Analysis of a class of spectral volume methods for linear scalar hyperbolic conservation laws

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Abstract: In the paper, we study the spectral volume (SV) methods for scalar hyperbolic conservation laws with a class of subdivision points under the Petrov-Galerkin framework. Due to the strong connection between the DG method and the SV method with the appropriate choice of the subdivision points, it is natural to analyze the SV method in the Galerkin form and derive the analogous theoretical results as in the DG method. This paper considers a class of SV methods, whose subdivision points are the zeros of a specific polynomial with a parameter in it. Properties of the piecewise constant functions under this subdivision, including the orthogonality between the trial solution space and test function space, are provided. With the aid of these properties, we are able to derive the energy stability, optimal a priori error estimates of SV methods with arbitrary high order accuracy. We also study the superconvergence of the numerical solution with the correction function technique, and show the order of superconvergence would be different with different choices of the subdivision points. In the numerical experiments, by choosing different parameters in the SV method, the theoretical findings are confirmed by the numerical results.

Key Words: Spectral volume method; Energy stability; Petrov-Galerkin method.

1 Introduction

Hyperbolic systems of conservation laws are a class of partial differential equations which arise in several areas of continuum physics, such as the description of the conservation of mass, momentum and energy in mechanical systems. The solutions to the hyperbolic conservation laws are often with low regularity and may even contain discontinuities, which is quite challenging to manipulate. As it is almost impossible to write down the explicit formula of the exact solution, the numerical approximation would be a natural choice to compute the solution. In the past several decades, numerical algorithms of conservation laws have been extensively investigated. One of the well-known methods is the Godunov method [18], which gives rise to many successive numerical methods, such as the monotonic upstream-centered scheme for conservation laws (MUSCL) [36, 37], generalized Riemann problems (GRP) [1, 42], total variation diminishing (TVD) methods [21], essentially non-oscillatory (ENO) [22] and weighted essentially non-oscillatory (WENO) [25], discontinuous

Galerkin (DG) methods [12, 11, 10, 9, 13], spectral volume/difference (SV/SD) methods [39, 26], flux reconstruction (FR) or correction procedure via reconstruction (CPR) [23, 40, 17], just to name a few.

Among various kinds of algorithms above, some of them have close connections, such as the DG methods, SV/SD methods and FR/CPR methods. For the linear problems, with careful choices of the solution points and flux reconstruction functions, the DG method can be viewed as a special case of the SV method, while the SV method can be incorporated in the FR/CPR framework. we refer the readers to [43, 34, 15, 28, 41] for more details. The SV method was first proposed in [39] to solve the hyperbolic conservation laws on unstructured grids, in which the numerical solution is represented by piecewise constant functions on the control volumes (CVs) in the finite volume (FV) manner, and it is more efficient than the traditional FV method in terms of both memory and CPU requirements. During the past two decades, the SV method has been used widely for a variety of problems such as shallow water equations, Navier-Stokes equations, advection-diffusion equation and electromagnetic field, see e.g. [7, 27, 30, 20, 14] and the references cited therein. Also, the mathematical theory for the SV method has been extensively investigated since it was born. In [43], Zhang and Shu performed Fourier type analysis on the SV method and obtained the stability, accuracy and convergence. In [31, 33, 32], Van den Abeele et al. analyzed the SV method in 1D and 2D by using dispersion and dissipation analysis, and by using the matrix method for the SV method on tetrahedral grids. It is worth noting that Abeele et al. pointed out that the SV method and the SD method are equivalent in 1D [34], thus the theory of the SD method could apply to the SV method at least in one dimension. Van den Abeele et al. studied the accuracy and stability of the SD method based on wave propagation analysis in [35]. Jameson gave a proof of the stability of the SD method in a Sobolev-type norm, provided that the interior flux points are the zeros of the corresponding Legendre polynomial [24]. More recently, Cao and Zou in [6] analyzed the SV method for 1D linear scalar hyperbolic equations and obtained the L^2 stability, error estimates and superconvergence for two kinds of interior flux points.

