Noise-Sustained Spiral Waves: Effect of Spatial and Temporal Memory

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We have investigated the constructive roles of temporal noise, spatial disorder, and spatiotemporal fluctuation on the formation of spiral waves. Noise-sustained spiral waves are observed, the order of which passes a maximum with the increment of noise intensity, showing spatiotemporal coherent resonance. Moreover, we find that spatiotemporal fluctuation is much more favorable for the ordering process than temporal-noise only or spatial-disorder only. We demonstrate that this phenomenon shows the disadvantageous effect of spatial or temporal memory of noise.

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The constructive roles of noise in nonlinear systems have gained extensive attention in recent years. The most famous phenomena include stochastic resonance [1] and noise-induced transitions [2]. Recently, a new trend has been the study of noise-constructive effects in spatially extended systems, such as spatiotemporal stochastic resonance (STSR) [3], array-enhanced coherent resonance (AECR) [4], noise-enhanced signal propagation [5], noise-enhanced synchronization [6], noise-sustained wave propagation [7,8], etc.

In a spatially extended system, noise can exist in three different ways, i.e., time dependent, spatial dependent, and spatiotemporal dependent. We may call these three types of noise by temporal noise, spatial disorder, and spatiotemporal fluctuation, respectively. Consider a dynamic system evolving on a two-dimensional surface with a control parameter p subjected to noise. For the case of temporal noise, the surface is homogeneous with respect to p, but the value of p changes randomly with time; for spatial disorder, the surface is highly heterogeneous, but the surface remains static during the time evolution; for spatiotemporal fluctuation, however, the system evolves on a randomly changing heterogeneous surface. In the literature, it is shown that all three types of noise can play constructive roles. For temporal noise, examples include noise-induced pattern transitions [2] and AECR [4]. For spatial disorder, one finds that it can eliminate oscillator death [9], tame spatiotemporal chaos into order [10], and enhance the velocity of wave propagation [8]. For spatiotemporal fluctuation, many similar results have also been reported including STSR [3], noise-supported traveling waves [8], noise-sustained target waves and pulsating waves, noise-enhanced lifetime of scroll rings [11], etc.

Though temporal noise, spatial disorder, or spatiotemporal fluctuation show similar constructive effects, the relationship and difference among their mechanisms have not been clearly clarified yet. Actually, we may characterize these three types of noise by spatial or temporal *memory*. For spatial disorder, once the initial quenched disorder is introduced it will never change, i.e., the system has an eternal memory of the initial disorder. For temporal noise, the lattice is homogenous so that the system has a spatial memory of the noise on the whole surface. For spatiotemporal fluctuation, the system's dynamic features change randomly for each lattice site at each time step such that there is no spatial or temporal memory. A short temporal memory implies that the system's structure changes fast when the system evolves, and a small spatial memory means that the medium is highly heterogeneous. Since in real systems dynamic-induced structure change could be very important and spatial heterogeneity is rather ubiquitous [12], it is therefore a quite important issue to study the effect of spatial and temporal memory of the noise.

In this Letter, we investigated noise-sustained spiral waves using the three types of noises discussed above. We choose spiral waves because it is of ubiquitous importance in physical, chemical, or biological systems, including the famous Belousov-Zhabotinsky reaction [8] and cardiac tissues [13]. We find that, for properly chosen initial condition and control parameters, spiral waves cannot sustain in the absence of noise. However, when the control parameter is subjected to temporal noise, spatial disorder, or spatiotemporal fluctuation, spiral waves can survive. For all three cases, there is an optimal noise level where the noise-sustained spiral waves are most "ordered." A qualified measure, which may be termed as *weighted windowing spatial autocorrelation* (WWSAC), is introduced to characterize the order of the spiral waves. This measure shows a clear maximum with the increment of noise intensity, showing clear evidence for *spatiotemporal coherent resonance* (STCR), which is a counterpart of coherent resonance [14] or internal signal stochastic resonance [15] in spatially extended systems. It is interesting that spatiotemporal fluctuation is found to be much more constructive in the forming of ordered spiral waves than spatial-disorder only or temporal-noise only. This implies that both spatial memory and temporal memory are unfavorable for the noise-sustained spiral waves. This fact is further demonstrated by the investigation of WWSAC, which decreases monotonically with the increment of the spatial or temporal memory of the noise.

