Power spectra of totally asymmetric exclusion processes on lattices with a junction

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Received 10 October 2014
Accepted 22 June 2015
Published 24 July 2015

In this paper, we have studied the power spectra of the entire particle occupancy in the totally asymmetric exclusion processes on lattices with a junction, at which two lanes I and II merge into lane III. We have performed the investigations, utilizing both Monte Carlo simulations and theoretical analysis based on linearized Langevin equations. The effect of lane length, particle entry rates at lanes I and II, and particle exit rate at lane III have been discussed. The transition from damped periodic oscillation to irregular oscillation as well as disappearance of oscillation has been observed. In particular, when lanes I, II, III are simultaneously in low density (LD) state or high density (HD) state, and the length $L_1$ of lane I is not remarkably different from the length $L_2$ of lane II, the damped periodic oscillations could be observed and the minima of the power spectra could be calculated via weighted average. However, when $L_1$ is notably different from $L_2$, the oscillation becomes irregular.

Keywords: ASEP; power spectra; linearized Langevin equations.

PACS Nos.: 05.40.-a, 05.60.Cd, 05.70.Ln, 87.10.+e.

1. Introduction

In recent years, the asymmetric simple exclusion process (ASEP) has attracted the interests of physicists, because it plays an important role in understanding various nonequilibrium phenomena in chemistry, physics and biology.1−3 In this model, the stochastic dynamics of multi-particle transport across one-dimensional lattices is described, in which particles interact only through a hard core exclusion potential. The ASEP has a number of important applications in biological transport, kinetics of protein synthesis, biopolymerization, car traffic, etc.4−12 Moreover, ASEP has exhibited many nonequilibrium phenomena, e.g. boundary induced phase transition, phase separation, spontaneous symmetry breaking, localized shock.13−22

In the simplest limit of an ASEP, which is called the totally asymmetric simple exclusion process (TASEP), particles move only in one direction. With open boundaries,
three stationary phases are identified, determined by the injection rate $\alpha$ at the entrance and the extraction rate $\beta$ at the exit. With small injection rate and large extraction rate: $\alpha < \beta$ and $\alpha < 0.5$, the system is found in a low density (LD) entry limited phase. When $\alpha > \beta$ and $\alpha < 0.5$, the system is in a high density (HD) exit limited phase. When $\alpha > 0.5$ and $\beta > 0.5$ the system is determined by processes in the bulk, and we have a maximal-current (MC) phase. Finally, when $\alpha = \beta < 0.5$, the system is in coexistence state in which the domain wall separating the LD and the HD performs a random walk in the system, and the system exhibits a linear density profile.

Recently, Adams et al. have investigated power spectra of the time series of the total number of particles $N(t)$ in a TASEP lattice, which provides insight into the time correlation of $N(t)$. It is found that in the MC phase and on the coexistence line between LD and HD phases, the power spectrum displays algebraic decay. However, deep within the HD and LD phases, the pronounced oscillations, which damp into power laws, have been found. The physical origins of the oscillations can be traced to the time it takes a fluctuation to propagate throughout the entire lattice.

Cook and Dong have generalized the power spectra analysis to the TASEP with a defect. The disappearance of oscillations has been observed when the defect is located at the center of the system. When the defect is off-center, oscillations are restored. Cook and Zia have studied the power spectra of a constrained TASEP, taking into account the finite supply of particles for the TASEP. Large suppressions at low frequencies have occurred due to the added constraint. Angel and Zia have further extended the power spectra analysis to study the total occupation number in a segment in a zero-range process on a ring. Two distinct components of damped oscillations have been found and the essential origin of both components is shown in a simple pedagogical model.

In order to describe some more complex realistic processes, many extensions of the original TASEP have been studied. Junctions, in which two or more lanes merge into one lane, have been frequently observed in reality, such as (i) decrease of the number of protofilaments on a microtubule and (ii) merging of two or more roads. Since a microtubule junction might lead to motor protein crowding phenomena that are responsible for many human diseases and a road junction might lead to traffic congestion, the TASEP with a junction has been widely investigated.

This paper is motivated by the works mentioned above and studies the power spectra of the TASEP with a junction. Both theoretical analysis and extensive computer Monte Carlo simulations have been carried out to explore the power spectrum of the total number of the particles in the entire system.

The paper is organized as follows. The model and the theoretical analysis are presented in next section. Results were discussed and compared with the simulation ones in Sec. 3. Finally, conclusion is given Sec. 4.

2. The Model and Theoretical Analysis

The sketch of the model is shown in Fig. 1. The system consists of three lanes I, II and III. At the junction, lanes I and II merge together to form lane III. The length of the
three lanes are \( L_1, L_2 \) and \( L_3 \), respectively. Particles are injected from the left end of lanes I and II with rate \( \alpha_1 \) and \( \alpha_2 \), respectively, and removed at the right end of lane III with rate \( \beta \), and move along the lattice with unit rate.

The random sequential updating scheme is used and we define one Monte Carlo step (MCS) as making \( L + 2 \) attempts to update the lattice so that on average one chance is given to update each lattice site as well as introduce a new particle at both lanes I and II. Here, \( L = L_1 + L_2 + L_3 \).

