Strong asymmetric coupling of multilane PASEPs
Qi-Hong Shi, Rui Jiang*, Mao-Bin Hu, Qing-Song Wu

State Key Laboratory of Fire Science and School of Engineering Science, University of Science and Technology of China, Hefei 230026, People’s Republic of China

A R T I C L E   I N F O

Article history:
Received 25 February 2012
Received in revised form 17 May 2012
Accepted 6 June 2012
Available online 27 July 2012
Communicated by A.R. Bishop

Keywords:
ASEP
Phase diagram
Vertical cluster mean field theory

A B S T R A C T

This Letter has firstly investigated three-lane PASEPs in which particles on lane \( i \) could move forward with rate \( p_i \) and backward with rate \( q_i \). Particles could also jump fully asymmetrically from lane \( i \) to lane \( i+1 \). It is found that in the case that \( p_i > q_i \) and \( p_i - q_i \) are not equal to each other, the phase diagram in three-lane system has three different geometric structures. The principles of phase diagram structure have been generalized to multilane PASEPs. The case that \( p_i - q_i \) could be equal has also been studied. Unfortunately, the phase diagram structure is complex and generalization to multilane PASEPs is difficult. A vertical cluster mean-field analysis has been carried out, which shows good agreement with simulations.

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In the past years, asymmetric simple exclusion process (ASEP) has attracted interests of many researchers because of its critical role for understanding various non-equilibrium phenomena in chemistry, physics and biology [1–4]. The ASEP is defined on a one-dimensional lattice of size \( L \), on which each site can be either empty or occupied by one particle. In the bulk, if the target site is empty, particles can hop from site \( i \) with rate \( p \) to site \( i+1 \) or with rate \( q \) to site \( i-1 \). The ASEP is called totally ASEP (TASEP) if \( p = 0 \) or \( q = 0 \), otherwise it is named partially ASEP (PASEP). Mechanisms of many non-equilibrium processes, such as biological transport, kinetics of protein synthesis and biopolymerization, car traffic, and hopping of quantum dots, have been investigated due to the successful description via ASEP [5–13].

Recently many works have been carried out to extend single lane ASEP to two-lane and multilane ASEP to describe, e.g., the motion of molecular motors along protofilament, the extraction of membrane tubes by molecular motors, car traffic on multilane road, pedestrian traffic in channel, macroscopic clustering phenomena, spin transport and various systems of oppositely moving particles [14–40]. Among them, Cai et al. have studied asymmetric coupling in multiple TASEPs [23], and have obtained how the phase diagram structure depends on the lane number.

In this Letter we extend the work of Ref. [23] to study asymmetric coupling in multiple PASEPs. We consider a system of \( N \) parallel one-dimensional lattices of length \( L \) (Fig. 1). At every time step a site is randomly chosen. The particle dynamics in the bulk of the system are as follows. Particles on lane \( i \) \( (1 < N) \) will jump to lane \( i+1 \) with rate \( 1 \) if the corresponding site on lane \( i+1 \) is empty, otherwise they will move on lane \( i \) with rate \( p_i \) toward right and rate \( q_i \) toward left. Particles on lane \( N \) move with rate \( p_N \) toward right and rate \( q_N \) toward left without lane changing. At the entrances, particles are injected with rate \( \alpha \). At the exits, particles are removed with rate \( \beta \). Note that in the special case \( N = 2 \), the model reduces to two-lane model, which has been studied in Ref. [40].

Firstly we consider a three-lane PASEP. We carry out a vertical cluster mean field (MF) analysis of the system, which was firstly proposed by Kolomeisky group [24–27] and explicitly takes into account the correlations inside the vertical cluster of lattice site and is proved to be an appropriate and convenient theoretical tool for analyzing the multilane ASEP. Let us define \( P_{\tau_1 \tau_2 \tau_3} \) as a probability to find a vertical cluster in which the sites belonging to lanes 1, 2, 3 are in states \( \tau_1, \tau_2, \tau_3 \), respectively. Here

* Corresponding author. Tel.: +86 551 3600127; fax: +86 551 3606459.
E-mail address: rjiang@ustc.edu.cn (R. Jiang).

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http://dx.doi.org/10.1016/j.physleta.2012.06.040
Solution 2: $\tau_i = 0$ or 1 means that the site on lane $i$ is empty or occupied. We have

$$P_{111} + P_{110} + P_{101} + P_{100} + P_{011} + P_{010} + P_{001} + P_{000} = 1$$

due to the conservation of probability.