Our study is based on the well-known Barkley model shown below [16]:

$$\frac{\partial u}{\partial t} = \frac{1}{\varepsilon}u(1-u)\left(u - \frac{v+b}{a}\right) + D\nabla^2 u,$$

$$\frac{\partial v}{\partial t} = u - v.$$

Here variables u and v are activator and inhibitor species, a, b, and ε are parameters, and D is the diffusion coefficient of u. The parameter ε is typically small so that the time scale of u is much faster than that of v. For different parameter values, the medium can be excitable, oscillatory, or bistable such that the model can be used to model the pattern formation processes in many systems. In our study, we fix $\varepsilon = 1/300$, $b = 0.016$, $D = 1.0$, and choose a as the varying control parameter. Numerical simulation is performed on an $L \times L$ two-dimensional lattice with zero-flux boundary condition using the algorithm proposed by Barkley. We choose $L = 60$ (note that the choice of a larger lattice size does not change the qualitative results of the present work but requires more computer time). A specific initial condition is adopted as a seed for spiral waves: $u(i, j) = v(i+2, j) = 0.7$ for $L/2 - 2 \le i \le L/2 + 2$, $1 \le j \le L/2$, where $u(i, j)$ and $v(i, j)$ denote the values of u and v at lattice site (i, j), respectively. For the parameters and initial-boundary conditions chosen above, spiral waves can form in a wide range of parameter a. However, when a exceeds a certain critical value, no spiral tips can be formed and the system falls into the homogeneous stable state $u(i, j) = v(i, j) = 0$ for all the lattice sites. We numerically find that the critical value a_0 approximately equals 1.059. One should note that this critical value is sensible to the chosen parameters, initial conditions, and the lattice size. Noise is then added to the system through the modulation of the parameter a. For the case of temporal noise, parameter a randomly changes with time t but it is the same for all the lattice sites, i.e., $a_{ij}(t) = a_0 + \sigma\xi(t)$, where $a_{ij}(t)$ denotes the value of a on site (i, j) at time t, $\xi(t)$ is Gaussian white noise satisfying $\langle\xi(t)\rangle = 0$, $\langle\xi(t)\xi(t')\rangle = \delta(t - t')$, and σ is the noise intensity. For spatial disorder, a changes from site to site on the lattice but does not change with t, i.e., $a_{ij}(t) = a_0 + \sigma\xi_{ij}$ with $\langle\xi_{ij}\rangle = 0$ and $\langle\xi_{ij}\xi_{i'j'}\rangle = \delta_{ii'}\delta_{jj'}$. As to spatiotemporal fluctuation, we thus have $a_{ij}(t) = a_0 + \sigma\xi_{ij}(t)$, where $\xi_{ij}(t)$ satisfies $\langle\xi_{ij}(t)\rangle = 0$ and $\langle\xi_{ij}(t)\xi_{i'j'}(t')\rangle = \delta(t - t')\delta_{ii'}\delta_{jj'}$.

When noise is absent, as described above, no spiral waves can sustain for $a_0 = 1.059$. When a small amount of noise is added, we find that spiral waves can sustain only *occasionally.* When the noise level is large, noise-sustained wave propagation is observed but the spiral waves may break up into pieces. For an intermediate noise level, "good" spiral waves can sustain for a long time. In Fig. 1, some typical snapshots of noise-sustained spiral waves are shown for $\sigma = 0.05$ and $\sigma = 0.2$. All three types of noise show similar qualitative behaviors. The fact that there exists an optimal noise level for the noise-sustained spiral waves shows the characteristic of STCR.