All measurements are taken after 100K MCS to ensure that the system has reached the steady state. The total number of particles in the system \( N(t) \) is measured every 10 MCS and we label these by \( t = 1, 2, \ldots, T \) where \( T = 10^4 \) (10^6 MCS). The results are averaged over 100 runs. Since the three lanes can be viewed as three separate TASEPs, we only focus on the power spectrum of the entire system \( I(f) \), which is defined as

\[
I(f) = \langle |\tilde{N}(f)|^2 \rangle, \quad (1)
\]

where \( \langle \cdots \rangle \) indicates the average over 100 runs and

\[
\tilde{N}(f) = \frac{1}{T} \sum_{t=1}^{T} N(t) e^{ift} \quad (2)
\]

with

\[
f = \frac{2\pi m}{T}; \quad m = 1, 2, \ldots. \quad (3)
\]

Now, we carry out the theoretical analysis. Using the approach proposed in Ref. 24, we begin with the Langevin equation for discrete time, \( \tau \), labeling each attempt at updating. Thus, \( \tau = 1, 2, \ldots, K \), with

\[
K \equiv 10(L + 2)T. \quad (4)
\]

We treat the three lanes separately:

**Lane I**

\[
\partial_\tau \rho(1, \tau) = \frac{\alpha_1(1 - \rho(1, \tau)) - \rho(1, \tau)(1 - \rho(2, \tau))}{L + 2} + \xi_1(\tau) - \xi(1, \tau), \quad (5)
\]
\[ \partial_{\tau} \rho(x, \tau) = \frac{\rho(x - 1, \tau)(1 - \rho(x, \tau)) - \rho(x, \tau)(1 - \rho(x + 1, \tau))}{L + 2} + \xi(x - 1, \tau) - \xi(x, \tau) \quad \{1 < x < L_1\}, \]

\[ \partial_{\tau} \rho(L_1 + 1, \tau) = \frac{\rho(L_1 + 1, \tau)(1 - \rho(L_1 + 1, \tau)) - \rho(L_1 + 1, \tau)(1 - \rho(L_1 + 2, \tau))}{L + 2} + \xi(L_1 - 1, \tau) - \xi(L_1, \tau). \]

Lane II

\[ \partial_{\tau} \rho(L_1 + 1, \tau) = \frac{\alpha_2(1 - \rho(L_1 + 1, \tau)) - \rho(L_1 + 1, \tau)(1 - \rho(L_1 + 2, \tau))}{L + 2} + \xi_2(\tau) - \xi(L_1 + 1, \tau), \]

\[ \partial_{\tau} \rho(x, \tau) = \frac{\rho(x - 1, \tau)(1 - \rho(x, \tau)) - \rho(x, \tau)(1 - \rho(x + 1, \tau))}{L + 2} + \xi(x - 1, \tau) - \xi(x, \tau) \quad \{L_1 + 1 < x < L_1 + L_2\}, \]

\[ \partial_{\tau} \rho(L_1 + L_2, \tau) = \frac{\rho(L_1 + L_2 - 1, \tau)(1 - \rho(L_1 + L_2, \tau)) - \rho(L_1 + L_2, \tau)(1 - \rho(L_1 + L_2 + 1, \tau))}{L + 2} + \xi(L_1 + L_2 - 1, \tau) - \xi(L_1 + L_2, \tau). \]

Lane III

\[ \rho(L_1, \tau)(1 - \rho(L_1 + L_2 + 1, \tau)) + \rho(L_1 + L_2, \tau) \times (1 - \rho(L_1 + L_2 + 1, \tau)) - \rho(L_1 + L_2 + 1, \tau) \]

\[ \partial_{\tau} \rho(L_1 + L_2 + 1, \tau) = \frac{\rho(L_1 + L_2 + 1, \tau)(1 - \rho(L_1 + L_2 + 2, \tau))}{L + 2} \times (1 - \rho(L_1 + L_2 + 2, \tau)) + \xi(L_1, \tau) + \xi(L_1 + L_2, \tau) - \xi(L_1 + L_2 + 1, \tau), \]

\[ \partial_{\tau} \rho(x, \tau) = \frac{\rho(x - 1, \tau)(1 - \rho(x, \tau)) - \rho(x, \tau)(1 - \rho(x + 1, \tau))}{L + 2} + \xi(x - 1, \tau) - \xi(x, \tau) \quad \{L_1 + L_2 + 1 < x < L\}, \]

\[ \partial_{\tau} \rho(L, \tau) = \frac{\rho(L - 1, \tau)(1 - \rho(L, \tau)) - \beta \rho(L, \tau)(1 - \rho(L, \tau))}{L + 2} + \xi(L - 1, \tau) - \xi(L, \tau), \]

where the sites on lane I are numbered 1, \ldots, L_1, sites on Lane II are numbered \(L_1 + 1, \ldots, L_1 + L_2\), and sites on lane III are numbered \(L_1 + L_2 + 1, \ldots, L\). \(\xi\) is assumed to be homogeneous, uncorrelated white noise associated with hopping from one site to its next site with zero mean and variance \(A\) such that

\[ A_1 \delta_{x,x'} \delta_{\tau,\tau'} \quad x \in [1, L_1], \]

\[ \langle \xi(x, \tau)\xi(x', \tau') \rangle = A_2 \delta_{x,x'} \delta_{\tau,\tau'} \quad x \in [L_1 + 1, L_1 + L_2], \]

\[ A_3 \delta_{x,x'} \delta_{\tau,\tau'} \quad x \in [L_1 + L_2 + 1, L]. \]
\(\xi_1, \xi_2\) are assumed to be noise hopping into Lane I and Lane II, respectively, and
\(\langle \xi_1(\tau)\xi(x', \tau') \rangle = 0, \ \langle \xi_2(\tau)\xi(x', \tau') \rangle = 0, \ \langle \xi_1(\tau)\xi_1(\tau') \rangle = A_1\delta_{\tau, \tau'}, \ \langle \xi_2(\tau)\xi_2(\tau') \rangle = A_2\delta_{\tau, \tau'}\). The \(1/(L+2)\) on the right-hand side is due to the probability of choosing a particular site to update. For simplicity, we assume that the density is flat in the three lanes. Thus

\[
\tilde{\rho}(x) = \tilde{\rho}_1 \quad x \in [1, L_1]; \\
\tilde{\rho}(x) = \tilde{\rho}_2 \quad x \in [L_1 + 1, L_1 + L_2]; \\
\tilde{\rho}(x) = \tilde{\rho}_3 \quad x \in [L_1 + L_2 + 1, L].
\]

(15)

(16)

(17)

