Chapter 6  
Dimensional Regularization. Ultraviolet and Infrared Divergences

Abstract  The cornerstone of Quantum Field Theory is renormalization. We shall speak more about in the next chapters. Before, it is necessary to discuss the method. The best and most simple is, of course, dimensional regularization (doesn’t break the symmetries, doesn’t violate the Ward Identities, preserves Lorentz invariance, etc.). When explained consistently, it becomes very simple and clear. Here, we shortly discuss ultraviolet (UV) and infrared (IR) divergences with a few examples. However, in Chap. 8, we shall extensively treat one-loop two and three-point functions and analyse many more examples of IR and UV divergences.

6.1 Master Integral

When calculating higher order quantum corrections (loop diagrams) one will find that, normally, the loop integrals are divergent. In order to make sense out of this divergent integrals and be able to properly define finite observables one has to regularize the divergent integrals (make them finite) and renormalize. In this chapter we shall treat the regularization procedure. In the next chapter we shall talk about renormalization.

In order to regularize the loop integrals, one can use different approaches i.e., cut-off, Wilson regularization, dimensional regularization, etc. This last approach is the one we shall treat in this chapter. It is the most adequate due to the fact that it preserves all the symmetries of the theory. It consists in considering that the space-time dimension is not 4 but \( D = 4 + 2\epsilon \) and work in the limit \( \epsilon \to 0 \). With this consideration all loop-integrals are finite. The general \( D \)-dimensional integral that we will always relate to (as we shall shortly see in the next sections) is

\[
J(D, \alpha, \beta, a^2) \equiv \int \frac{d^Dk}{(2\pi)^D} \frac{(k^2)^\alpha}{(k^2 - a^2)^\beta}
\]  

(6.1)

where \( D \) is the number of space-time dimensions, as we already mentioned. We can easily demonstrate (see Appendix A) that it can be written in terms of the Euler Gamma function as

© Springer International Publishing Switzerland 2016
V. Ilisie, Concepts in Quantum Field Theory, UNITEXT for Physics, DOI 10.1007/978-3-319-22966-9_6
\[
J(D, \alpha, \beta, a^2) = \frac{i}{(4\pi)^{D/2}} (a^2)^{D/2} (-a^2)^{\alpha-\beta} \frac{\Gamma(\beta - \alpha - D/2) \Gamma(\alpha + D/2)}{\Gamma(\beta) \Gamma(D/2)}
\]

(6.2)

For \( z \) a complex number with \( \text{Re}(z) > 0 \) we have the following property of \( \Gamma(z) \)

\[
\Gamma(z + 1) = z \Gamma(z).
\]

(6.3)

For \( n = 0, 1, 2, 3, \ldots \), it takes the form of the usual factorial function

\[
\Gamma(n + 1) = n!.
\]

(6.4)

The function \( \Gamma(z) \) has simple poles at \( z = 0, -1, -2, \ldots \) In the region \(-1 < \epsilon < 0\) with \( \epsilon \in \mathbb{R} \) and \( |\epsilon| \ll 1 \), we have the following Laurent expansion up to \( O(\epsilon) \)

\[
\Gamma(-\epsilon) = -\frac{1}{\epsilon} - \gamma_E + O(\epsilon),
\]

(6.5)

where \( \gamma_E = 0.57721 \ldots \) is the Euler-Mascheroni constant. Taking \( D = 4 + 2\epsilon \) with \(-1 < \epsilon < 0\), \( |\epsilon| \ll 1 \) (UV-divergent integrals in \( D = 4 \) dimensions are convergent in \( D < 4 \) dimensions) and power-expanding in \( \epsilon \), it is straightforward to find the expression for the UV divergent function \( J(D, 0, 2, a^2) \):

\[
J(D, 0, 2, a^2) = \frac{-i}{(4\pi)^2} \left( \frac{1}{\epsilon} + \gamma_E - \ln(4\pi) + \ln(a^2) + O(\epsilon) \right).
\]

