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Explosive Synchronization and Emergence of Assortativity on Adaptive Networks *

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We report an explosive transition from incoherence to synchronization of coupled phase oscillators on adaptive networks, following an Achlioptas process based on dynamic clustering information. During each adaptive step of the network topology, a portion of the links is randomly removed and the same amount of new links is generated following the so-called product rules (PRs) applied to the dynamic clusters. Particularly, two types of PRs are considered, namely, the min-PR and max-PR. We demonstrate that the synchronization transition becomes explosive in both cases. Interestingly, we find that the min-PR rule can lead to disassortativity of the network topology, while the max-PR rule leads to assortativity.

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The explosive transition of network structure has gained considerable attention[1−13] since the pioneering work by Achlioptas *et al.*[1] Therein, they proposed a stochastic percolation procedure using product rule (PR) yielding a first-order percolation transition which is characterized by the sharp emergence of a giant cluster. Friedman and Landsberg[2] demonstrated that this simple strategy is sufficient for changing the behavior of the percolation transition. Following the work of Achlioptas *et al.*, the PR rule has been applied to percolation on many network models, as well as regular lattices.[3−8] Meanwhile, many other studies tried to propose alternative rules to obtain explosive percolation. For instance, a weighted rule where bonds are occupied according to a certain probability,[9,10] an acceptance method where new bonds are selected randomly and occupied according to a certain weight,[11] etc. The explosive character of the percolation transition on the human protein homology network was also reported.[12] Although there are some debates on the nature of the explosive percolation transition very recently,[13] this phenomenon is undoubtedly very interesting and important and deserves more studies.

Notice that, however, almost all the explosive phase transitions reported so far only concern the phase transition of the topological structure of a network, i.e., the percolation transition. For many real-world networks, there are phase transitions not only of network structure but also of dynamics taking place on them, which are often associated with the formation of a giant dynamic cluster (DC). Examples include synchronization of coupled phase oscillators,[14] epidemic spreading and consistent opinion formation,[15,16] order-disorder transition in the Ising model,[17] etc. In many systems, such dynamic phase transitions are of second order, i.e., the largest dynamic cluster grows continuously as the control parameter changes. Meanwhile, a well-defined order parameter increases continuously from 0. Therefore, an interesting question arises: If we apply the PR rule to the evolution of dynamic clusters, can such dynamic phase transitions be explosive or not?

We study the synchronization process of coupled phase oscillators on adaptive networks, where the network topology coevolves with the dynamics by applying PR based on the dynamic clustering information. We assume that the network changes via the natural death of old links and birth of new ones. Specifically, the new links are added following the dynamic PRs: For each new link, we choose the one which minimizes (min-PR) or maximizes (max-PR) the product of the sizes of the two DCs it joins. We present evidences that explosive synchronization does happen, i.e., the size of the largest DC or the order parameter increases discontinuously at the onset of synchronization. In addition, we have also investigated the characteristic topological features of the resulting networks, finding that the min-PR (max-PR) can automatically lead to disassortativity (assortativity), respectively. Therefore, our model may also be interpreted as a possible mechanism that explains the emergence of assortativity on real networks, which is a common property of real networks.[18]

Here we consider the Kuramoto model, which has been widely used to study synchronization of coupled phase oscillators.[19] The Kuramoto model consists of a population of N coupled phase oscillators, described by time-dependent phase $\theta_i(t)$ and natural frequencies ω_i, which are picked up from a given probability distribution $g(\omega)$. The dynamic equation reads

$$\dot{\theta}_i = \omega_i + \sigma \sum_{i=1}^{N} a_{ij} \sin(\theta_j - \theta_i), \quad i = 1, \cdots, N, \quad (1)$$

where σ is the coupling strength and a_{ij} is the ele-

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ment of adjacency matrix of the underlying network, which is $a_{ij} = 1$ if nodes i and j are connected and $a_{ij} = 0$ otherwise. For this model, synchronization is conveniently measured by the time-averaged order parameter,

$$R = \lim_{t\to\infty} \frac{1}{t}\int_0^t r\,(t) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \frac{1}{N}\sum_{j=1}^{N} e^{i(\theta_j(t)-\psi(t))},$$

where $\psi(t) = \frac{1}{N}\sum_i \theta_i(t)$ is the average phase and the order parameter r measures the coherence of oscillators. Clearly, $r = 0$ if all the oscillators are totally incoherent and $r = 1$ if the oscillators are completely synchronized. In the thermodynamic limit of $N \to \infty$, a continuous transition from incoherence ($R = 0$) to synchrony ($R > 0$) occurs via a supercritical bifurcation as the coupling strength σ passes through a critical value σ_c.

