

Electrical resistivity of a fractal network: The scattering of extended electronic states by both fractons and phonons

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We calculate the electrical resistivity (ER) ρ of a fractal network from the view of the scattering of extended electronic states with both phonons and fractons and obtain different dependences of ER on the fractal dimensionality d_f , temperature T , fracton dimensionality \bar{d}_f , and characteristic length l_c , for different-order interactions and different Euclidean dimensionalities d . As to the first interaction, ρ is proportional to T at the high- T limit, as known, and $\rho \sim aT + bT^{3\bar{d}_f/d_f + \bar{d}_f - 1}$ at some low- T ranges. In the second-order case, ρ is a constant at the high- T limit, which is consistent with some recent experiments. In particular, we find that before a special fractal dimensionality d_f^0 , there exists a minimum in the ρ - T curve, while after it ρ is a monotonically increasing function of T . The form of the ρ - d_f curve also shows different characteristics when d changes from 2 to 3. Finally, we discuss the percolating network and obtain scalar laws and scalar exponents.

I. INTRODUCTION

The vibrational elementary excitation as a fractal network transforms from phonons at low frequencies to fractons at high frequencies with the crossover frequency ω_c . The phonons and fractons have different density of states $N(\omega)$, dispersion laws $\omega \sim q$, and wave functions $\Phi_\alpha(\mathbf{R})$. For phonons ($\omega < \omega_c$),

$$N_{\text{ph}}(\omega) = \frac{d\omega^{d-1}}{\omega_D^d}, \tag{1a}$$

$$\omega = v_s q, \tag{1b}$$

$$\Phi_\alpha^{\text{ph}}(\mathbf{R}_\mu) = \frac{1}{\sqrt{G}} \exp[-i\mathbf{q} \cdot \mathbf{R}_\mu], \tag{1c}$$

while for fractons ($\omega > \omega_c$), according to Alexander *et al.*,¹⁻³

$$N_{\text{fr}}(\omega) = \frac{\bar{d}_f \omega^{\bar{d}_f - 1}}{\omega_F^{\bar{d}_f}}, \tag{2a}$$

$$\omega = [(l_c q)^{d_f/\bar{d}_f}] \omega_c, \tag{2b}$$

$$\Phi_\alpha^{\text{fr}}(\mathbf{R}_\mu) = c |\mathbf{R}_\mu|^{(d-d_f)/2} l_{\omega_\alpha}^{-d_f/2} \times \exp \left[-\frac{1}{2} \left[\frac{\mathbf{R}_\mu}{l_{\omega_\alpha}} \right]^{d_\phi} \right]. \tag{2c}$$

Here, $N_{\text{ph}}(\omega)$ and $N_{\text{fr}}(\omega)$ are both normalized to unity and v_s is the velocity of sound. G is the volume of the system, l_c is the characteristic length, d_f is the fractal dimensionality, \bar{d}_f is the fracton dimensionality, and l_{ω_α} is the localization length of the α th mode. ω_D and ω_F are Debye frequencies of phonon and fractons, respectively, which satisfy the relationships¹

$$\omega_D = \left[\frac{l_c}{a_0} \right] \omega_c, \tag{3a}$$

$$\omega_F = \left[\frac{l_c}{a_0} \right]^{d_f/\bar{d}_f} \omega_c, \tag{3b}$$

where a_0 is the lattice constant.

The interaction of the vibrational state with the electronic states near the Fermi surface plays an important role in the electrical resistivity. It is well known that the theory of electron-phonon interactions has given a good explanation of the ρ - T relationship for crystalline materials, that is, the T^5 law at low-enough temperatures and a linear relationship at the high- T limit. However, for disordered alloys and other noncrystalline materials, things become much more complicated. Some materials show minimum phenomena at normal temperatures⁴ but others do not. Also, at the high- T limit, we find for some high- T_c superconductors that the linear law still exists⁵ but for some disordered alloys ρ is a constant instead.⁶ Perhaps the transformation from phonons to fractons can explain this. In the following parts, we will calculate the resistivity from the viewpoint of the interaction of the extended electron with both fractons and phonons.

II. METHOD OF CALCULATION

We can write the electronic-vibrational state interaction potential as follows:

$$V_p = - \sum_{\mu} U_{\mu} \cdot \nabla V(\mathbf{r} - \mathbf{R}_{\mu}), \tag{4a}$$

where $V(\mathbf{r})$ is the electron potential energy in the ion field, \mathbf{R}_{μ} is the equilibrium coordinate of the μ th atom, and U_{μ} is its displacement off \mathbf{R}_{μ} , where⁷

$$U_{\mu} = \sum_{\alpha} \left[\frac{\hbar}{2g\omega_{\alpha}} \right]^{1/2} [\Phi_{\alpha}(\mathbf{R}_{\mu})b_{\alpha} + \Phi_{\alpha}^*(\mathbf{R}_{\mu})b_{\alpha}^{\dagger}] \mathbf{e}_{\alpha}. \tag{4b}$$

b_α^\dagger (b_α) is the creation (annihilation) operator of the α th vibrational model and g is the density of mass of the system. We can obtain $\Phi_\alpha(\mathbf{R}_\mu)$ from (1c) or (2c) and \mathbf{e}_α is a unit vector in the polarization direction. The interaction Hamiltonian may be written as