(6.6)

It will be useful to define the following quantity:

\[
\tilde{\epsilon}^\epsilon \equiv \frac{1}{\epsilon} + \gamma_E - \ln(4\pi).
\]

(6.7)

Because \( a \) in (6.6) has energy dimensions within the logarithm, we multiply and divide the RHS of (6.6) by \( \mu^{2\epsilon} \), with \( \mu \) a parameter with energy dimensions (called the renormalization scale), and use the expansion

\[
\mu^{-2\epsilon} = 1 - 2\epsilon \ln(\mu) + O(\epsilon^2),
\]

(6.8)

to finally obtain the following simple expression for \( J(D, 0, 2, a^2) \)

\[
J(D, 0, 2, a^2) = \frac{-i}{(4\pi)^2} \mu^{2\epsilon} \left[ \frac{1}{\tilde{\epsilon}^\epsilon} + \ln \left( \frac{a^2}{\mu^2} \right) \right] + O(\epsilon).
\]

(6.9)
We can easily relate all the UV divergent integrals to this one by using the recursion properties of the Gamma function so practically one does not have to integrate ever again over the four-momentum (except for some special cases where IR divergences are also present). Some useful results are presented next.

### 6.2 Useful Results

Using the Gamma function recursion properties we can find very useful the following relations

\[
\begin{align*}
J(D, 0, 1, a^2) &= \frac{a^2}{D/2 - 1} J(D, 0, 2, a^2) \\
J(D, 1, 1, a^2) &= \frac{a^4}{D/2 - 1} J(D, 0, 2, a^2) \\
J(D, 1, 2, a^2) &= \frac{a^2 D}{D - 2} J(D, 0, 2, a^2) \\
J(D, 2, 2, a^2) &= \frac{a^4 (D + 2)}{D - 2} J(D, 0, 2, a^2) \\
J(D, 1, 3, a^2) &= \frac{D}{4} J(D, 0, 2, a^2)
\end{align*}
\]  

(6.10)

As we shall see, all these results will be very useful in Chap. 8. One can construct similar recursion relations for any values of \(\alpha\) and \(\beta\) of our function \(J(D, \alpha, \beta, a^2)\) with no additional complications.

It is worth mentioning that the previous relations are general, meaning that they are valid for any dimension \(D\) (\(D < 4\) or \(D > 4\)). This can turn out to be useful also when treating IR divergences (IR divergent integrals in \(D = 4\) dimensions are convergent in \(D > 4\) dimensions).

An example of integral that is finite in \(D = 4\) dimensions is the following:

\[
J(4, 0, 3, a^2) = -\frac{i}{32\pi^2} \frac{1}{a^2}.
\]  

(6.11)

When IR divergences are present we have to be a little bit more careful (with this integral) as we shall shortly see with an explicit example. Some other useful generic results in \(D\) dimensions for the Dirac gamma matrices are the following:

\[
\begin{align*}
g^{\mu\nu} g_{\mu\nu} &= D I_D, \\
\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu &= (2 - D) \gamma^\nu, \\
\gamma^\mu \gamma^\nu \gamma^\sigma \gamma_\mu &= 4 g^{\nu\sigma} I_D + (D - 4) \gamma^\nu \gamma^\sigma, \\
\gamma^\mu \gamma^\nu \gamma^\sigma \gamma^\rho \gamma_\mu &= -2 \gamma^\rho \gamma^\sigma \gamma^\nu - (D - 4) \gamma^\nu \gamma^\sigma \gamma^\rho.
\end{align*}
\]  