Assuming that far away from the boundaries, the system is uniform along the lattices, the dynamics of the bulk vertical clusters are thus governed by master equations. We give one example of master equations as following and other equations could be obtained similarly

$$\frac{dP_{111}}{dt} = (p_3 + q_3)P_{111}^0 + (p_1 + q_1 + p_3 + q_3)P_{110}P_{011} + (p_1 + q_1 + p_2 + q_2 + p_3 + q_3)P_{110}P_{001} - (p_1 + q_1 + p_2 + q_2 + p_3 + q_3)P_{111}P_{000} - (p_2 + q_2 + p_3 + q_3)P_{111}P_{100} - (p_1 + q_1 + p_3 + q_3)P_{111}P_{010} - (p_1 + q_1 + p_2 + q_2)P_{111}P_{001}.$$  

When the system reaches stationary state, we have $\frac{dP_{111}}{dt} = 0$. Solving the master equations and considering particle conservation, it is straightforward to obtain three solutions:

Solution 1: $P_{101} = P_{100} = P_{111} = P_{110} = P_{010} = P_{011} = P_{000} = 0$.

Solution 2: $P_{101} = P_{100} = P_{111} = P_{110} = P_{010} = P_{000} = 0$.

Solution 3: $P_{101} = P_{100} = P_{110} = P_{000} = P_{001} = P_{010} = 0$.

Solution 1 means that lane 1 and lane 2 are empty, solution 2 means that lane 1 is empty and lane 3 is fully occupied, while solution 3 means that lane 2 and lane 3 are fully occupied.

The stationary currents in the bulk of the system are expressed as:

$$J_{\text{bulk},1} = (P_{111} + P_{110})(1 - P_{111} - P_{110} - P_{101} - P_{100}) \times (p_1 - q_1),$$

$$J_{\text{bulk},2} = (P_{111} + P_{101})(1 - P_{111} - P_{110} - P_{011} - P_{010}) \times (p_2 - q_2),$$

$$J_{\text{bulk},3} = (P_{111} + P_{101} + P_{011} + P_{001})(p_3 - q_3) \times (1 - P_{111} - P_{101} - P_{011} - P_{001}).$$

The particle currents at the entrance of the two lanes are given by the following expressions:

$$J_{\text{entr},1} = \alpha(1 - P_{111} - P_{110} - P_{101} - P_{100}) \times (p_1 - q_1),$$

$$J_{\text{entr},2} = \alpha(1 - P_{111} - P_{110} - P_{011} - P_{010}) \times (p_2 - q_2),$$

$$J_{\text{entr},3} = \alpha(1 - P_{111} - P_{101} - P_{011} - P_{001}) \times (p_3 - q_3).$$

At the exit, the currents are

$$J_{\text{exit},1} = \beta(P_{111} + P_{110}),$$

$$J_{\text{exit},2} = \beta(P_{111} + P_{011}),$$

$$J_{\text{exit},3} = \beta(P_{111} + P_{101} + P_{011} + P_{001}).$$

Firstly, we consider the situation when the system is described by solution 1. The expressions for the particle currents are significantly simplified,

$$J_{\text{bulk},1} = 0, \quad J_{\text{bulk},2} = 0, \quad J_{\text{bulk},3} = P_{001}(1 - P_{001})(p_3 - q_3),$$

$$J_{\text{entr},1} = \alpha, \quad J_{\text{entr},2} = \alpha, \quad J_{\text{entr},3} = \alpha(1 - P_{001}),$$

$$J_{\text{exit},1} = 0, \quad J_{\text{exit},2} = 0, \quad J_{\text{exit},3} = \beta P_{001}.$$  

If we view the vertical clusters (001) as “particles” and vertical clusters (000) as “holes”, the three-lane system can be simplified into one-lane PASEP with forward rate $p_3$ and backward rate $q_3$.

Naturally, there will be three phases LD$_3$, HD$_3$ and MC$_3$. Here $X_i$ means that lane 1 is in phase $X$, lane $j$ with $j < i$ is empty, and lane $j$ with $j > i$ is fully occupied.

$$J_{\text{entr},1} + J_{\text{entr},2} + J_{\text{entr},3} = J_{\text{bulk},1} + J_{\text{bulk},2} + J_{\text{bulk},3}. $$

Substituting Eq. (9) into Eq. (10) we have

$$\alpha_{\text{eff},3}(1 - P_{001}) = 2\alpha \alpha(1 - P_{001}) = P_{001}(1 - P_{001})(p_3 - q_3).$$