$$H' = \int \Psi^\dagger(\mathbf{r}) V_P \Psi(\mathbf{r}) d^3r,$$

in which $\Psi^\dagger(\mathbf{r})$ and $\Psi(\mathbf{r})$ are the field operators of the electron. As is well understood, the electronic states that can contribute to electrical resistivity are near the Fermi surface. So the electron wave function can be regarded as a plane wave. Hence the interaction Hamiltonian can be rewritten as

$$\mathbf{H}_{\text{el-ph}} = \frac{1}{i} \frac{N}{G^{3/2}} \sum_{\mathbf{k}, \mathbf{q}}^{\omega < \omega_c} \left[\frac{\hbar}{2g\omega_\alpha} \right]^{1/2} V(\mathbf{q}) \times [a_{\mathbf{k}+\mathbf{q}}^\dagger a_{\mathbf{k}} (b_{\mathbf{q}} + b_{\mathbf{q}}^\dagger)] (\mathbf{e}_\alpha \cdot \mathbf{q}), \quad (5a)$$

$$\mathbf{H}_{\text{el-fr}} = \frac{1}{i} \frac{1}{G} \sum_{\mathbf{k}, \mathbf{k}', \alpha}^{\omega > \omega_c} \left[\frac{\hbar}{2g\omega_\alpha} \right]^{1/2} V(\mathbf{k}-\mathbf{k}') I(\mathbf{k}-\mathbf{k}') \times [a_{\mathbf{k}}^\dagger a_{\mathbf{k}'} (b_\alpha + b_\alpha^\dagger)] [\mathbf{e}_\alpha \cdot (\mathbf{k}-\mathbf{k}')], \quad (5b)$$

where N is the total number of atoms, $a_{\mathbf{k}}^\dagger, a_{\mathbf{k}'}$ are the creation and annihilation operators of the electron, and

$$V(\mathbf{k}-\mathbf{k}') = \int V(\mathbf{x}) \exp[i(\mathbf{k}-\mathbf{k}') \cdot \mathbf{x}] d^3x,$$

$$I(\mathbf{k}-\mathbf{k}') = \sum_{\mu} \Phi_\alpha(\mathbf{R}_\mu) \exp[i(\mathbf{k}-\mathbf{k}') \cdot \mathbf{R}_\mu].$$

The factor $|I(\mathbf{k}-\mathbf{k}')|^2$ is very important. Following the procedure of Alexander and Orbach⁷ (see Appendix), we obtain

$$|I(\mathbf{k}-\mathbf{k}')|^2 = \frac{(I_{\omega_\alpha} \cdot |\mathbf{k}-\mathbf{k}'|)^{d_f}}{a_0^{d_f}}. \quad (6)$$

For a fractal system, the translation symmetry and the reciprocal space do not exist anymore. But we can see from (5) that $\mathbf{H}_{\text{el-ph}}$ and $\mathbf{H}_{\text{el-fr}}$ depend only on the change of the electronic wave vector $\mathbf{k}-\mathbf{k}'$ instead of \mathbf{k} or \mathbf{k}' , respectively. Hence, in the following parts, we rewrite $\mathbf{k}-\mathbf{k}'$ as \mathbf{q} and introduce an integral about ω instead of the summation over \mathbf{q} as follows:

$$\left[\frac{N}{G} \right]^{-1} \sum_{\mathbf{q}} \rightarrow \int_0^{\omega_c} N_{\text{ph}}(\omega) d\omega + \int_{\omega_c}^{\omega_F} N_{\text{fr}}(\omega) d\omega. \quad (7)$$

Both sides of (7) are normalized to unit. Also, we will substitute \mathbf{k} by \mathbf{k}_F where \mathbf{k}_F is the electronic wave vector at the Fermi surface and will only consider the longitudinal boson so that $\mathbf{e}_\alpha \cdot \mathbf{q}/|q|$ or $\mathbf{e}_\alpha \cdot (\mathbf{k}-\mathbf{k}')/|\mathbf{k}-\mathbf{k}'|$ equals 1.

III. THE FIRST-ORDER INTERACTION

When we talk about the first-order interaction, what we consider is the process of emission or absorption of a fracton or phonon by the electron. By solving the Boltzmann transport equation to first order, we obtain⁸

$$\left[\frac{1}{\tau_F} \right]^I = \left[\frac{1}{\tau_F} \right]_{\text{ph}}^I + \left[\frac{1}{\tau_F} \right]_{\text{fr}}^I, \quad (8a)$$

where τ_F is the electronic relaxation time at the Fermi surface, the superscript I denotes the first-order interaction, and

$$\left[\frac{1}{\tau_F} \right]_{\text{ph}}^I = \left[\frac{2m^* \pi V^2}{\hbar^2 M} \right] \left[\frac{N^2}{k_F^3 v_s^3} \right] \left[\frac{d}{\omega_D^d G^3} \right] \times \int_0^{\omega_c} \frac{\omega^{d+1} e^{\hbar\omega/k_B T}}{e^{2\hbar\omega/k_B T} - 1} d\omega, \quad (8b)$$

$$\left[\frac{1}{\tau_F} \right]_{\text{fr}}^I = \left[\frac{2m^* \pi V^2}{\hbar^2 M} \right] \left[\frac{N}{k_F^3 G^3} \right] \left[\frac{\bar{d}_f}{\omega_F^{\bar{d}_f} l_c^3 \omega_c^{3\bar{d}_f/d_f}} \right] \times \int_{\omega_c}^{\omega_F} \frac{\omega^{3\bar{d}_f/d_f + \bar{d}_f - 2} e^{\hbar\omega/k_B T}}{e^{2\hbar\omega/k_B T} - 1} d\omega. \quad (8c)$$