(6.12)
where $I_D$ is the identity matrix in $D$ dimensions. The standard convention is to take its trace equal to four:

$$Tr(I_D) = 4. \quad (6.13)$$

Many more relations of this type involving gamma matrices are present all over the literature. Some other interesting results that involve the function $J(D, \alpha, \beta, a^2)$ are the following:

$$\int \frac{d^Dk}{(2\pi)^D} \frac{(k^2)^\alpha k^\mu}{(k^2 - a^2)^\beta} = 0, \quad (6.14)$$

$$\int \frac{d^Dk}{(2\pi)^D} \frac{(k^2)^\alpha k^\mu k^\nu}{(k^2 - a^2)^\beta} = \frac{g^{\mu\nu}}{D} J(D, \alpha + 1, \beta, a^2) \quad (6.15)$$

Using the last result we can straightforwardly deduce another relation:

$$\int \frac{d^Dk}{(2\pi)^D} \frac{(k \cdot p)^2}{(k^2 - a^2)^\beta} = \int \frac{d^Dk}{(2\pi)^D} \frac{k^\mu k^\nu p_\mu p_\nu}{(k^2 - a^2)^\beta} g^{\mu\nu} k^2 \quad (6.16)$$

$$= \frac{1}{D} \int \frac{d^Dk}{(2\pi)^D} \frac{k^2 p^2}{(k^2 - a^2)^\beta} \quad (6.16)$$

$$= \frac{p^2}{D} J(D, 1, \beta, a^2).$$

Thus, we have found the following equality in $D$ dimensions

$$\int \frac{d^Dk}{(2\pi)^D} \frac{(k \cdot p)^2}{(k^2 - a^2)^\beta} = \int \frac{d^Dk}{(2\pi)^D} \frac{(p^2 / D)k^2}{(k^2 - a^2)^\beta}. \quad (6.17)$$

The reader is highly encouraged to find the generalization of this result for $(k \cdot p)^\alpha$ with $\alpha$ an arbitrary positive integer.

### 6.3 Example: Cancellation of UV Divergences

The following is a simple but very useful example of how the divergent parts of an integral cancel and lead to a finite final result. Consider

$$G^{\mu\nu} = \int \frac{d^Dk}{(2\pi)^D} \frac{4k^\mu k^\nu - g^{\mu\nu} k^2}{[k^2 - a^2]^3}. \quad (6.18)$$
The calculation is straightforward. Using (6.10) we obtain

\[ G^{\mu\nu} = g^{\mu\nu} \left( \frac{4}{D} - 1 \right) J(D, 1, 3, a^2) \]
\[ = g^{\mu\nu} \left( \frac{4}{D} - 1 \right) \frac{D}{4} J(D, 0, 2, a^2) \]
\[ = g^{\mu\nu} \left( -\frac{\epsilon}{2} \right) J(D, 0, 2, a^2) + O(\epsilon^2) \]
\[ = g^{\mu\nu} \frac{i}{32\pi^2} + O(\epsilon). \tag{6.19} \]

Taking \( D \to 4 \) we obtain:

\[
G^{\mu\nu} = \int \frac{d^Dk}{(2\pi)^D} \frac{4k^\mu k^\nu - g^{\mu\nu}k^2}{[k^2 - a^2]^3} = g^{\mu\nu} \frac{i}{32\pi^2}. \tag{6.20}
\]

It is a common mistake to think that because the final result is finite in 4 dimensions we could have made directly the substitution \( k^\mu k^\nu \to g^{\mu\nu}k^2/4 \) instead of \( k^\mu k^\nu \to g^{\mu\nu}k^2/D \). Wrong! This integral consists in the sum of two parts that diverge in 4 dimensions. Only after these two parts are summed, the final result turns out to be finite. Thus, the substitution \( k^\mu k^\nu \to g^{\mu\nu}k^2/D \) is only valid if it gives rise to a finite result. In our case both parts are finite for \( D < 4 \) (and not \( D = 4 \)) dimensions and therefore, we must maintain \( D = 4 + 2\epsilon \) until the sum of both parts is performed.