The fluctuations of \(\varphi(x, \tau)\) about its average are defined as

\[
\varphi(x, \tau) = \rho(x, \tau) - \bar{\rho}_1 \quad x \in [1, L_1]; \\
\varphi(x, \tau) = \rho(x, \tau) - \bar{\rho}_2 \quad x \in [L_1 + 1, L_1 + L_2]; \\
\varphi(x, \tau) = \rho(x, \tau) - \bar{\rho}_3 \quad x \in [L_1 + L_2 + 1, L].
\]

(18)

(19)

(20)

If we ignore the interactions, then we keep only terms linear in \(\varphi\). Thus, the Langevin equations could be rewritten as

Lane I

\[
\frac{\partial}{\partial \tau} \varphi(1, \tau) = \frac{-\varphi(1, \tau) + \bar{\rho}_1 \varphi(2, \tau)}{L + 2} + \xi_1(\tau) - \xi(1, \tau),
\]

(21)

\[
\frac{\partial}{\partial \tau} \varphi(x, \tau) = \frac{(1 - \bar{\rho}_1)\varphi(x - 1, \tau) - \varphi(x, \tau) + \bar{\rho}_1 \varphi(x + 1, \tau)}{L + 2}
+ \xi(x - 1, \tau) - \xi(x, \tau) \quad \{1 < x < L_1\},
\]

(22)

\[
(1 - \bar{\rho}_1)\varphi(L_1 - 1, \tau) - (1 - \bar{\rho}_3 + \bar{\rho}_1)\varphi(L_1, \tau)
+ \xi(L_1 - 1, \tau) - \xi(L_1, \tau).
\]

(23)

Lane II

\[
\frac{\partial}{\partial \tau} \varphi(L_1 + 1, \tau) = \frac{-\varphi(L_1 + 1, \tau) + \bar{\rho}_2 \varphi(L_1+2, \tau)}{L + 2} + \xi_2(\tau) - \xi(L_1 + 1, \tau),
\]

(24)

\[
\frac{\partial}{\partial \tau} \varphi(x, \tau) = \frac{(1 - \bar{\rho}_2)\varphi(x - 1, \tau) - \varphi(x, \tau) + \bar{\rho}_2 \varphi(x + 1, \tau)}{L + 2}
+ \xi(x - 1, \tau) - \xi(x, \tau) \quad \{L_1 + 1 < x < L_1 + L_2\},
\]

(25)

\[
(1 - \bar{\rho}_2)\varphi(L_1 + L_2 - 1, \tau) - (1 - \bar{\rho}_3 + \bar{\rho}_2)\varphi(L_1 + L_2, \tau)
+ \xi(L_1 + L_2 - 1, \tau) - \xi(L_1 + L_2, \tau).
\]

(26)
Lane III

\[
\partial_\tau \varphi(L_1 + L_2 + 1, \tau) = \frac{(1 - \bar{\rho}_3)\varphi(L_1, \tau) + (1 - \bar{\rho}_3)\varphi(L_1 + L_2, \tau) - (1 - \bar{\rho}_3 + \bar{\rho}_1 + \bar{\rho}_2)\varphi(L_1 + L_2 + 1, \tau)}{L + 2} + \frac{\bar{\rho}_3\varphi(L_1 + L_2 + 2, \tau)}{L + 2} - \frac{\bar{\rho}_1(\bar{\rho}_3 - \bar{\rho}_1) + \bar{\rho}_2(\bar{\rho}_3 - \bar{\rho}_2)}{L + 2} + \xi(L_1, \tau) \\
+ \xi(L_1 + L_2, \tau) - \xi(L_1 + L_2 + 1, \tau),
\]

(27)

\[
\partial_\tau \varphi(x, \tau) = \frac{(1 - \bar{\rho}_3)\varphi(x - 1, \tau) - \varphi(x, \tau) + \bar{\rho}_3\varphi(x + 1, \tau)}{L + 2} + \xi(x - 1, \tau) - \xi(x, \tau) \quad \{L_1 + L_2 + 1 < x < L\},
\]

(28)

\[
\partial_\tau \varphi(L, \tau) = \frac{(1 - \bar{\rho}_3)\varphi(L - 1, \tau) - \varphi(L, \tau)}{L + 2} + \xi(L - 1, \tau) - \xi_3(L, \tau).
\]

(29)

Making the Fourier transform

\[
\tilde{\xi}_1(\omega) = \frac{1}{K} \sum_\tau e^{i\omega\tau} \xi_1(\tau) \quad \tilde{\xi}_2(\omega) = \frac{1}{K} \sum_\tau e^{i\omega\tau} \xi_2(\tau),
\]

(30)

\[
\tilde{\xi}(x, \omega) = \frac{1}{K} \sum_\tau e^{i\omega\tau} \xi(x, \tau) \quad \tilde{\varphi}(x, \omega) = \frac{1}{K} \sum_\tau e^{i\omega\tau} \varphi(x, \tau)
\]

(31)

with

\[
\omega = \frac{2\pi n}{K} \quad n \in [1, K].
\]

(32)

Equations (21)--(29) change into

Lane I

\[
\varsigma(\omega)\tilde{\varphi}(1, \omega) - \frac{\bar{\rho}_1}{L + 2} \tilde{\varphi}(1, \omega) = \tilde{\eta}(1, \omega),
\]

(33)

\[
-\frac{(1 - \bar{\rho}_1)}{L + 2} \tilde{\varphi}(x - 1, \omega) + \varsigma(\omega)\tilde{\varphi}(x, \omega) - \frac{\bar{\rho}_1}{L + 2} \tilde{\varphi}(x + 1, \omega)
\]

\[
= \tilde{\eta}(x, \omega) \quad \{1 < x < L_1\},
\]

(34)

\[
-\frac{(1 - \bar{\rho}_1)}{L + 2} \tilde{\varphi}(L_1 - 1, \omega) + \left(\varsigma(\omega) + \frac{-\bar{\rho}_3 + \bar{\rho}_1}{L + 2}\right) \tilde{\varphi}(L_1, \omega)
\]

\[
- \frac{\bar{\rho}_1}{L + 2} \tilde{\varphi}(L_1 + L_2 + 1, \omega) = \tilde{\eta}(L_1, \omega).
\]