Here, M is the mass of the atom and V^2 is the average of $V^2(\mathbf{q})$ over all \mathbf{q} . We assume that this procedure does not result in qualitative errors. Noting that $v_s \approx a_0 \omega_D$, $N = gG/M$, and $\rho = m^*/ne^2 \tau_F$, where m^* and n are the effective mass and density of the electron, and substituting (3), we finally get

$$\rho^I = \rho_{\text{ph}}^I + \rho_{\text{fr}}^I, \quad (9a)$$

where

$$\rho_{\text{ph}}^I = A \left[\frac{V^2 d}{G \omega_F} \right] \left[\lambda^{d_f + d_f / \bar{d}_f - 3d_f / d - 2d} \right] \times \int_0^1 x^{d+1} P(\beta x) dx, \quad (9b)$$

$$\rho_{\text{fr}}^I = A \left[\frac{V^2 \bar{d}_f}{G \omega_F} \right] \left[\frac{a_0^{2d-d_f}}{G} \right] \left[\lambda^{d_f / \bar{d}_f - d_f - 3d_f / d} \right] \times \int_1^{\lambda^{d_f / \bar{d}_f}} x^{3\bar{d}_f/d_f + \bar{d}_f - 2} P(\beta x) dx. \quad (9c)$$

Here, λ is defined as l_c/a_0 , A is a constant irrelevant to d_f , $P(\beta x) \equiv e^{\beta x}/(e^{2\beta x} - 1)$, $\beta \equiv \hbar\omega_c/k_B T$, and $x = \omega/\omega_c$. We have also used the relationships $k_F \propto n^{1/d}$, the density of mass of a fractal network $g = (l_c/a_0)^{d_f} M/l_c^d$, and $n = NZ/G$ in which Z is the number of valence electrons. Hence we can obtain the following results from (8) and (9).

(1) At the high- T limit when $T \gg \hbar\omega_F/k_B$, it is explicit that $\rho_{\text{ph}}^I \sim T$ and $\rho_{\text{fr}}^I \sim T$ so that $\rho^I \sim T$, which is consistent with the result in crystalline materials. When $T \gg \hbar\omega_c/k_B$ and $T \ll \hbar\omega_F/k_B$, we find $\rho_{\text{ph}}^I \sim T$ and $\rho_{\text{fr}}^I \sim T^{3\bar{d}_f/d_f + \bar{d}_f - 1}$ so that

$$\rho^I \sim aT + bT^{3\bar{d}_f/d_f + \bar{d}_f - 1},$$

where a, b are coefficients changing with d_f, \bar{d}_f , and l_c . The dependence of the resistivity on T is shown in Figs. 1 and 2. There, T_c is defined as $\hbar\omega_c/k_B$ and the value of resistivity is just a relative number. This is also the case

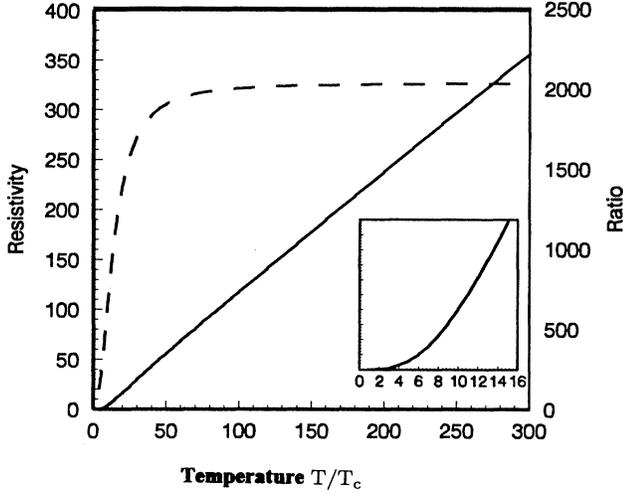


FIG. 1. The first-order interaction. The dependence of resistivity and ratio on temperature. $d=2$, $l_c/a_0=30.0$, $d_f=1.50$, and $\bar{d}_f=\frac{4}{3}$. The inset gives some more detail about the resistivity below $T=16T_c$. In Figs. 1–11, the dashed line denotes the value on the ratio axis, and the solid line the relative value of the resistivity.

in other figures. It can be seen that they are very similar to those in Ref. 9.

(2) ρ^I is related not only to T but also to d_f , \bar{d}_f , and l_c . Letting G be l_c^d , and neglecting the difference between a_0^d and \hat{a}_0^d , we perform our calculation of ρ^I at different d_f and obtain the curves in Figs. 3 and 4. When $d=2$, ρ^I is a monotonically decreasing function of interesting dimensionalities. But there is a minimum in the ρ^I-d_f curve when $d=3$. Here we have used the $\bar{d}_f=\frac{4}{3}$ conjecture on a percolative network which is not true in other cases. It must be emphasized that more correct quantitative results depend on a more correct relationship between d_f and \bar{d}_f .