To further demonstrate the characteristic of STCR, a qualified measure of the order of the noise-sustained waves rather than a qualitative description would be more helpful. In the present work, we introduce a measure termed as weighted windowing spatial autocorrelation. The calculation of the WWSAC for a given noise intensity contains three major steps. At first, after some transient time $t_{\rm trans}$, we need to calculate the order measure $C(t)$ of the snapshot on the lattice at each sampling time t. Then this order measure is averaged over a long time T to get the averaged order measure $C_r = \langle C(t)\rangle_t$ for this run. Finally, we repeat these two steps and average C_r over 50 different runs weighted by the lifetime of each run to get the WWSAC:

$$\mathrm{WWSAC} = \frac{1}{50T}\sum_{r=1}^{50} t_r C_r.$$

Here t_r is the lifetime of the rth run which is defined as the time the noise-sustained waves survive on the lattice after the transient time has passed. Note that, if the waves still survive after $T + t_{\rm trans}$, the lifetime will be recorded as T. In our simulation, we choose $t_{\rm trans}$ to be several cycles of the resulting spiral waves and $T = 80t_{\rm trans}$. However, in the first step, one encounters difficulty in defining such a single measure $C(t)$ to characterize the order of a snapshot at time t. Turning to Fig. 1, it is obvious that panels in the first row are more ordered than the corresponding panels in the second row, and they are also more ordered than the homogeneous state where no ordered pattern exists. One notes that the good

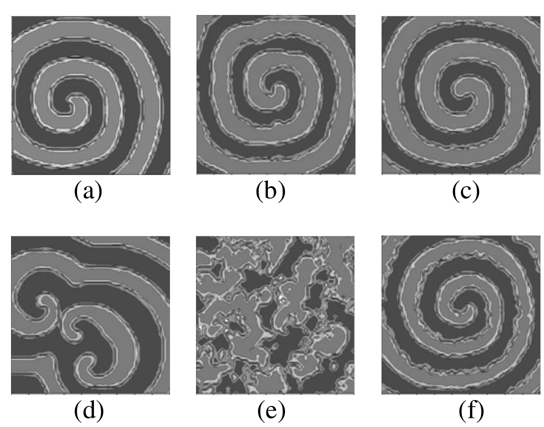


FIG. 1. Typical snapshots of noise-sustained spiral waves. The first row: $\sigma = 0.05$; the second row: $\sigma = 0.2$. (a),(d): temporal noise; (b),(e): spatial disorder; (c),(f): spatiotemporal fluctuation.

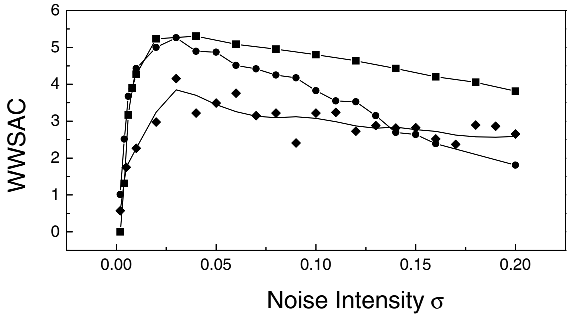


FIG. 2. Variation of WWSAC with noise intensity. Diamonds: temporal noise; circles: spatial disorder; solid squares: spatiotemporal fluctuation. The lines are drawn to guide the eyes.

spirals in panels 1(a)–1(c) are more periodic in space than those shown in panels 1(d) and 1(e). Therefore one might expect that a measure of the spatial periodicity of the snapshots could be used as a reasonable measure of its order. However, since the periodicity of panels 1(a)–1(c) exists only in the radial direction across the spiral core, which is near the center of the lattice, it makes sense to calculate the periodicity in a horizontal strip window across the lattice center. As in Ref. [14], we use the autocorrelation function to characterize the intensity of this periodicity. These considerations finally lead to the following definition of $C(t)$:

$$C(t) = \sum_{k=1}^{L-1} c^2(k, t),$$

$$\text{where } c(k, t) = \frac{\frac{1}{L}\sum_{i=1}^{L-k} \tilde{u}(i, t)\tilde{u}(i + k, t)}{\frac{1}{L}\sum_{i=1}^{L} \tilde{u}(i, t)^2},$$

$$\tilde{u}(i, t) = \frac{1}{L_w} \sum_{j=-(L_w-1)/2}^{j=(L_w-1)/2} u(i, L/2 + j) - \bar{u},$$