(35)

Lane II

\[
\varsigma(\omega)\tilde{\varphi}(L_1 + 1, \omega) - \frac{\bar{\rho}_2}{L + 2} \tilde{\varphi}(L_1 + 2, \omega) = \tilde{\eta}(L_1 + 1, \omega),
\]

(36)

\[
-\frac{(1 - \bar{\rho}_2)}{L + 2} \tilde{\varphi}(x - 1, \omega) + \varsigma(\omega)\tilde{\varphi}(x, \omega) - \frac{\bar{\rho}_2}{L + 2} \tilde{\varphi}(x + 1, \omega)
\]

\[
= \tilde{\eta}(x, \omega) \quad \{L_1 + 1 < x < L_1 + L_2\},
\]

(37)
The equations could be written more compactly as

\[
\frac{(1 - \bar{\rho}_3)}{L + 2} \tilde{\varphi}(L_1 + L_2 - 1, \omega) + \left(\varsigma(\omega) + \frac{-\bar{\rho}_3 + \bar{\rho}_2}{L + 2}\right) \tilde{\varphi}(L_1 + L_2, \omega)
\]
\[
- \frac{\bar{\rho}_2}{L + 2} \tilde{\varphi}(L_1 + L_2 + 1, \omega) = \tilde{\eta}(L_1 + L_2, \omega).
\]  

(Lane III)

\[
- \frac{(1 - \bar{\rho}_3)}{L + 2} \tilde{\varphi}(L_1, \omega) - \frac{(1 - \bar{\rho}_3)}{L + 2} \tilde{\varphi}(L_1 + L_2, \omega) + \left(\varsigma(\omega) + \frac{-\bar{\rho}_3 + \bar{\rho}_1 + \bar{\rho}_2}{L + 2}\right)
\]
\[
\times \tilde{\varphi}(L_1 + L_2 + 1, \omega) - \frac{\bar{\rho}_3}{L + 2} \tilde{\varphi}(L_1 + L_2 + 2, \omega) = \tilde{\eta}(L_1 + L_2 + 1, \omega),
\]  

(39)

\[
- \frac{(1 - \bar{\rho}_3)}{L + 2} \tilde{\varphi}(x - 1, \omega) + \varsigma(\omega) \tilde{\varphi}(x, \omega) - \frac{\bar{\rho}_3}{L + 2} \tilde{\varphi}(x + 1, \omega)
\]
\[
= \tilde{\eta}(x, \omega) \quad \{L_1 + L_2 + 1 < x < L\},
\]  

(40)

\[
- \frac{(1 - \bar{\rho}_3)}{L + 2} \tilde{\varphi}(L - 1, \omega) + \varsigma(\omega) \tilde{\varphi}(L, \omega) = \tilde{\eta}(L, \omega),
\]  

(41)

where

\[
\varsigma(\omega) = e^{i\omega} - 1 + \frac{1}{L + 2},
\]  

(42)

\[
\tilde{\eta}(x, \omega) = \begin{cases} 
\xi_1(\omega) - \xi(1, \omega) & \{x = 1\} \\
\xi_2(\omega) - \xi(L_1 + 1, \omega) & \{x = L_1 + 1\} \\
\xi(L_1, \omega) + \xi(L_1 + L_2, \omega) - \xi(L_1 + L_2 + 1, \omega) & \{x = L_1 + L_2 + 1\} \\
\xi(x - 1, \omega) - \xi(x, \omega) & \{\text{otherwise}\}
\end{cases}
\]  

(43)

The equations could be written more compactly as

\[ M \tilde{\varphi} = \tilde{\eta}. \]  

(44)

The \(L \times L\) matrix \(M\) has entries

\[
M_{ii} = \varsigma(\omega) \quad i \in [1, L_1] \cup [L_1 + 1, L_1 + L_2] \cup [L_1 + L_2 + 2, L],
\]  

(45)

\[
M_{ii} = \varsigma(\omega) + \frac{\bar{\rho}_1 - \bar{\rho}_3}{L + 2} \quad i = L_1,
\]  

(46)

\[
M_{ii} = \varsigma(\omega) + \frac{\bar{\rho}_2 - \bar{\rho}_3}{L + 2} \quad i = L_1 + L_2,
\]  

(47)

\[
M_{ii} = \varsigma(\omega) + \frac{\bar{\rho}_1 + \bar{\rho}_2 - \bar{\rho}_3}{L + 2} \quad i = L_1 + L_2 + 1,
\]  

(48)

\[
M_{ii+1} = -\frac{\bar{\rho}_1}{L + 2} \quad i \in [1, L_1],
\]  

(49)

\[
M_{ii+1} = -\frac{\bar{\rho}_2}{L + 2} \quad i \in [L_1 + 1, L_1 + L_2],
\]  

(50)
\[ M_{ii+1} = -\frac{\bar{\rho}_3}{L+2} \quad i \in [L_1 + L_2 + 1, L - 1], \tag{51} \]
\[ M_{ii-1} = -\frac{1 - \bar{\rho}_1}{L+2} \quad i \in [2, L_1], \tag{52} \]
\[ M_{ii+1} = -\frac{1 - \bar{\rho}_2}{L+2} \quad i \in (L_1 + 1, L_1 + L_2], \tag{53} \]
\[ M_{ii+1} = -\frac{1 - \bar{\rho}_3}{L+2} \quad i \in [L_1 + L_2 + 1, L], \tag{54} \]
\[ M_{L_1(L_1+L_2+1)} = -\frac{\bar{\rho}_1}{L+2} \quad M_{(L_1+L_2+1)L_1} = -\frac{1 - \bar{\rho}_3}{L+2}. \tag{55} \]