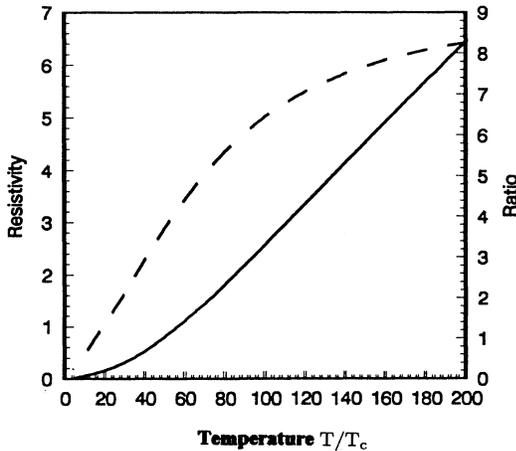


FIG. 2. The first-order interaction. The dependence of resistivity and ratio on temperature. $d=3$, $l_c/a_0=30.0$, $d_f=2.20$, and $\bar{d}_f=\frac{4}{3}$.

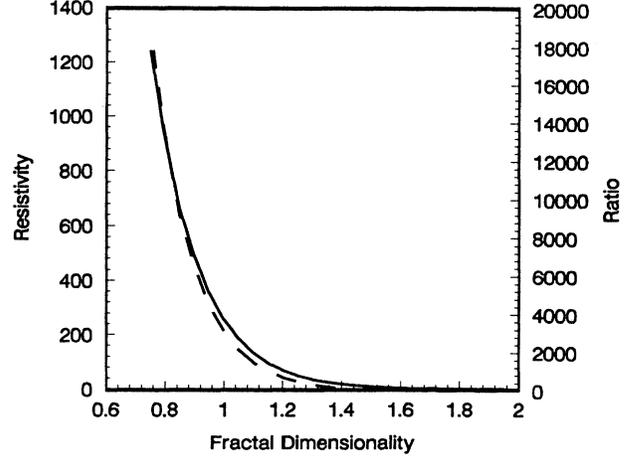


FIG. 3. The first-order interaction. The dependence of resistivity and ratio on fractal dimensionality. $d=2$, $l_c/a_0=30.0$, $T/T_c=100$, and $\bar{d}_f=\frac{4}{3}$.

(3) To show the relative contribution of phonon or fracton to ρ^I , we defined a factor $R \equiv \rho_{\text{ph}}^I/\rho_{\text{fr}}^I$. The variation of R with T or d_f is also shown in Figs. 1–4. We now find a special d_{fo}^I , before which $R \gg 1$ and after which $R \ll 1$. This expresses the fact that when the system becomes more and more distant from the Euclidean space, fractons play a more and more important role in the resistivity.

In particular, at the high-temperature limit, from (9),

$$\rho_{\text{ph}}^I \sim \frac{V^2}{\omega_F G^2} \lambda^{2d_f/\bar{d}_f + d_f - d - 3d_f/d}, \quad (10a)$$

$$\rho_{\text{fr}}^I \sim \frac{V^2}{\omega_F G^2} \lambda^{3(1-d_f/d)}; \quad (10b)$$

hence

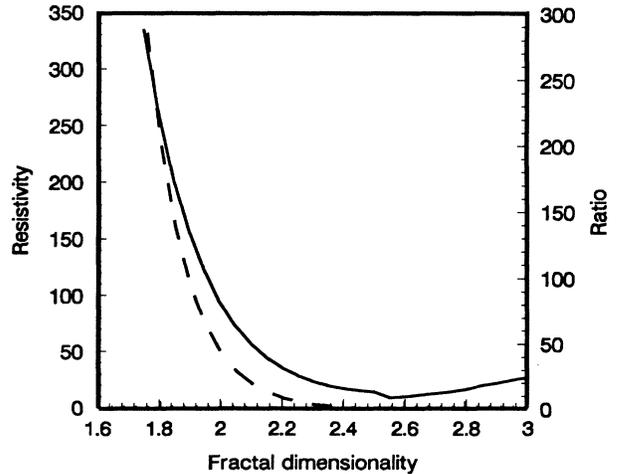


FIG. 4. The first-order interaction. The dependence of resistivity and ratio on fractal dimensionality. $d=3$, $l_c/a_0=30.0$, $T/T_c=1000$, and $\bar{d}_f=\frac{4}{3}$.

$$R \sim \lambda^{d+3-d_f-2d_f/\bar{d}_f}. \quad (10c)$$

Still using the conjecture $\bar{d}_f = \frac{4}{3}$, we obtain $d_{fo}^I \approx 2.0$ for $d=2$ and $d_{fo}^I \approx 2.4$ for $d=3$. That tells us that on a two-dimensional precolative network ($d_f=1.9$), fractons contribute much more to ρ^I than do phonons. But this is not the case when $d=3$ ($d_f=2.5$).

IV. THE SECOND-ORDER INTERACTION

As to the second-order interaction, we consider the two processes shown in Fig. 5. In Fig. 5(a), a boson (fracton and phonon) \mathbf{q} is first emitted by the electronic state \mathbf{k} and then absorbed by the electronic state $\mathbf{k}'=\mathbf{k}-\mathbf{q}$. In Fig. 5(b), $A-\mathbf{q}$ boson is emitted by the electronic state $\mathbf{k}-\mathbf{q}$ and then annihilated by \mathbf{k} .