### 6.4 Feynman Parametrization

Usually we don’t find simple propagators as in (6.1), therefore we have to perform some manipulations over the denominators. The standard procedure is using the Feynman parametrization:

\[
\frac{1}{A^\alpha B^\beta} = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha) \Gamma(\beta)} \int_0^1 dx x^{\alpha-1}(1-x)^{\beta-1}, \tag{6.21}
\]
\[
\frac{1}{A^\alpha B^\beta C^\gamma} = \frac{\Gamma(\alpha + \beta + \gamma)}{\Gamma(\alpha) \Gamma(\beta) \Gamma(\gamma)} \int_0^1 dx \int_0^1 dy \ x^{\alpha-1}(1-x)^{\beta-1}(1-y)^{\gamma-1}
\times \frac{(xy)^{\alpha-1}[x(1-y)]^{\beta-1}(1-x)^{\gamma-1}}{[Ax + Bx(1-y) + C(1-x)]^{\alpha+\beta+\gamma}}, \tag{6.22}
\]
For \( \alpha = \beta = \gamma = 1 \) we simply get:

\[
\frac{1}{AB} = \int_0^1 dx \frac{1}{[Ax + B(1-x)]^2},
\]

\[
\frac{1}{ABC} = \int_0^1 dx \int_0^1 dy \frac{2x}{[Ay + Bx(1-y) + C(1-x)]^3}.
\]

An useful alternative for (6.24) is the following:

\[
\frac{1}{ABC} = 2 \int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{\delta(1-x-y-z)}{[Ax + By + Cz]^3} = \int_0^1 dy \int_0^{1-y} dz \frac{2}{[A(1-y-z) + By + Cz]^3}.
\]

For a generic \( n \)-point function one can use the generalized Feynman parametrization given by:

\[
\frac{1}{A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_n}} = \frac{\Gamma(\alpha)}{\Gamma(\alpha_1) \Gamma(\alpha_2) \ldots \Gamma(\alpha_n)} \int_0^1 dx_1 \int_0^1 dx_2 \ldots \int_0^1 dx_n \\
\times \frac{x_1^{\alpha_1-1} x_2^{\alpha_2-1} \ldots x_n^{\alpha_n-1}}{(x_1 A_1 + x_2 A_2 + \cdots + x_n A_n)^\alpha} \\
\times \delta(1-x_1-x_2-\cdots-x_n),
\]

where we have defined \( \alpha \equiv \alpha_1 + \alpha_2 + \cdots + \alpha_n \). Integrating over the \( \delta \)-function, we find

\[
\frac{1}{A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_n}} = \frac{\Gamma(\alpha)}{\Gamma(\alpha_1) \Gamma(\alpha_2) \ldots \Gamma(\alpha_n)} \\
\times \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \ldots \int_0^{1-x_1-x_2-\cdots-x_{n-2}} dx_{n-1} \\
\times \frac{x_1^{\alpha_1-1} x_2^{\alpha_2-1} \ldots x_{n-1}^{\alpha_{n-1}-1} (1-x_1-\cdots-x_{n-1})^{\alpha_{n-1}-1}}{(x_1 A_1 + x_2 A_2 + \cdots + x_{n-1} A_{n-1} + (1-x_1-\cdots-x_{n-1}) A_n)^\alpha}.
\]

(6.27)

Next we shall see a few rather simple examples of how these parametrizations can be used. However in Chap. 8, we shall use these parametrizations to calculate more complicated UV and IR divergent integrals.

It is worth mentioning that one can consider the analytical prolongation to the complex plane and use this last parametrization (6.27) for non-integer powers.
of propagators. They appear when one wants integrate over the four-momentum, logarithmic functions that depend on the four-momentum. At the end of Chap. 8 we shall also take a simple two-loop example to see how this can be done.