Other elements \( M_{ij} \) are zero. Note that different from the matrix \( M \) in Ref. 24, here the matrix \( M \) is not a tridiagonal one due to the presence of the two terms in Eq. (55), which makes the power spectra much more complicated. Also note that the element \( M_{L_1(L_1+1)} = 0 \) and \( M_{(L_1+1)L_1} = 0 \). By inverting the matrix \( (M^{-1} = S) \), we have the \( \tilde{\varphi} \)s in terms of the \( \tilde{\eta} \)s, namely \( \tilde{\varphi}(x, \omega) = \sum_{y=1}^{L} S_{xy} \tilde{\eta}(y, \omega). \) \( \tilde{N} \) is related to \( \tilde{\varphi} \) by summing over the spatial coordinate. Thus, we have
\[
\tilde{N} = \frac{(\bar{\rho}_1/L+2)\tilde{\varphi}(1, \omega)+(\bar{\rho}_2/L+2)\tilde{\varphi}(L_1 + 1, \omega) + (1 - \bar{\rho}_2/L+2)\tilde{\varphi}(L, \omega) + \tilde{\xi}(L, \omega) - \tilde{\xi}_2(\omega) - \tilde{\xi}_1(\omega)}{1 - e^{i\omega}}, \tag{57} \]
for the entire lattice. Since the simulation data are recorded at every \( l \) MCS \( (l = 10 \) in this paper),\(^{24} \) we still need to sum over the harmonics \( I(\omega = 2\pi n/K) \) to make meaningful comparisons with the simulation data. As pointed out in Ref. 25, the difference between the two depends on a straightforward sum over the \( T/l \) harmonics of \( f \). Thus, the final result, written in terms of \( m \), is
\[
I(m) = \sum_{z=0}^{l-1} I(\omega_{m,z}), \tag{58} \]
where
\[
\omega_{m,z} = \frac{2\pi}{T} \left( m + \frac{zT}{l} \right). \tag{59} \]

The variances \( A_1, A_2 \) and \( A_3 \) can be calculated from the microscopic description\(^{36} \)
\[
A_1 = \frac{\bar{\rho}_1(1 - \bar{\rho}_1)}{L+2}, \quad A_2 = \frac{\bar{\rho}_2(1 - \bar{\rho}_2)}{L+2}, \quad A_3 = \frac{\bar{\rho}_3(1 - \bar{\rho}_3)}{L+2}. \tag{60} \]
Thus,

\[
I(\omega_{m,z}) = \frac{1}{T(2 - 2 \cos \omega_{m,z})} \times \left[ \left( \frac{\bar{\rho}_1}{L + 2} \right)^2 \langle \tilde{\varphi}(1, \omega_{m,z}) \tilde{\varphi}^*(1, \omega_{m,z}) \rangle \\
+ \left( \frac{\bar{\rho}_2}{L + 2} \right)^2 \langle \tilde{\varphi}(L_1 + 1, \omega_{m,z}) \tilde{\varphi}^*(L_1 + 1, \omega_{m,z}) \rangle \\
+ 2\bar{\rho}_1 \bar{\rho}_2 \frac{L}{L + 2} \mathcal{R} \left[ \langle \tilde{\varphi}(1, \omega_{m,z}) \tilde{\varphi}^*(L_1 + 1, \omega_{m,z}) \rangle \right] \\
+ \left( \frac{1 - \bar{\rho}_3}{L + 2} \right)^2 \langle \tilde{\varphi}(L, \omega_{m,z}) \tilde{\varphi}^*(L, \omega_{m,z}) \rangle \\
+ 2\bar{\rho}_1 (1 - \bar{\rho}_3) \frac{L}{L + 2} \mathcal{R} \left[ \langle \tilde{\varphi}(L_1 + 1, \omega_{m,z}) \tilde{\varphi}^*(L, \omega_{m,z}) \rangle \right] \\
+ \left( \frac{2\bar{\rho}_1}{L + 2} \right) \mathcal{R} \left[ \langle \tilde{\varphi}(1, \omega_{m,z}) (\tilde{\xi}^* (L, \omega_{m,z}) - \tilde{\xi}_2^* (\omega_{m,z}) - \tilde{\xi}_1^* (\omega_{m,z})) \rangle \right] \\
+ \left( \frac{2\bar{\rho}_2}{L + 2} \right) \mathcal{R} \left[ \langle \tilde{\varphi}(L_1 + 1, \omega_{m,z}) (\tilde{\xi}^* (L, \omega_{m,z}) - \tilde{\xi}_2^* (\omega_{m,z}) - \tilde{\xi}_1^* (\omega_{m,z})) \rangle \right] \\
+ \left( \frac{2(1 - \bar{\rho}_R)}{L + 2} \right) \mathcal{R} \left[ \langle \tilde{\varphi}(L, \omega_{m,z}) (\tilde{\xi}^* (L, \omega_{m,z}) - \tilde{\xi}_2^* (\omega_{m,z}) - \tilde{\xi}_1^* (\omega_{m,z})) \rangle \right] \\
+ A_3 + A_2 + A_1 \right],
\]

(61)

where

\[
\langle \tilde{\varphi}(a, \omega_{m,z}) \tilde{\xi}_1^* (\omega_{m,z}) \rangle = S_{a1} \times A_1, 
\]

(62)

\[
\langle \tilde{\varphi}(a, \omega_{m,z}) \tilde{\xi}^* (L_1, \omega_{m,z}) \rangle = -S_{aL_1} \times A_1 + S_{a(L_1 + L_1 + 1)} \times A_1, 
\]

(63)

\[
\langle \tilde{\varphi}(a, \omega_{m,z}) \tilde{\xi}_2^* (\omega_{m,z}) \rangle = S_{a(L_1 + 1)} \times A_2, 
\]

(64)

\[
\langle \tilde{\varphi}(a, \omega_{m,z}) \tilde{\xi}^* (L_1 + L_2, \omega_{m,z}) \rangle = -S_{a(L_1 + L_2)} \times A_2 + S_{a(L_1 + L_1 + 1)} \times A_2, 
\]