The interaction matrix element can be written as⁴

$$\begin{aligned} \Theta_{\mathbf{k},\mathbf{k}'} &= \frac{\hbar V^2}{2MG^2} \left[\sum_q^{\text{ph}} \frac{Nq^2}{\omega_q} + \sum_q^{\text{fr}} \frac{q^2 a_0^d}{\omega_q a_0^{d_f}} \right] \\ &\times \left[\frac{1-f_0(\mathbf{k}')}{E(\mathbf{k})-E(\mathbf{k}')-\hbar\omega_q} - \frac{f_0(\mathbf{k}')}{E(\mathbf{k}')-E(\mathbf{k})-\hbar\omega_q} \right] \\ &\times [n(\omega_q)+1], \end{aligned} \quad (11)$$

where $f_0(\mathbf{k})$ and $n(\omega_q)$ are the equilibrium Fermi and Bose distribution functions, respectively. We now make an assumption that the electron only transports between the energy surface E_F and E_F-kT .⁴ Hence,

$$\begin{aligned} \Theta_{\mathbf{k},\mathbf{k}'} &= \frac{\hbar V^2}{2MG^2} \left[\sum_q \frac{Nq^2}{\omega_q} + \sum_q \frac{q^2 a_0^d}{\omega_q a_0^{d_f}} \right] \\ &\times \frac{e^{\hbar\omega/k_B T}}{e^{\hbar\omega/k_B T}-1} \frac{1}{k_B T + \hbar\omega}. \end{aligned} \quad (12)$$

Here we have made an approximation $f_0(\mathbf{k}') \approx 1$. Using $1/\tau_F \propto |\Theta_{\mathbf{k},\mathbf{k}'}|^2$ and $\rho = m^*/ne^2\tau_F$, we obtain

$$(\sqrt{\rho})^{\text{II}} = (\sqrt{\rho})_{\text{ph}}^{\text{II}} + (\sqrt{\rho})_{\text{fr}}^{\text{II}}, \quad (13a)$$

where

$$\begin{aligned} (\sqrt{\rho})_{\text{ph}}^{\text{II}} &= B \left[\frac{dV^2}{\omega_F^2 G} \right] \left[\lambda^{2d_f/\bar{d}_f+3d_f/2-5d/2-2} \right] \\ &\times \int_0^1 x^d F(\beta x) dx, \end{aligned} \quad (13b)$$

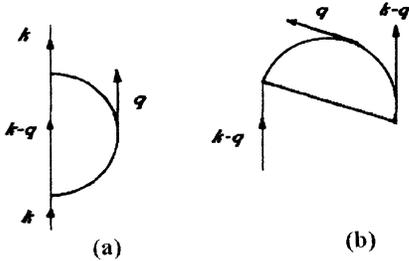


FIG. 5. Diagram of the second-order interaction between the electron and the fracton or phonon.

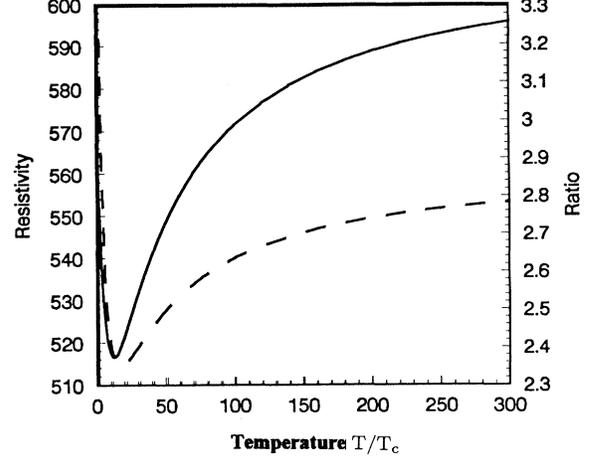


FIG. 6. The second-order interaction. The dependence of resistivity and ratio on temperature. $d=2$, $l_c/a_0=30.0$, $d_f=1.50$, and $\bar{d}_f=\frac{4}{3}$.

$$\begin{aligned} (\sqrt{\rho})_{\text{fr}}^{\text{II}} &= B \left[\frac{\bar{d}_f V^2}{\omega_f^2} \right] \left[\frac{a_0^{2d-d_f}}{G^2} \right] \\ &\times \left[\lambda^{2d_f/\bar{d}_f-d_f/2-d/2-2} \right] \\ &\times \int_1^{\lambda^{d_f/\bar{d}_f}} x^{2d_f/\bar{d}_f+\bar{d}_f-2} F(\beta x) dx, \end{aligned} \quad (13c)$$

in which

$$F(\beta x) \equiv \frac{e^{\beta x}}{(e^{\beta x}-1)(x+1/\beta)}.$$

β and x are defined as before. From (13) we have some conclusions different from the first-order results above.

(1) In the high- T limit when $T \gg \hbar\omega_F/k_B$ now

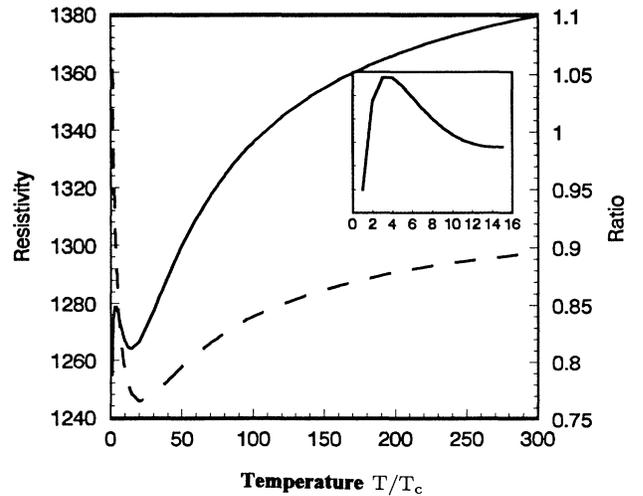


FIG. 7. The second-order interaction. The dependence of resistivity and ratio on temperature. $d=2$, $l_c/a_0=30.0$, $d_f=1.65$, and $\bar{d}_f=\frac{4}{3}$. The inset also gives some more detail about the resistivity below $T=16T_c$.