In our model, the co-evolution dynamics are described as follows. We start from an Erdös-Rényi random network with N nodes and the connectivity probability p and run the dynamics via numerical simulation of Eq. (1) with a time step Δt. The node frequencies are randomly picked up from a Gaussian distribution with zero mean value and unit variance. After every m dynamic time step, the network evolves one adaptive step. Since network topology usually changes much more slowly than the dynamics on it, we set $m \gg 1$. For each adaptive step, we consider two processes: the natural death and biased birth of links. We assume that the number of dead links and new links are the same, given by n, to keep the total number of links unchanged. The former process is realized simply by randomly deleting links, while we apply Achlioptas dynamics to add new links: For each new link, we first randomly choose two pairs of candidate nodes which are not yet connected and then link the pair which minimizes (min-PR) or maximizes (max-PR) the product of the sizes of dynamic clusters it joins. For comparison, we have also performed simulations for the case without preference (no-PR) when adding a new link.

To define a DC, we introduce the time-averaged frequency[20] of oscillator i as

$$\bar{\omega}_i = \frac{1}{m\Delta t}\int_{t_0-m\Delta t}^{t_0} \dot{\theta}_i(t)dt,$$

where t_0 is the time when we run the adaptive step. We can say that two oscillators i and j are in the same DC if $|\bar{\omega}_j - \bar{\omega}_i| \le \delta_0$, where δ_0 is a given small threshold value. Then, the size of a DC can be conveniently measured by the number of nodes within it. In the following, we fix the time step $\Delta t = 0.005$, connectivity probability $p = 0.15$, time steps between successive adaptive step $m = 400$, the number of deleted/added links $n = 0.1pC_N^2$ and frequency threshold inside a DC $\delta_0 = 0.001$. The network size is $N = 500$ if not otherwise stated. The robustness of our main results against the choice of the above parameters will be discussed later.

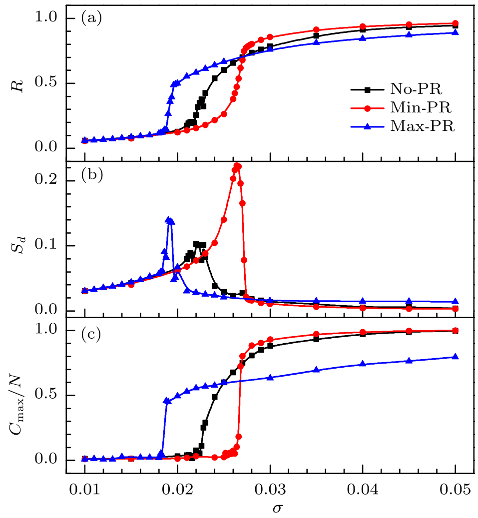

Fig. 1. (Color online) Dependences of (a) R, (b) S_d and (c) $C_{\max}/N$ (see the text for definition of these parameters) on the coupling strength σ.

In Fig. 1(a), the ensemble-averaged order parameter $\langle R\rangle$ as a function of the coupling strength σ is present. Here $\langle\cdot\rangle$ stands for averaging over 30 different initial network realizations. In the thermodynamic limit $N \to \infty$, $\langle R\rangle$ is zero for $\sigma < \sigma_c$, and increases from 0 at σ_c. For a finite system, however $\langle R\rangle$ cannot be exactly zero for $\sigma < \sigma_c$ due to fluctuations. Hence to locate the transition point, one can calculate the standard deviation S_d of r over time and ensemble which usually shows a clear-cut peak at σ_c. We use this as a criterion to define σ_c for finite systems in the present work. In Fig. 1(b), S_d is plotted as a function of σ. Apparently, the peak for both the cases of min-PR and max-PR are much more pronounced and sharp than that for the no-PR case. This observation hints that the synchronization transition with PRs may become explosive compared to that without PR. To be consistent with this, the ratio of the largest dynamic cluster size, $C_{\max}$, to the total number of nodes N, increases sharply from 0 as shown in Fig. 1(c).