(65)

\[
\langle \tilde{\varphi}(a, \omega_{m,z}) \tilde{\xi}^* (L, \omega_{m,z}) \rangle = -S_{aL} \times A_3; 
\]

(66)

\[
\langle \tilde{\varphi}(a, \omega_{m,z}) \tilde{\xi}^* (b, \omega_{m,z}) \rangle = \begin{cases} \\
A_1 \times (S_{a(b + 1)} - S_{ab}) & b \in [1, L_1 - 1] \\
A_2 \times (S_{ab} - S_{a(b - 1)}) & b \in [L_1 + 1, L_1 + L_2 - 1] \\
A_3 \times (S_{a(b - 1)} - S_{a(b - 2)}) & b \in [L_1 + L_2 + 1, L - 1] 
\end{cases}, 
\]

(67)

\[
\langle \tilde{\varphi}(a, \omega_{m,z}) \tilde{\varphi}^* (b, \omega_{m,z}) \rangle = A_1 \times \left[ 2 \sum_{x=1}^{L_1} S_{ax} S_{bx}^* - \sum_{x=2}^{L_1} (S_{ax} S_{b(x-1)}^* + S_{a(x-1)} S_{bx}^*) \right] \\
+ A_2 \times \left[ 2 \sum_{x=L_1 + 1}^{L_1 + L_2} S_{ax} S_{bx}^* - \sum_{x=L_1 + 2}^{L_1 + L_2} (S_{ax} S_{b(x-1)}^* + S_{a(x-1)} S_{bx}^*) \right] 
\]

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When the sum of the inflow rate $J_1 + J_2$ is smaller than $J_3$, then lanes I–III are all in LD.

- When $J_1 + J_2 = 0.25$ and $\beta > 0.5$, then lane III will be in MC, and lanes I and II will be in the coexistence state.
- When $\beta < 0.5$, $J_1 + J_2 > J_3$, $J_1 > J_3/2$, $J_2 > J_3/2$, then lanes I–III are all in HD.
- When $\beta > 0.5$, $J_1 + J_2 > J_3$, $J_1 > J_3/2$, $J_2 > J_3/2$, then lanes I and II are in HD, and Lane III is in MC.
- When $\beta < 0.5$, $J_1 + J_2 > J_3$, $J_1 > J_3/2$, $J_2 < J_3/2$, then lane II is in LD, lanes I and III are in HD.
- When $\beta > 0.5$, $J_1 + J_2 > J_3$, $J_1 > J_3/2$, $J_2 < J_3/2$, then lane II is in LD, lane I is in HD and lane III is in MC.

### 3.2. Lanes I and II in LD

This subsection focuses on the situation that lanes I and II are in LD phase. In this case, lane III must be in LD. Firstly, we study the case that $\alpha_1 = \alpha_2 = \alpha$ and $L_1 = L_2 = L'$. Figure 2 shows that the power spectrum exhibits damped periodic oscillation as in a single lane TASEP. With the increase of $L'$, the period decreases. Figure 3 shows that with the increase of $\alpha$, the period decreases and the oscillation amplitude decreases as well. When $\alpha = 1/2 - \sqrt{2}/4$, lane III will be in MC state and lanes I and II will be in the coexistence state. In this case, Monte Carlo simulation shows that the damped oscillation structure disappears, see Fig. 3(d). However, since it is assumed that the density profile is flat in the lanes, the theory is not able to predict the power spectra related to coexistence state.
Since each lane may be treated as an independent TASEP, the minima of the power spectrum are at $nTv = L_0$ in lanes I and II and at $nTv = L_3$ in lane III. Here, $v$ is propagation speed of fluctuations: Thus, $v_1 = 1 - 2\rho_1$ and $v_3 = 1 - 2\rho_3$, $\rho_1$ and $\rho_3$ are bulk densities on lanes I and III. Since both lanes I and III are in LD phase, one has

$$\frac{1 + \sqrt{1 - 4(\alpha_1(1 - \alpha_1) + \alpha_2(1 - \alpha_2))}}{2}, \quad L_1 = L_2 = L' = 300 \text{ and } L_3 = 600.$$ 

(a) $L_0 = 300$, (b) $L_0 = 600$ and (c) $L' = 900$. The black solid lines show the simulation results, the red lines show the theoretical results, the dashed lines show the expected minima of $I(m)$.

Since each lane may be treated as an independent TASEP, the minima of the power spectrum are at $nTv_1/L'$ in lanes I and II and at $nTv_3/L_3$ in lane III. Here, $v$ is propagation speed of fluctuations: Thus, $v_1 = 1 - 2\rho_1$ and $v_3 = 1 - 2\rho_3$, $\rho_1$ and $\rho_3$ are bulk densities on lanes I and III. Since both lanes I and III are in LD phase, one has

$$\frac{1 + \sqrt{1 - 4(\alpha_1(1 - \alpha_1) + \alpha_2(1 - \alpha_2))}}{2}, \quad L_1 = L_2 = L' = 300 \text{ and } L_3 = 600.$$ 

(a) $\alpha = 0.01$, (b) $\alpha = 0.02$, (c) $\alpha = 0.05$, (d) $\alpha = 0.1$ and (e) $\alpha = 0.5 - \sqrt{2}/4$. The dashed lines show the expected minima of $I(m)$.
\[ v_1 = 1 - 2\alpha \quad \text{and} \quad v_3 = \sqrt{1 - 8\alpha(1 - \alpha)} \].

In the junction system, the fluctuations could be generated at both entrances of lanes I and II. The time needed to transverse the system is equal for the fluctuations generated at both entrances, which equals to \( L_0 / (v_1 + L_3/v_3) \). Therefore, the minima of \( I(\omega) \) are expected to be at \( nT \frac{L_0}{L_0 + L_3/v_3} \), which is in good agreement with the theoretical results.