Thus we can obtain

$$\alpha_{\text{eff},3} = P_{001}(p_3 - q_3) = \frac{p_3 - q_3 + \alpha}{2} - \sqrt{(p_3 - q_3 + \alpha)^2 - 12\alpha(p_3 - q_3)}, \quad (12)$$

For the high density HD$_3$ phase, the exit process determines the particle dynamics. Thus

$$\beta_{\text{eff},3} = \beta. \quad (13)$$

The LD$_3$ phase is restricted by $\alpha_{\text{eff},3} < \beta_{\text{eff},3} < \beta_{\text{eff},3} < (p_3 - q_3)/2$, which leads to

$$p_3 - q_3 + \alpha - \sqrt{(p_3 - q_3 + \alpha)^2 - 12\alpha(p_3 - q_3)} < \min[\beta, (p_3 - q_3)/2]$$

and the density on lane 3

$$\rho_3 = P_{001} = \frac{p_3 - q_3 + \alpha}{2(p_3 - q_3)} - \frac{\sqrt{(p_3 - q_3 + \alpha)^2 - 12\alpha(p_3 - q_3)}}{2(p_3 - q_3)}.$$  

This phase (HD$_3$) exists under the condition

$$\beta < \min[\alpha_{\text{eff},3}, (p_3 - q_3)/2].$$

The density on lane 3 of this phase are given by

$$\rho_3 = P_{001} = 1 - \beta/(p_3 - q_3).$$

The MC$_3$ phase is specified by conditions

$$\alpha_{\text{eff},3} > (p_3 - q_3)/2, \quad \beta > (p_3 - q_3)/2$$

with $\rho_3 = 1/2$.

For solutions 3 and 2, we can assume clusters (111) or (011) as "particles" and clusters (011) or (001) as "holes", respectively. Similarly we could have
Monte Carlo time steps and the previous $2.5 \times 10^{10}$ Monte Carlo time steps are abandoned in order to obtain stationary states. We simulated system of size $L = 1000$ for each lane, and in several cases we have also checked our simulations for lattices with $L = 10000$. The phase diagram structure of system is determined by values $p_1 - q_1, p_2 - q_2, p_3 - q_3$ based on our analysis. With "particle–hole" symmetry, we only need to consider three cases including $(p_1 - q_1 < p_2 - q_2 < p_3 - q_3), (p_3 - q_3 < p_1 - q_1 < p_2 - q_2)$, and $(p_2 - q_2 < p_1 - q_1 < p_3 - q_3)$. Density profiles and phase diagrams are shown in Figs. 2 and 3. One can see that simulation results are in good agreement with theoretical predictions.

We note that in Fig. 3(a), phases LD$_2$, HD$_2$, MC$_2$ are embedded in region LD$_3$, and phases HD$_3$, MC$_3$ are embedded in region HD$_2$. In Fig. 3(b), phases LD$_1$, HD$_1$, MC$_1$ (LD$_3$, HD$_3$, MC$_3$) are embedded in HD$_2$ (LD$_2$) respectively. The geometric structure of diagram Fig. 3(c) is more complicated. Firstly phases LD$_2$, HD$_2$, MC$_2$ are embedded in region LD$_1$, then phases LD$_1$, HD$_1$, MC$_1$ are embedded in region HD$_3$.

We generalize the principles of phase diagram structure to multilane PASEPs. For a randomly chosen lane $i$, search within $(i + 1 \rightarrow H)$ to find the nearest lane $j_1$ with $p_{j_1} - q_{j_1} > p_i - q_i$ and search within $(i \rightarrow i - 1)$ to find the nearest lane $j_2$ with $p_{j_2} - q_{j_2} > p_i - q_i$. If $p_{j_1} - q_{j_1} < p_{j_2} - q_{j_2}$, phases LD$_1$, HD$_1$, MC$_1$ are embedded in region HD$_3$, otherwise if $p_{j_1} - q_{j_1} > p_{j_2} - q_{j_2}$, phases LD$_1$, HD$_1$, MC$_1$ are embedded in region LD$_3$. If $j_2$ $(j_1)$ does not exist, phases LD$_1$, HD$_1$, MC$_1$ are embedded in region HD$_3$ (LD$_3$). If $p_i - q_i$ is larger than $p_j - q_j$ of any other lane $j$, phases LD$_i$, HD$_i$, MC$_i$ are not embedded in other phase.
multilane PASEPs can be calculated by the six-lane PASEP. As an example, Fig. 4 shows the phase diagram of a six-lane system obtained by MF analysis. Monte Carlo simulation shows good agreement with our analysis. (a) $p_1 = 0.9$, $p_2 = 0.5$, $p_3 = 0.8$, $p_4 = 1$, $p_5 = 0.7$, $p_6 = 0.6$, $q_1 = q_2 = q_3 = q_4 = q_5 = q_6 = 0$; (b) is the amplified graph of area marked by dashed line in (a).