Although the above results suggest that the synchronization transition may become discontinuous, more evidence is necessary. To this end, we have used the fact that for a finite system with first-order like transition, the probability distribution of the order parameter within the transition region should be bimodal rather than unimodal. Accordingly, we have calculated the distribution of r over different initial network realizations and time at different coupling strengths and the results are shown in Fig. 2. Clearly, if σ is away from the transition point σ_c, the distributions are all unimodal whether PRs are applied or not. However, in the very vicinity of σ_c, bimodal distributions are found for both min-PR and max-PR cases, while the distribution is still unimodal for the no-PR case. The peak at a small r value indicates an incoherent state and the peak at large r denotes a phase-locked synchronization state. The presence of two main peaks shows the coexistence of the incoherent and synchronized states, which is a characteristic of the first-order or discontinuous transition for a finite system.[21] Therefore, we believe that the transition from incoherence to synchronization becomes explosive under both min-PR and max-PR dynamics.

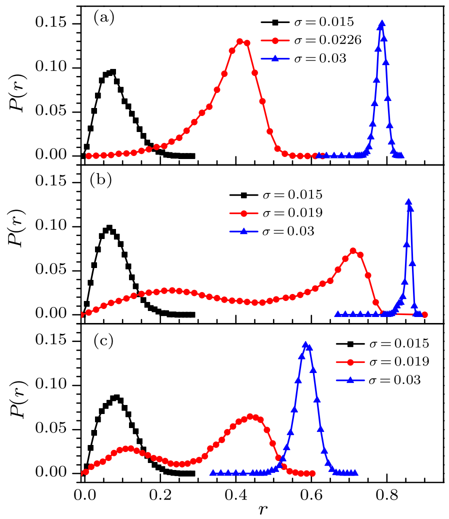

Fig. 2. (Color online) Distribution of the order parameter r for (a) no-PR, (b) min-PR and (c) max-PR. Within the transition range, the distribution shows bimodal shape for the latter two cases. Note that in (b), the $P(r)$ data shown for $\sigma = 0.03$ has been divided by 2.

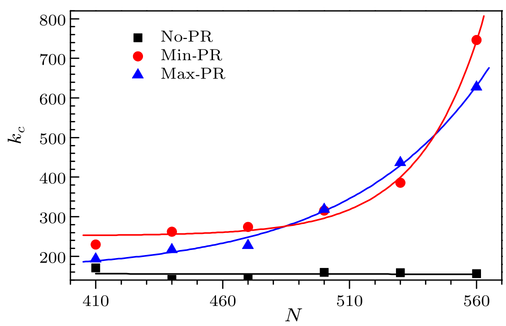

Fig. 3. (Color online) Dependence of k_c on system size N under no-PR, min-PR and max-PR dynamics. Symbols: simulation results. Red and blue lines: fitting by exponential growth. Black line: Drawn to guide the eyes.

The above results are obtained for a network of size $N = 500$. Rigorously speaking, to reach the conclusion undoubtedly, one should perform simulations on networks of much larger sizes. However, direct simulation of the coevolution dynamics on a network of a very large size is rather expensive. Nevertheless, one may design some alternative approaches to further demonstrate the explosive feature of the transition without simulations on very large networks. We have carefully investigated the fast transition region for different system sizes. One expects that the transition will become extremely sharp as the system size goes to infinity. To show this, we have introduced a parameter, k_c, to characterize the sharpness of transition, which may be explicitly defined as $k_c = \lim_{\sigma\to\sigma_c} (R-R_c)/(\sigma-\sigma_c)$. The dependences of k_c on the system size N are shown in Fig. 3. We can see that, for the min-PR and max-PR cases, k_c increases monotonically with the system size N and both the curves can be well-fitted by exponential growth functions. For the no-PR case, however, k_c is nearly unchanged with N. Therefore, an infinite slope will present as $N \to \infty$ in both the min-PR and max-PR situations, which gives further evidence that the synchronization transition does become discontinuous.

As stated above, the network topology coevolves with the dynamics. Can the topology exhibit some new features while the synchronization of oscillators changes from a continuous transition to a discontinuous one? Interestingly, we find that this is true: the dynamic PRs can lead to the emergence of degree correlations on the resulting network, either assortativity (max-PR) or disassortativity (min-PR). As we know, degree correlation is a ubiquitous property of many real-world networks. For assortative (disassortative) networks, nodes with large degrees tend to connect with nodes with large (small) degrees, respectively. Usually, one may use the assortativity coefficient a_c to measure the degree correlation of a network, which is defined as

$$a_c = \frac{M^{-1}\sum_i j_i k_i - [M^{-1}\sum_i \frac{1}{2}(j_i+k_i)]^2}{M^{-1}\sum_i \frac{1}{2}(j_i^2+k_i^2) - [M^{-1}\sum_i \frac{1}{2}(j_i+k_i)]^2},$$
$$\text{for} \quad i = 1, \cdots, M,$$