$\bar{u}$ is the average value of $u(i, j)$ inside the strip window $1 \leq i \leq L, L/2 - (L_w - 1)/2 \leq j \leq L/2 + (L_w - 1)/2$, L_w is the width of the window which is set to be 3 in the present work. One notes that the window width L_w cannot be too large since no periodicity exists for a horizontal window far from the lattice center.

For small noise level, there are lots of runs for which the lifetime t_r is 0 so that the WWSAC is small. For large noise level, the periodicity of the waves is lost and C_r is small for each run so that the WWSAC is also small. Therefore one expects the existence of an optimal noise level when the WWSAC reaches its maximum. Figure 2 depicts the dependence of the WWSAC with noise intensity for temporal noise, spatial disorder, and spatiotemporal fluctuation. All three curves show a clear maximum which demonstrates the occurrence of STCR.

In Fig. 1, one also notes that the spiral waves induced by spatiotemporal fluctuation as shown in panel 1(f) are more ordered than those induced by temporal noise in panel 1(d) or spatial disorder in panel 1(e) for the same noise intensity $\sigma = 0.2$. This effect is further confirmed in Fig. 2. As already discussed in the introductory paragraphs, this seems to implicate that introduction of spatial or temporal memory to the noise may be disadvantageous to the noise-sustained spiral waves as represented in the scheme shown in Fig. 3.

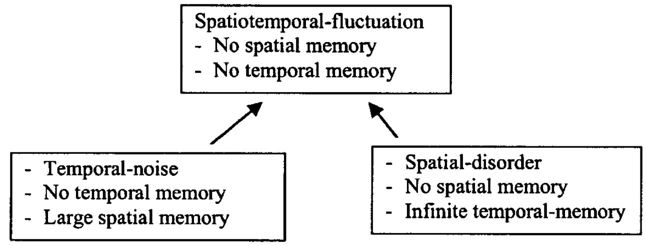


FIG. 3. Relationship between noise type and the order of noise-sustained spiral waves. The two arrows show the direction of increasing order with the decrement of spatial memory (left arrow) or temporal memory (right arrow).

To demonstrate this point, we can add spatial or temporal memory to the spatiotemporal fluctuation and study the effects. Note that, without any spatial or temporal memory, the noise item $\xi_{ij}(t)$ added to a_0 will refresh every time step for each lattice site. If a finite temporal memory m_t is added, the noise items on the lattice sites will refresh only at time km_t with k an integer. On the other hand, if a spatial memory m_s is introduced, the lattice will be divided into $(L/m_s)^2$ blocks. Inside each block the noise is the same, while different blocks have different noise intensity. By adding infinite m_t to spatiotemporal fluctuation, one recovers the case of spatial disorder; while for temporal noise, m_s equals the lattice size L. In Fig. 4, some typical snapshots for different spatial or temporal memory are shown. It is obvious