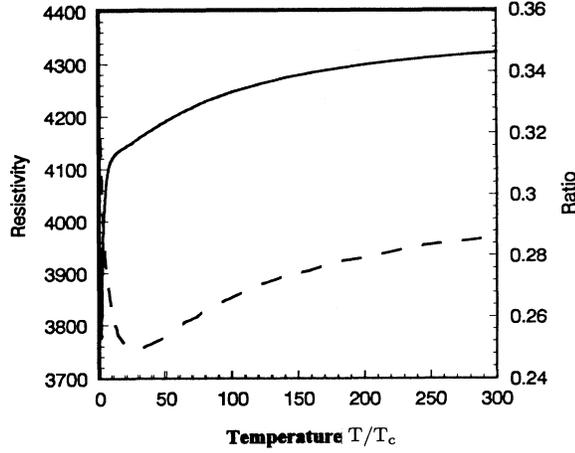


FIG. 8. The second-order interaction. The dependence of resistivity and ratio on temperature. $d=2$, $l_c/a_0=30.0$, $d_f=1.80$, and $\tilde{d}_f=\frac{4}{3}$.

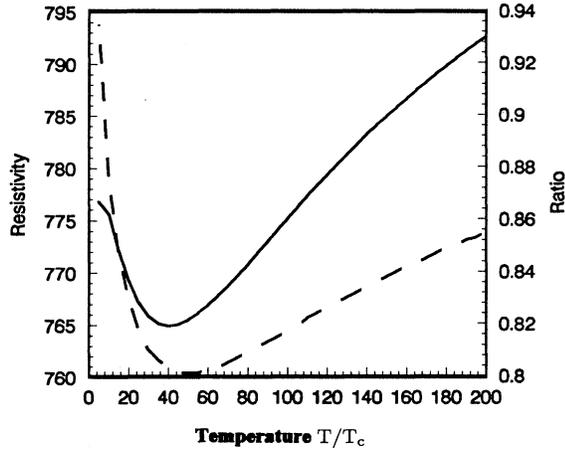


FIG. 9. The second-order interaction. The dependence of resistivity and ratio on fractal dimensionality. $d=3$, $l_c/a_0=30.0$, $d_f=2.10$, and $\tilde{d}_f=\frac{4}{3}$.

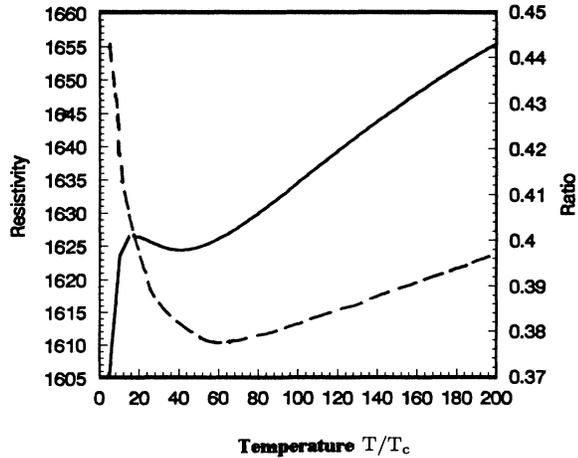


FIG. 10. The second-order interaction. The dependence of resistivity and ratio on fractal dimensionality. $d=3$, $l_c/a_0=30.0$, $d_f=2.20$, and $\tilde{d}_f=\frac{4}{3}$.

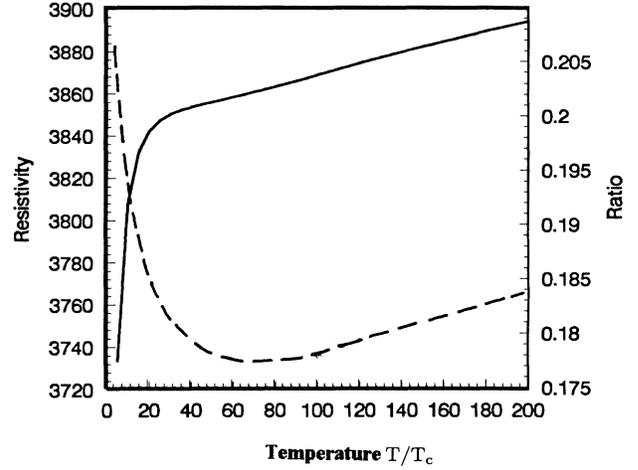


FIG. 11. The second-order interaction. The dependence of resistivity and ratio on fractal dimensionality. $d=3$, $l_c/a_0=30.0$, $d_f=2.30$, and $\tilde{d}_f=\frac{4}{3}$.

$\rho^{\text{II}} \sim \text{const}$ instead of $\rho^{\text{II}} \propto T$. This result is consistent with the experiments done by Lin and Yu.⁶ They found that for the disordered alloy $\text{Ti}_x\text{Al}_{1-x}$ ($x \leq 0.135$) the electrical resistivity is a constant at a high enough T .