### 6.5 Example: UV Pole

Let’s consider the following integral:

\[
I^\mu = \int \frac{d^Dk}{(2\pi)^D} \frac{k^\mu}{((p+k)^2 - m^2)k^2}. \tag{6.28}
\]

Using the parametrization in (6.23) with \( A = (p+k)^2 - m^2 \) and \( B = k^2 \) we obtain

\[
\frac{1}{((p+k)^2 - m^2)k^2} = \int_0^1 dx \frac{1}{[((p+k)^2 - m^2)x + k^2(1-x)]^2} = \int_0^1 dx \frac{1}{[(k + px)^2 - a^2]^2}, \tag{6.29}
\]

where we have defined \( a^2 \equiv -p^2x(1-x) + m^2x \). Therefore \( I^\mu \) takes the form

\[
I^\mu = \int_0^1 dx \int \frac{d^Dk}{(2\pi)^D} \frac{k^\mu}{((k + px)^2 - a^2)^2} = \int_0^1 dx \int_0^1 dx \frac{k^\mu - xp^\mu}{(2\pi)^D [k^2 - a^2]^2} = -p^\mu \int_0^1 xdx J(D, 0, 2, a^2)
\]

\[
= p^{\mu} \frac{i}{(4\pi)^2} \mu^2 \left[ \frac{1}{\hat{\epsilon}} + \ln \left( \frac{a^2}{\mu^2} \right) \right] = p^{\mu} \frac{i}{(4\pi)^2} \mu^2 \left[ \frac{1}{2\hat{\epsilon}} + \int_0^1 xdx \ln \left( \frac{-p^2x(1-x) + m^2x}{\mu^2} \right) \right] \tag{6.30}
\]

(to get to the second line we shifted the integration variable \( k \rightarrow k - x p \)).

### 6.6 Example: IR Poles

Consider the following integral

\[
I = \int \frac{d^Dk}{(2\pi)^D} \frac{1}{k^2(k + p_2)^2(k - p_3)^2}, \tag{6.31}
\]
Dimensional Regularization. Ultraviolet and Infrared Divergences

with $p_2^2 = p_3^2 = 0$. Thus our denominator in the limit of small $k$ behaves like

$$\frac{1}{k^2(k^2 + 2p_2 \cdot k)(k^2 - 2p_3 \cdot k)} \xrightarrow{k^2 \ll 1} \frac{1}{k^2(2p_2 \cdot k)(2p_3 \cdot k)}.$$  (6.32)

We can observe two types of infrared divergences present in the denominator of (6.32):

1. $k \to 0$, which is called soft divergence,
2. $p_3 \cdot k$ or $p_2 \cdot k \to 0$, which is called collinear divergence.  (6.33)

The integral that we are treating here will turn out to be only IR divergent, without UV divergences (a case where we find both types of divergences will be treated in Chap. 8). In order to treat these IR divergences, we shall consider the space-time dimensions to be $D = 4 + 2\epsilon'$ with $\epsilon' \in \mathbb{R}$, $\epsilon' > 0$ and $\epsilon' \ll 1$ (the IR divergent integrals in 4 dimensions are convergent in $D > 4$ dimensions, as we have already mentioned). Using the Feynman parametrization (6.24) with $A = k^2 + 2p_2 \cdot k$, $B = k^2 - 2p_3 \cdot k$ and $C = k^2$, we find

$$I = \int_0^1 dx \int_0^1 dy \int \frac{dDk}{(2\pi)^D} \frac{2x}{[(k + p_2 xy - p_3 x(1 - y))^2 - a^2]^3}$$

$$= \int_0^1 dx \int_0^1 dy \int \frac{dDk}{(2\pi)^D} \frac{2x}{[(k^2 - a^2)^3]}$$

$$= 2 \int_0^1 dx \int_0^1 dy x J(D, 0, 3, a^2),$$  (6.34)

where $a^2 = -2(p_2 \cdot p_3)x^2y(1 - y)$. Taking $D \to 4$ as in (6.11) would give rise to an infinity when integrating over the Feynman parameters, as we will see in a moment. Let’s write $J(D, 0, 3, a^2)$ in the form of (6.2):