Next, we study the situation that \( L_1 = L_2 = L' \) but \( \alpha_1 \neq \alpha_2 \). It is found that the damped periodic oscillation is still observed, see Fig. 4. The time needed to transverse the system equals to \( L'/v_1 + L_3/v_3 \) for fluctuations generated at entrance of lane I and equals to \( L'/v_2 + L_3/v_3 \) for fluctuations generated at entrance of lane II. Here, \( v_1 = 1 - 2\alpha_1 \), \( v_2 = 1 - 2\alpha_2 \) and \( v_3 = \sqrt{1 - 4[\alpha_1(1 - \alpha_1) + \alpha_2(1 - \alpha_2)]} \), which can be obtained as mentioned before. We approximate the average time needed for the fluctuations to transverse the system by a weighted average of the two times \( \frac{\gamma_1 L'/v_1 + \gamma_2 L'/v_2}{\gamma_1 + \gamma_2} + L_3/v_3 \), Here, \( \gamma_1 \) and \( \gamma_2 \) are two weights, representing the occurrence probability of the fluctuations on lanes I and II. Since each particle induces a fluctuation, we can approximate the weights by \( \gamma_1 = \rho_1 \) and \( \gamma_2 = \rho_2 \). Here, \( \rho_1 \) and \( \rho_2 \) are bulk densities on lanes I and II. Since both lanes I and II are in LD phase, \( \rho_1 = \alpha_1 \) and \( \rho_2 = \alpha_2 \). The minima of \( I(\omega) \) are therefore expected to be at \( nT / (\frac{\alpha_1 L'/v_1 + \alpha_2 L'/v_2}{\alpha_1 + \alpha_2} + L_3/v_3) \), which is also in good agreement with the theoretical results.
Finally, we study the situation that \( \alpha_1 = \alpha_2 = \alpha \) but \( L_1 \neq L_2 \). Figure 5(a) shows that when \( L_1 \) and \( L_2 \) are notably different, the damped oscillation has become very irregular. On the other hand, when \( L_1 \) is not remarkably different from \( L_2 \), the damped periodic oscillation still can be observed, and the minima are at

\[
\beta = \frac{\sqrt[4]{1 - 4\alpha_1(1-\alpha_2) + \alpha_1(1-\alpha_2)^2}}{2}, \quad L_3 = 600 \quad \text{and} \quad L_1 \neq L_2.
\]

\((a) \quad L_1 = 300, L_2 = 600, \quad (b) \quad L_1 = 300, L_2 = 330.\) The dashed lines show the expected minima of \( I(m) \).
\[ nT = \frac{L_1}{v_1} + \frac{L_2}{v_2} + \frac{L_3}{v_3}, \text{ see Fig. 5(b).} \] With other parameters fixed, the critical value between the regular and irregular oscillation is \( L_2 \approx 360 \).

To understand why irregular oscillations occur when \( L_1 \) is remarkably different from \( L_2 \), we have compared the power spectra of three single TASEPs: One with length \( L = 100 \) and entry rate 0.01, the second with length \( L = 1000 \) and entry rate 0.01, and the third with length \( L = 100 \) and entry rate 0.1. This is because lanes I and II can be viewed as two single TASEPs linked in parallel via the junction. Note that differently, lane I (or lane II) and lane III are linked in series via the junction.

As shown in Fig. 6, the amplitude of \( I(m) \) of the first TASEP and the second one is approximately of the same order. As a result, when the two TASEPs are linked in parallel, the power spectrum becomes irregular since the period of the two power spectra is significantly different from each other. On the other hand, the amplitude of \( I(m) \) of the third TASEP is about one order larger than that of the first one. Therefore, the power spectra of the third TASEP dominates when it is linked to the first one in parallel. Therefore, the periodic oscillation is still observed.

### 3.3. Lanes I and II in HD

This subsection studies the situation that lanes I and II are in HD phase. In this case, Lane III could be in either HD or MC. Figure 7 shows the results in which

\[ \alpha_1 = \alpha_2 = \frac{1}{2}, L_1 = L_2 = L_3 = 300. \] (a) \( \beta = 0.01 \), (b) \( \beta = 0.1 \) and (c) \( \beta = 0.5 \). The dashed lines show the expected minima of \( I(m) \).
\( L_1 = L_2 = L' \). Similarly, the power spectrum exhibits damped periodic oscillation, and the minima of \( I(\omega) \) is at \( \left. \frac{\nu_1 T}{L_{1/2} \nu_{1/2}/v_3} \right| \). Here, \( v_3 = 1 - 2\rho_3 = 2\beta - 1 < 0 \) and \( v_1 = 1 - 2\rho_1 = -\sqrt{1 - 2\beta(1 - \beta)} < 0 \), because fluctuations are generated from the exit of Lane III and propagate from downstream to upstream. With the increase of \( \beta \), the period decreases and the oscillation amplitude decreases as well, see Figs. 7(a) and 7(b). When \( \beta = 0.5 \), Lane III transits into MC, and the damped oscillation structure disappears, see Fig. 7(c). This is because \( v_3 = 0 \) when lane III is in MC state, which means that fluctuations cannot propagate in MC state. Thus, \( L_3/v_3 \to \infty \) and the minima of \( I(\omega) \left. \right|_{L_{1/2} / v_{1/2} + L_{1/2} / v_{3}} \to 0 \).

Figure 8 shows the results in which \( L_1 \neq L_2 \). Similarly, when \( L_1 \) and \( L_2 \) are notably different, the damped oscillation has become irregular, see Fig. 8(a). When
1 is not remarkably different from \( L_2 \), the damped periodic oscillation still can be observed, and the minima are at 
\[ |nT/(L_1/v_1 + L_2/v_2 + L_3/v_3)| \], see Fig. 8(b). Here, 
\[ v_2 = v_1 = -\sqrt{1 - 2\beta(1 - \beta)}. \] With other parameters fixed, the critical value between the regular and irregular oscillation is \( L_2 \approx 240 \).