(2) We have performed the calculation of ρ^{II} as a function of the temperature. What interests us most is that, when d_f increases, the form of the $\rho^{\text{II}}-T$ curve changes explicitly. At first, there is a minimum in the $\rho^{\text{II}}-T$ curve later ρ^{II} is a monotonically increasing function of T . We have also defined a factor $R' \equiv (\sqrt{\rho})_{\text{fr}}^{\text{II}} / (\sqrt{\rho})_{\text{ph}}^{\text{II}}$ ($R' \sim \lambda^{2d-d_f-2d_f/d_f+2}$ at high T with $d_{f0}^{\text{II}} \simeq 2.4$ for $d=2$) and the variation of ρ^{II} and R' with T are shown in Figs. 6–11.

(3) From (13), we know that for a second-order interaction, ρ^{II} is a monotonically increasing function of d_f , different from the case of the first-order interaction.

V. DISCUSSIONS ON THE PERCOLATIVE NETWORK

It is well known that the percolative network has fractal properties near the percolation threshold P_c . So we think its electric resistivity comes from the contribution of both fractons and phonons. We now use what we have obtained above for a two-dimensional system. From (10b) and (13c), we have

$$\rho_{\text{fr}}^{\text{I}} \sim \frac{V^2}{G^2 \omega_F} \lambda^{3(1-d_f/d)}, \quad (14a)$$

$$\sqrt{\rho_{\text{fr}}^{\text{II}}} \sim \frac{V^2}{G^2 \omega_F^2} \lambda^{(d_f-d)/2}. \quad (14b)$$

Here we have neglected the contribution of the phonon just because we have $d_{f0}^{\text{I}} \simeq 2.0$ and $d_{f0}^{\text{II}} \simeq 2.4$ when $d=2$. Now we assume the following scalar laws:

$$\omega_F \sim |P - P_c|^a, \quad (15a)$$

$$\omega_c \sim |P - P_c|^b, \quad (15b)$$

$$l_c \sim |P - P_c|^{-\nu}, \quad (15c)$$

$$V \sim |P - P_c|^{-c}, \quad (15d)$$

where P is the occupation probability. Noting that

$$\rho \sim |P - P_c|^{-t}, \quad \omega_F = \omega_c \lambda^{d_f/\bar{d}_f},$$

and using the scalar law of fractons¹⁰ $\omega_c \sim \lambda^{-d_f/\bar{d}_f}$, we obtain from (14a) that

$$t = 3b \left[\frac{\bar{d}_f}{d_f} - \frac{\bar{d}_f}{d} \right] + 2c. \quad (16)$$

Recently, Royer, Benoit, and Poussiguet have calculated b using the spectral moments method. They got $b \approx 0.7$ for $d=2$.¹¹ From percolative theory, we have $t=1.1$, $\bar{d}_f=1.36$, and $d_f=1.9$ for $d=2$. So it is easy to conclude that $c=0.5$. If we apply this to (14b), we have $\rho_{\text{fr}}^{\text{II}} \sim |P - P_c|^{-2.1}$ ($d=2$). We can also say that if c can be determined exactly by another method, perhaps Eq. (16) has provided another method to calculate the scalar exponent t .

VI. DISCUSSION AND CONCLUSION

There are two important conclusions from this paper. First, the electric resistivity ρ is now not only a function of temperature as known for crystalline materials, but also a function of the fractal dimensionality d_f , the fracton dimensionality \bar{d}_f , and the characteristic length l_c , as has been shown in (9) and (13). More correct variations of ρ with these parameters depend on the implicit functional relationship among them and the latter is still an unsolved question. Secondly, we have provided a more reasonable way to calculate electrical resistivity, that is, we have considered phonon and fractons simultaneously. We predicted that the variation of ρ with temperature for a fractal network has some complicated characteristics as shown in Sec. IV. We also discussed the percolating network in Sec. V. Whether the scalar laws there exist is still left unproven. But in any case, it is an illustration of fracton theory. Finally, we must emphasize that our results depend greatly on the density of states and dispersion law of the fracton. The relations we have used in (3) are provided by Alexander *et al.*¹ but their correct forms are still worth disputing.

ACKNOWLEDGMENTS

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APPENDIX

The factor $|I(\mathbf{k}-\mathbf{k}')|^2$ has been quoted in different forms in different references. In Ref. 4, it is

$$|I(\mathbf{k}-\mathbf{k}')|^2 = (I_{\omega_\alpha} |\mathbf{k}-\mathbf{k}'|)^{d_f}. \quad (\text{A1})$$

However, in Ref. 5,

$$|I(\mathbf{k}-\mathbf{k}')|^2 = \frac{1}{|\mathbf{k}-\mathbf{k}'|^{d_f}}. \quad (\text{A2})$$

So we performed the calculation carefully following the

procedure provided by Ref. 4 and what we obtained is shown in (6).

The calculation is based on the superlocalization of the fracton wave function introduced by Alexander *et al.*,¹

$$\Phi_\alpha^{\text{fr}}(\mathbf{R}_\mu) = c |\mathbf{R}_\mu|^{d-d_f/2} I_{\omega_\alpha}^{-d_f/2} \exp \left[-\frac{1}{2} \left(\frac{\mathbf{R}_\mu}{I_{\omega_\alpha}} \right)^{d_\phi} \right].$$