$$I = \frac{-i}{(4\pi)^{D/2}} \Gamma(3 - D/2) \int_0^1 dx \int_0^1 dy x (a^2)^{D/2 - 3}$$

$$= \frac{-i}{(4\pi)^{D/2}} \Gamma(3 - D/2) (-2p_2 \cdot p_3)^{D/2 - 3} \int_0^1 dxx^{D - 5}$$

$$\times \int_0^1 dy y^{D/2 - 3} (1 - y)^{D/2 - 3}.$$  (6.35)

Using the Euler Beta function (A.17) we obtain the following result for our integral:
\[ I = \frac{-i}{(4\pi)^{D/2}} \Gamma(3 - D/2)(-2p_2 \cdot p_3)^{D/2 - 3} \frac{\Gamma(D - 4) \Gamma(D/2 - 2) \Gamma(D/2 - 2)}{\Gamma(D - 3) \Gamma(D - 4)} \]
\[ = \frac{-i}{(4\pi)^{D/2}} \frac{\Gamma(3 - D/2)}{\Gamma(D - 3)} (-2p_2 \cdot p_3)^{D/2 - 3} \frac{\Gamma(D/2 - 2) \Gamma(D/2 - 2)}{\Gamma(D/2 - 2) \Gamma(D/2 - 2)} \]
\[ = \frac{-i}{(4\pi)^{2}} (-2p_2 \cdot p_3)^{-1} (4\pi)^{-\epsilon'} (-2p_2 \cdot p_3)^{\epsilon'} \frac{\Gamma(1 - \epsilon')}{\Gamma(1 + 2\epsilon')} \Gamma(\epsilon') \Gamma(\epsilon') \]
\[ = \frac{i}{(4\pi)^{2}} \frac{1}{2p_2 \cdot p_3} \frac{(-2p_2 \cdot p_3)^{-1}}{4\pi} \epsilon' \frac{\Gamma(1 - \epsilon')}{\Gamma(1 + 2\epsilon')} \Gamma^2(\epsilon'). \quad (6.36) \]

Of course, one could further expand using:

\[ \Gamma(\epsilon') = \frac{1}{\epsilon'} - \gamma_E + \frac{1}{12} (\pi^2 + 6\gamma_E^2) \epsilon' + O(\epsilon'^2), \quad (6.37) \]
\[ \Gamma(1 \mp \epsilon') = 1 \pm \gamma_E \epsilon' + \frac{1}{12} (\pi^2 + 6\gamma_E^2) \epsilon'^2 + O(\epsilon'^3), \quad (6.38) \]
\[ (-1)^\epsilon' = e^{\pm i\pi \epsilon'} = 1 \pm i\pi \epsilon' - \frac{\pi^2 \epsilon'^2}{2}. \quad (6.39) \]

Thus, we observe that our result (6.36) is proportional to IR poles of the form $1/\epsilon'$ and $1/\epsilon'^2$. This is why taking the limit $D \to 4$ as in (6.11), in (6.34) would have been wrong.

One could argue that in the first example we also have a propagator of the type $\sim 1/k^2$, which goes to infinity as $k$ goes to zero. Therefore one should find IR divergences in this case also. It turns out, however, that it is not the case. It is clear that when integrating over $x$, the expression (6.30) does not diverge. This expression is divergent if $m^2 = p^2 = 0$, however this specific case will be treated in Chap. 8. Thus, any potential IR divergence must be treated carefully. As we can see, after doing the calculation, the IR divergence might actually not be there.

**Further Reading**

M. Kaku, *Quantum Field Theory: A Modern Introduction*
M. Srednicki, *Quantum Field Theory*
S. Pokorsky, *Gauge Field Theories*
F. Mandl, G.P. Shaw, *Quantum Field Theory*