Furthermore, similar as in three-lane system, $\alpha_{\text{eff},i}$, $\beta_{\text{eff},i}$ in multilane PASEPs can be calculated by

\begin{align}
\alpha_{\text{eff},i} &= \frac{p_i - q_i + \alpha}{2} - \frac{\sqrt{(p_i - q_i + \alpha)^2 - 4i\alpha(p_i - q_i)}}{2}, \\
\beta_{\text{eff},i} &= \frac{p_i - q_i + \beta}{2} - \frac{\sqrt{(p_i - q_i + \beta)^2 - 4(N - i + 1)\beta(p_i - q_i)}}{2}
\end{align}

and boundaries could be obtained as prescribed in Eqs. (23) and (24). We have checked our analysis via simulations, which is found to be correct. As an example, Fig. 4 shows the phase diagram of a six-lane PASEP.

Next we consider the case that $p_1 > q_i$ but $p_i - q_i$ could be equal. In the subcase that $p_1 - q_1 = p_2 - q_2 = p_3 - q_3$, the situation is the same as that discussed in Ref. [23], via a rescaling of time.

Now we study the subcase that two of $p_i - q_i$ are equal. Due to particle-hole symmetry, we need to consider four situations, i.e., (i) $p_1 - q_1 > p_2 - q_2 = p_3 - q_3$; (ii) $p_1 - q_1 < p_2 - q_2 = p_3 - q_3$; (iii) $p_2 - q_2 < p_1 - q_1 = p_3 - q_3$; (iv) $p_2 - q_2 > p_1 - q_1 = p_3 - q_3$. Fig. 5 shows four different phase diagram structures corresponding to the four situations. As expected, when $p_1 - q_1 = p_j - q_j$, coexistence phase $\text{MC}_{i,j}$ might be observed, which means that the left bulk is in $\text{MC}_i$ and the right bulk is in $\text{MC}_j$, see Fig. 6. However, we would like to mention that there is no coexistence phenomenon in situation (iv). Due to the complexity of the phase diagram structure, generalization to multilane PASEPs is difficult.

In conclusion, this Letter has studied three-lane PASEPs with asymmetric strong coupling. A vertical cluster mean-field analysis has been carried out, which shows good agreement with simulation results. In the case that $p_i > q_i$ and $p_i - q_i$ are not equal to each other, three phase diagram structures have been observed. We have generalized our discussion to $N$-lane system, which is found to be correct via verification of simulation. When $p_i - q_i$ could be equal, the situation is different. The case that all $p_i - q_i$ are equal to each other is the same as discussed in Ref. [23] via rescaling of time. When two of them are equal, four phase diagram structures are identified, in which phase coexistence phenomenon has been observed in three of them. Due to complexity of phase diagram structures, generalization to multilane PASEPs is difficult and needs to be investigated carefully in future work.
Fig. 5. Phase diagram of a three-lane system obtained by MF analysis. Monte Carlo simulation shows good agreement with our analysis. (a) $p_1 = 0.8$, $p_2 = p_3 = 0.6$, $q_1 = q_2 = q_3 = 0$; (b) $p_1 = 0.8$, $p_2 = p_3 = 1$, $q_1 = q_2 = q_3 = 0$; (c) $p_1 = 0.9$, $p_2 = p_3 = 1$, $q_1 = q_2 = q_3 = 0$; (d) $p_2 = 1$, $p_1 = p_3 = 0.8$, $q_1 = q_2 = q_3 = 0$.

Fig. 6. (Color online.) Density profiles of typical coexistence phases. (a) Phase MC1, $\alpha = 0.6$, $\beta = 2.2$, $p_1 = p_2 = 0.6$, $q_1 = q_2 = q_3 = 0$; (b) phase MC2, $\alpha = \beta = 0.4$, $p_1 = 0.8$, $p_2 = p_3 = 0.6$, $q_1 = q_2 = q_3 = 0$; (c) phase MC3, $\alpha = 0.7$, $p_1 = 0.9$, $p_1 = p_3 = 1$, $q_1 = q_2 = q_3 = 0$.

Acknowledgements

This work is supported by the National Basic Research Program of China (No. 2012CB725404), the NNSFC (Nos. 11072239 and 71171185), and the Fundamental Research Funds for the Central Universities (No. WK2320000014).

References