrt{2}}{4n}, \quad \beta \geq \frac{1}{2}.$$ 

The (HD, LD, HD) phase:

$$\frac{1 + n - \sqrt{(1 + n)^2 - 4(1 + n^2)\beta(1 - \beta)}}{2(1 + n^2)} < \alpha_1 < \frac{1 - \sqrt{1 - 2\beta(1 - \beta)}}{2n}, \quad \beta < \frac{1}{2}.$$ 

The (LD, LD, LD) phase:

$$\alpha_1 < \frac{n + 1 - \sqrt{2n}}{2(1 + n^2)} \quad \text{for} \quad \beta > \frac{1}{2}.$$ 

$$\alpha_1 < \frac{1 + n - \sqrt{(1 + n)^2 - 4(1 + n^2)\beta(1 - \beta)}}{2(1 + n^2)} \quad \text{for} \quad \beta < \frac{1}{2}.$$ 

We have compared our analytical results with the simulation results in Figs. 2 and 3, and it is found they are in very good agreement.

\(^{c}\)Note that the other solution of $\alpha_{\text{eff}}$ needs to be discarded because $\alpha_{\text{eff}}$ should be smaller than $1/2$ in LD phase.
4. Density Profiles on Phase Boundaries

Now we focus on the phase boundaries. Firstly, phase transitions between (HD, HD, MC) and (HD, HD, HD), between (HD, LD, MC) and (HD, LD, HD) are of second order and thus does not need to be considered. On the other hand, boundary 1 between (LD, LD, LD) and (HD, LD, MC), boundary 2 between (HD, LD, MC) and (HD, HD, MC), boundary 3 between (HD, LD, HD) and (HD, HD, HD) are of first order due to density jump on chain I (boundary 1) or on chains II (boundaries 2 and 3). Consequently, we can expect a linear density profile on chain I (from $\alpha_1$ to $1 - \alpha_1$) or on chains II (from $\alpha_2$ to $1 - \alpha_2$). As shown in Fig. 6, the Monte Carlo simulation results are in good agreement with the theoretical predictions.

The boundary between (LD, LD, LD) and (HD, LD, HD) is also of first order. However, different from boundaries 1–3, density jump happens on both chain I and chain III. This introduces unnoticeable correlations near the junction point. As a
result, the linear density profile of mean field expectation strongly deviates from simulations, which can be seen from Fig. 7. In this case, the domain wall approach presented by Pronina and Kolomeisky\textsuperscript{18} takes correlations into account and could correctly predict the density profiles as shown below.

Let us denote the moving rates of the domain wall in chain I and chain III as $u_I$ and $u_{III}$, which can be determined by utilizing the expression

$$u_k = \frac{J_k}{\rho_+^k - \rho_-^k} \quad \text{for} \quad k = I, III,$$

where $J$ is bulk stationary state value of the current and $\rho_+, -$ denote density in the right (+) and left (−) of the domain wall. It is clear

$$J_I = \alpha_1(1 - \alpha_1), \quad J_{III} = \alpha_2(1 - \alpha_2), \quad J_{II} = \beta(1 - \beta),$$

$$\rho_+^I = \alpha_1, \quad \rho_-^I = 1 - \alpha_1, \quad \rho_+^{III} = \beta, \quad \rho_-^{III} = 1 - \beta.$$

From $J_{III} = J_I + J_{II}$ and $\alpha_2 = n\alpha_1$, we obtain

$$\beta = \frac{1 - \sqrt{1 - 4[(1 + n)\alpha_1 - (1 + n^2)\alpha_1^2]}}{2}$$

and

$$u_I = \frac{\alpha_1(1 - \alpha_1)}{1 - 2\alpha_1}, \quad u_{III} = \frac{(n + 1)\alpha_1 - (n^2 + 1)\alpha_1^2}{1 - 2\beta}. \quad (1)$$

At the junction, we have

$$\frac{u_IP_I}{L} = \frac{u_{III}P_{III}}{L}, \quad P_I + P_{III} = 1, \quad (2)$$
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here \(P_I\) and \(P_{III}\) are the probabilities of finding the domain wall at any position in chain I and chain III respectively.

Solving Eq. (2), we have

\[
P_I = \frac{u_{III}}{u_I + u_{III}}, \quad P_{III} = \frac{u_I}{u_I + u_{III}},
\]

which implies that the domain wall has different probabilities of being found in chain I and in chain III.

We introduce relative coordinate \(x = i/L\), where \(i\) is the site index. Thus, the case of \(0 < x \leq 1\) and \(1 < x \leq 2\) describe chain I and chain III respectively. Let \(x_{DW}\) denote position of the domain wall. Then we can obtain the probability distribution function of the domain wall

\[
\text{Prob}(x_{DW} < x) = P_I x, \quad 0 < x \leq 1
\]

\[
\text{Prob}(x_{DW} < x) = P_I + P_{III}(x - 1), \quad 1 < x \leq 2.
\]

Therefore, the density at any position is given by

\[
\rho(x) = \rho_I^k \text{Prob}(x_{DW} < x) + \rho_{III}^k \text{Prob}(x_{DW} > x), \quad k = I, III.
\]

Substituting Eqs. (1), (3)–(5) into Eq. (6), we have

\[
\rho(x)_I = \alpha_1 + \frac{(1 - 2\alpha_1)^2[(n + 1) - (n^2 + 1)\alpha_1]}{(1 - 2\alpha_1)(n + 1) - (n^2 + 1)\alpha_1 + (1 - \alpha_1)(1 - 2\beta)} x, \quad 0 < x \leq 1
\]

\[
\rho(x)_{III} = \beta + \frac{(1 - 2\alpha_1)((n + 1) - (n^2 + 1)\alpha_1)\sqrt{1 - 4[(n + 1)\alpha_1 - (n^2 + 1)\alpha_1^2]}}(1 - 2\alpha_1)(n + 1) - (n^2 + 1)\alpha_1 + (1 - \alpha_1)(1 - 2\beta)
\]

\[
+ \frac{(1 - 4[(n + 1)\alpha_1 - (n^2 + 1)\alpha_1^2])(1 - \alpha_1)}(1 - 2\alpha_1)(n + 1) - (n^2 + 1)\alpha_1 + (1 - \alpha_1)(1 - 2\beta)(x - 1),
\]

\[
1 < x \leq 2.
\]

The result from the domain wall approach is shown in Fig. 7, and it agrees well with simulations.

5. Conclusion

In this paper, we have investigated the effect of unequal injection rates in ASEP on lattices with a junction. The phase diagrams on \((\alpha_1, \alpha_2)\) plane with \(\beta\) fixed and on \((\alpha_1, \beta)\) plane with \(n = \alpha_2/\alpha_1\) fixed are presented. It is shown that there are five phases: (HD, HD, MC), (HD, LD, MC), (HD, HD, HD), (HD, LD, HD) and (LD, LD, LD), among which (HD, LD, MC) phase and (HD, LD, HD) phase will disappear when \(n = 1\). We have compared the mean field approximation of the density profiles and the phase boundaries with the Monte Carlo simulation results. It is found that good agreement could be achieved except the density profiles on the boundary between (LD, LD, LD) and (HD, LD, HD). This reveals that
the correlation is unnegligible in this case. The domain wall theory is adopted to calculate the density profile in this case and the results agree well with Monte Carlo simulations.

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