where M is the total number of links, j_i and k_i are the degrees of the two nodes at the ends of the ith edge. In Fig. 4, we present the curves of a_c, calculated after the system has reached the stationary state and averaged over time and different network realizations, as a function of the coupling strength σ. For the no-PR case, the adaptive step does not induce any degree correlation within the network thus a_c remains nearly zero characterizing a random network. When the min-PR or max-PR is applied, however, the dependence of a_c on σ is nontrivial. In the weak coupling region, a_c is also nearly zero implicating that no degree-correlation is present on the network. As the coupling strength approaches σ_c, a_c increases (decreases) rapidly until at σ_c it reaches a maximum (minimum) value, when the max-PR (min-PR) is applied in the adaptive step, respectively. With further increasing σ, a_c relaxes slowly. One expects that in the limit of infinite coupling strength, all the oscillators belong to a single DC and the PRs would not take effect such that the final network will lose degree correlation and a_c would be 0. Therefore, our model with PRs applied to the dynamic cluster can automatically lead to network assortativity or disassortativity.

One may understand the emergence of degree correlation qualitatively as follows. The key point is that nodes with larger degrees are coupled to the global mean field more strongly and henceforth are more likely to be inside a large DC. When the coupling strength is small, the oscillators are almost incoherent and few DCs are present. In this case, the preference rule for adding a new link, either max-PR or min-PR, is actually the same as the random selecting rule, such that a_c remains to be nearly 0. With increasing σ, DCs with various sizes appear on the network. As stated above, these DCs are more likely to be formed by nodes with larger degrees. Consider now that a new link is added with the preference rules. For the max-PR case, a new link tends to join two big DCs, thus is more likely to join two nodes with large degrees. Consequently, nodes with large degrees are connected together and the network will become assortative. In contrast, when min-PR is applied, the new link, among the two candidates, will join two DCs with smaller-size product. Therefore, the new link starting from a node with large degree is more likely to end at a node with small degree, which will finally result in disassortativity. If the coupling strength is strong enough, a giant DC appear on the network. In this case, the influence of the PRs will be less remarkable and the absolute value of a_c will decrease as shown in Fig. 4.

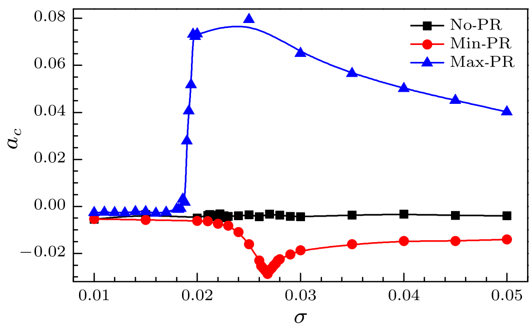

Fig. 4. Emergence of degree correlation on adaptive networks with PRs. When the coupling strength passes through the transition point, the min-PR rule leads to disassortativity and the max-PR to assortativity.

We have also investigated how the above results depend on the choices of other parameters. The parameter m measures the time scale of the network adaption. Our results show no quantitative difference given that $m \gg 1$ is well satisfied, e.g., $m > 100$. The number of removed or added links during each adaptive step, n, could be small or large: If n is small, the only difference is such that it takes longer time for the system to reach a stationary state. The frequency threshold value δ_0 to define a DC should be properly chosen. If δ_0 is too small, it is hard to find even one DC on the network. If it is too large, all the oscillators will be in the same DC and the PRs will not function. We have performed simulations for several δ_0 between 0.001 and 0.01 and found that the main conclusions of our work remain nearly the same.

In summary, we have studied the transition from incoherence to synchronization of coupled phase oscillators on adaptive networks. The network topology coevolves with the oscillator dynamics following the Achlioptas process based on dynamical clustering information. Each time a new link is added by max-PR or min-PR during the adaptive step. We find that both PRs can lead to explosive synchronization transition, which is different from the percolation transition reported by Achlioptas *et al.*[1] The difference is caused due to the fact that the dynamic property of coupled oscillators provides another competitive factor in our model. In the min-PR case, on the one hand, oscillators always tend to be phase locking near the transition point; on the other hand, the min-PR rule tries to desynchronize them. Contrarily, in the max-PR case, oscillators always tend to oscillate at their natural frequencies near the transition point but the max-PR rule tries to synchronize them. In addition, we find that max-PR (min-PR) can automatically lead to assortativity (disassortativity). As the dynamic on complex networks can exhibit many other types of dynamical phase transitions, e.g., oscillator death and coevolutionary networks are ubiquitous in nature. We believe our work may open more perspectives on the study of explosive phase transition on complex systems.

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