The main point is that, when $\mathbf{R}_\mu \leq 1/|\mathbf{k}-\mathbf{k}'|$, $\Phi_\alpha^{\text{fr}}(\mathbf{R}_\mu) \approx I_{\omega_\alpha}^{-d_f/2}$, and $\Phi_\alpha^{\text{fr}}(\mathbf{R}_\mu) \approx 0$ outside this range. We now divide the volume $I_{\omega_\alpha}^{d_f}$ into small volumes $1/|\mathbf{k}-\mathbf{k}'|^{d_f}$. The number is $I_{\omega_\alpha}^{d_f} |\mathbf{k}-\mathbf{k}'|^{d_f}$. In each of these small cells, $\Phi_\alpha^{\text{fr}}(\mathbf{R}_\mu) \approx I_{\omega_\alpha}^{-d_f/2}$. So,

$$\begin{aligned} \sum_\mu \Phi_\alpha^{\text{fr}}(\mathbf{R}_\mu) e^{i(\mathbf{k}-\mathbf{k}') \cdot \mathbf{R}_\mu} &= I_{\omega_\alpha}^{d_f} |\mathbf{k}-\mathbf{k}'|^{d_f} \left[I_{\omega_\alpha}^{-d_f/2} \sum_{\mu'} e^{i(\mathbf{k}-\mathbf{k}') \cdot \mathbf{R}_{\mu'}} \right] \\ &= I_{\omega_\alpha}^{d_f/2} |\mathbf{k}-\mathbf{k}'|^{d_f} \sum_{\mu'} e^{i(\mathbf{k}-\mathbf{k}') \cdot \mathbf{R}_{\mu'}}, \end{aligned}$$

where μ' denotes the summation over the small volume. Also,

$$\begin{aligned} |I(\mathbf{k}-\mathbf{k}')|^2 &= I_{\omega_\alpha}^{d_f} |\mathbf{k}-\mathbf{k}'|^{2d_f} \left[\sum_{\mu', \nu'} e^{i(\mathbf{k}-\mathbf{k}') \cdot (\mathbf{R}_{\mu'} - \mathbf{R}_{\nu'})} \right] \\ &= I_{\omega_\alpha}^{d_f} |\mathbf{k}-\mathbf{k}'|^{2d_f} \\ &\quad \times \left[\sum_{\mu', \nu'} \delta_{\mu', \nu'} + \sum_{\mu' \neq \nu'} e^{i(\mathbf{k}-\mathbf{k}') \cdot (\mathbf{R}_{\mu'} - \mathbf{R}_{\nu'})} \right] \\ &\approx I_{\omega_\alpha}^{d_f} |\mathbf{k}-\mathbf{k}'|^{2d_f} N'. \end{aligned}$$

Here N' is the number of atoms in the small volume, so

$$N' = \frac{1}{|\mathbf{k}-\mathbf{k}'|^{d_f} a_0^{d_f}},$$

where a_0 is the lattice constant. We finally obtain the result shown in (6). We can also detect the correctness of (6) from the viewpoint of dimensional analysis. First, $I_{\omega_\alpha}^{d_f} |\mathbf{k}-\mathbf{k}'|^{d_f}$ is a value on the order of unity, so for fractons

$$|I(\mathbf{k}-\mathbf{k}')|_{\text{fr}}^2 \approx \frac{1}{a_0^{d_f}} \approx \frac{N}{G}.$$

If we substitute $\Phi_\alpha^{\text{fr}}(\mathbf{R}_\mu)$ by $\Phi_\alpha^{\text{ph}}(\mathbf{R}_\mu)$, the new factor $|I(\mathbf{k}-\mathbf{k}')|_{\text{ph}}^2$ must be of the same dimension as $|I(\mathbf{k}-\mathbf{k}')|_{\text{fr}}^2$. Substituting

$$\Phi_\alpha^{\text{ph}}(\mathbf{R}_\mu) = \frac{1}{\sqrt{G}} e^{iq \cdot \mathbf{R}_\mu}$$

into

$$|I(\mathbf{k}-\mathbf{k}')|_{\text{ph}}^2 = \left[\sum_\mu \Phi_\alpha^{\text{ph}}(\mathbf{R}_\mu) e^{i(\mathbf{k}-\mathbf{k}') \cdot \mathbf{R}_\mu} \right]^2,$$

we get

$$|(\mathbf{k}-\mathbf{k}')|_{\text{ph}}^2 \approx \frac{N^2}{G},$$

which is N times $|I(\mathbf{k}-\mathbf{k}')|_{\text{fr}}^2$ but has no dimensional difference from it.

- ¹S. Alexander *et al.*, Phys. Rev. B **28**, 4615 (1983).
- ²R. Remmal and G. Toulouse, J. Phys. (Paris) Lett. **44**, L13 (1983).
- ³S. Alexander, O. Entin-Wohlman, and R. Orbach, Phys. Rev. B **32**, 6447 (1985).
- ⁴De-cheng Tian *et al.*, Phys. Rev. B **45**, 8116 (1992).
- ⁵Qing Jiang *et al.*, Phys. Rev. B **48**, 524 (1993).
- ⁶J. J. Lin and C. Yu, Phys. Rev. B **48**, 4864 (1993).
- ⁷S. Alexander and R. Orbach, Phys. Rev. B **34**, 2726 (1986).
- ⁸Joseph Callaway, *Quantum Theory of the Solid State* (Academic, San Diego, 1990), Chap. 7.
- ⁹Philip B. Allen and Werner W. Schulz, Phys. Rev. B **47**, 14 434 (1993).
- ¹⁰S. Alexander and R. Orbach, J. Phys. (Paris) Lett. **43**, L625 (1982).
- ¹¹E. Royer, C. Benoit, and G. Poussigue, J. Phys. Condens. Matter **4**, 561 (1992).