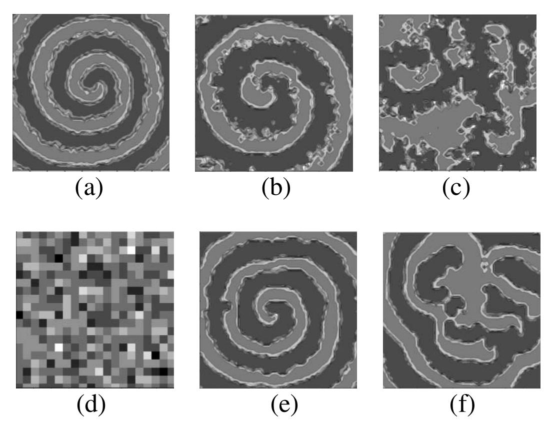
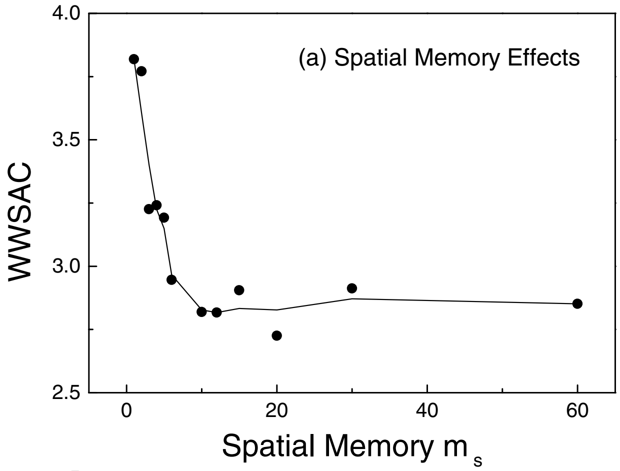


FIG. 4. Typical snapshots of noise-sustained spiral waves for different spatial or temporal memory for noise intensity 0.2. (a) The same as Fig. 1(f), $m_s = m_t = 0$; (b) $m_t = 5, m_s = 0$; (c) $m_t = 20, m_s = 0$; (d) an example lattice with spatial disorder memory $m_s = 3$; (e) $m_s = 3, m_t = 0$; (f) $m_s = 10, m_t = 0$.



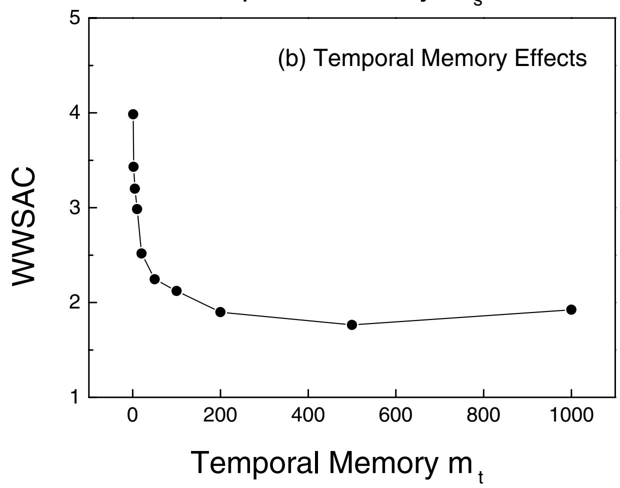


FIG. 5. Dependence of WWSAC on spatial or temporal memory. (a) Dependence on spatial memory with $m_t = 0$; (b) dependence on temporal memory with $m_s = 0$. Solid lines are drawn to guide the eyes. An increase of spatial or temporal memory leads to a sharp drop of the WWSAC.

that an increase of spatial or temporal memory will considerably reduce the order of noise-sustained spiral waves. This effect is further demonstrated by the sharp drop of WWSAC with m_s or m_t as shown in Fig. 5.

In conclusion, we have investigated the behavior of noise-sustained spiral waves using a two variable reaction diffusion model. We find that temporal noise, spatial disorder, and spatiotemporal fluctuation can all induce sustained spiral waves and there exists an optimal noise level when the noise-sustained spiral waves are most ordered. We introduce a qualified measure, WWSAC, to characterize the spatiotemporal order of the noisy waves. This measure shows a maximum at an intermediate noise level, showing the clear evidence of spatiotemporal coherent resonance. More interestingly, we find that spatiotemporal fluctuation is much more favorable for the ordering process than spatial-disorder only or temporal-noise only. This fact implies the disadvantageous effects of spatial or temporal memory.

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