### 3.4. Lane I in HD, lane II in LD

This subsection studies the situation that one of Lanes I and II is in HD, and the other lane is in LD. Without loss of generality, we assume that Lane I is in HD and Lane II is in LD. In this case, lane III could be in either HD or MC as in previous subsection.

Under this circumstance, the fluctuations could be generated at the entrance of lane II (which propagates downstream) and the exit of lane III (which propagates upstream). This makes the situation complicated. For example, Fig. 9(a) shows the results in which \( L_1 = L_2 = L_3 = 300, \) \( \alpha_2 = 0.04 \) and \( \beta = 0.1 \). In this case, the damped periodic oscillation could be observed. Figure 9(b) shows the results in which

![Fig. 8. (Continued)](b)

![Fig. 9. (Color online) Comparison of simulation and theoretical power spectra for \( \beta = 0.1, \alpha_2 = 0.04, \alpha_1 = \frac{1+\sqrt{1-4\beta(1-\beta)\alpha_2(1-\alpha_2)}}{2}, L_1 = L_2 = L', (a) L' = 300, L_3 = 300 and (b) L' = 700, L_3 = 100. In (a), the dashed lines showing the minima of \( I(m) \) are equidistant. In (b), the dashed lines are also equidistant, which clarifies that the oscillations are irregular.](a)
\[ L_1 = L_2 = 700 > L_3 = 100, \quad \alpha_2 = 0.04 \text{ and } \beta = 0.1. \] Figure 10(a) shows the results in which \( L_1 = L_2 = L_3 = 300, \alpha_2 = 0.04 \text{ and } \beta = 0.3. \) In the two cases, the oscillations become irregular. In Fig. 9(b), with other parameters fixed, the critical value between the regular and irregular oscillation is \( L_3 \approx 460. \) In Figs. 9(a) and 10(a), with other parameters fixed, the critical value is \( \beta \approx 0.17. \)

The irregular oscillations might be due to similar reasons as mentioned before. Since the fluctuations propagates downstream on lane II and propagates upstream on lane III, we can regard lanes II and III as linked in parallel. Therefore, when the amplitude of power spectra of the two lanes is approximately of the same order, irregular oscillations emerges.

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**Fig. 9.** (Continued)

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**Fig. 10.** (Color online) Comparison of simulation and theoretical power spectra for \( L_1 = L_2 = L_3 = 300, \alpha_2 = 0.04 \text{ and } \alpha_1 = \frac{1}{2} \sqrt{(1-\delta[1-\beta]^{-\alpha_2[1-\alpha_1]})}. \) (a) \( \beta = 0.3 \) and (b) \( \beta = 0.5. \) Panel (c) shows only the theoretical results in panel (a). In (b), the dashed lines show the expected minima of \( I(m). \) In (c), the dashed lines are equidistant, which clarifies that the oscillations are irregular.
Finally, when $\beta = 0.5$, lane III becomes MC state, but the damped periodic oscillations are still observed, see Fig. 10(b). This is different from the situation shown in Fig. 7(c), in which lane III is in MC and Lanes I and II are in HD. Here, the oscillations are feasible because fluctuations can propagate downstream from entrance of lane II, via the junction, and then propagate upstream to the entrance of lane I. Therefore, the minima are expected at $nT/(L_1/v_1 + L_2/v_2)$, which is also in good agreement with the theoretical results.

3.5. **Comparison with simulation results**

We have compared the theoretical results with the simulation ones in Figs. 2–5 and 7–10. In the simulations, we have made the density profile flat at the entrances of lanes I and II as well as at exit of lane III, via properly tuning $\alpha_1$, $\alpha_2$ and $\beta$. One can see that the theoretical results and the simulation ones are in good agreement when the system flow rate is low. With the increase of the system flow rate, the theoretical results begin to overestimate the oscillation amplitude.

Similar as in Ref. 24, the theory correctly predicts the minima location, but the damping of the oscillation is greater in the simulations. This is mainly because, as pointed out in Ref. 24, the theory is a linear one and assumes the interactions terms to be small, which is not the case when the system flow rate increases. Moreover, we have assumed that the density profile is flat in the lanes. However, boundary layers will form near the junction, and the boundary layer thickens with the increase of system flow rate.

Finally, we would like to mention that while theoretical analysis predicts that there are oscillations in Fig.10, the simulation results seem to indicate that there is no oscillation. Another possibility is that the oscillation amplitude is too small to be identified in simulation results. Therefore, in the future work, more efforts are needed to investigate this issue.

4. **Conclusion and Discussion**

Motivated by the oscillatory behaviors in the power spectra of an ordinary TASEP and the junction frequently observed in the biological and physical systems,
this paper investigates the power spectra associated with the total occupancy of particles in the TASEP on lattices with a junction, using both Monte Carlo simulations and theoretical analysis based on linearized Langevin equations. We have studied the effect of lane length $L_1, L_2, L_3$, particle entrance rates $\alpha_1, \alpha_2, \beta$ on the oscillatory behaviors.

Via the junction, the lattices are linked in parallel, which is different from lattices linked in series as studied previously. Mathematically, this makes the matrix $M$ in Eq. (44) a nontridiagonal one, and thus the power spectra becomes complicated.

The main findings are summarized as follows.

(i) When lanes I, II, III are simultaneously in LD or HD, and $L_1$ is not remarkably different from $L_2$, the damped periodic oscillations could be observed, and the minima of the power spectra could be calculated via weighted average. However, when $L_1$ is notably different from $L_2$, the oscillation becomes irregular. This is explained by studying the amplitude of the power spectra of single TASEP, since lanes I and II can be viewed as two single TASEPs linked in parallel via the junction.

(ii) When lane III is in MC, the oscillation behavior disappears when both lanes I and II are simultaneously in HD or coexistence state. This is because the fluctuations cannot propagate through the system.

Finally, when one of the lanes I and II is in LD and the other lane is in HD, the situation is complicated. The irregular oscillations might be due to similar reasons as mentioned before. Nevertheless, more investigations are still needed due to the deviations between simulation results and the theoretical analysis.

References