As mentioned before, there is a relation between the SV method and DG method, hence it is natural to expect some of the theoretical results in the DG method can be extended to the SV method. In fact, the authors in [6] studied the SV method for 1D linear scalar conservation laws with two specific distributions of subdivision points in the Galerkin form. In this paper, we consider the SV method with a broader class of the subdivision points, which are the zeros of a polynomial with a varying parameter in it, and show the energy stability and error estimates in the discontinuous Petrov-Galerkin (DPG) framework. Some review and development of DPG method can be found in e.g. [16, 3, 2]. By making use of the properties of the Legendre polynomials, we can show the distribution of the zeros of the given polynomial under a suitable range of the parameter, and the orthogonality of the basis functions between the trial solution space and test function space. We also show that the specific form of energy can be obtained *if and only if* the subdivision points are the zeros of the corresponding polynomial. Based on these facts, we are able to derive the energy stability, optimal a priori error estimates and superconvergence of the numerical solution. Note that when the parameter of the given polynomial is chosen appropriately, the energy norm is equal to the standard L^2 norm and the SV method reduces to the DG method, therefore all the theoretical analysis of DG method can apply. When the parameter varies, the energy norm is equivalent to the L^2 norm and our analysis shows that the order of the superconvergence will be different due to the exactness of the quadrature with the subdivision points as the quadrature points. To verify the theoretical findings, we conduct some numerical experiments to show the optimal error estimate, superconvergence with different choices of the subdivision and the polynomial degree of the trial

solution space.

The outline of the paper is organized as follows. In Section 2, we introduce the SV method for scalar conservation laws, and present one- and two-dimensional SV schemes as illustrating examples. In Section 3, we first show the subdivision points of the SV method, and some properties with these subdivision points in several lemmas. We then derive the energy stability and error estimates for linear scalar conservation laws in one and two dimensions with the help of these properties. Particularly, we show that the order of superconvergence will be one order higher only with a specific choice of the subdivision points. In Section 4, we present some numerical tests with various choices of the subdivision points and the polynomial degree of trial solution space. The concluding remarks are given in Section 5.

2   Spectral Volume Methods for Scalar Conservation Laws

Let us consider the scalar hyperbolic conservation laws as follows:

$$
\begin{cases}
u_t + \operatorname{div} \boldsymbol{f}(u) = 0, & (\boldsymbol{x}, t) \in \Omega \times (0, T], \\
u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), & \boldsymbol{x} \in \Omega,
\end{cases}
\tag{2.1}
$$

with periodic or compactly supported boundary conditions. The domain $\Omega \subset \mathbb{R}^d$, $\boldsymbol{x} = (x_1, x_2, \ldots, x_d)^T$ and $\boldsymbol{f}(u) = \big(f_1(u), f_2(u), \ldots, f_d(u)\big)^T$. Assume the partition of the computational domain is $\Omega = \cup_i T_i$, and each element T_i is divided into subcells named control volumes (CVs), denoted by $C_{i,\ell}$. Integrate (2.1) on $C_{i,\ell}$, with the divergence theorem we can obtain

$$
\int_{C_{i,\ell}} u_t \, \mathrm{d}\boldsymbol{x} + \int_{C_{i,\ell}} \operatorname{div} \boldsymbol{f}(u) \, \mathrm{d}\boldsymbol{x} = \int_{C_{i,\ell}} u_t \, \mathrm{d}\boldsymbol{x} + \oint_{\partial C_{i,\ell}} \boldsymbol{n} \cdot \boldsymbol{f}(u) \, \mathrm{d}S = 0,
\tag{2.2}
$$

where $\boldsymbol{n}$ is the unit outward normal of the boundary $\partial C_{i,\ell}$ of the CV $C_{i,\ell}$. Now we define the cell-averaged state variables as

$$
\overline{u}_{i,\ell} = \frac{1}{|C_{i,\ell}|} \int_{C_{i,\ell}} u(\boldsymbol{x}, t) \, \mathrm{d}\boldsymbol{x}
$$

where $|C_{i,\ell}|$ is the volume of $C_{i,\ell}$, then (2.2) becomes

$$
\frac{\mathrm{d}}{\mathrm{d}t} \overline{u}_{i,\ell} + \frac{1}{|C_{i,\ell}|} \oint_{\partial C_{i,\ell}} \boldsymbol{n} \cdot \boldsymbol{f}(u) \, \mathrm{d}S = 0.
$$

The spectral volume (SV) scheme for (2.1) is defined as follows: Seek $u_h(\cdot, t) \in V_h$ such that

$$
\frac{\mathrm{d}}{\mathrm{d}t} (\overline{u_h})_{i,\ell} + \frac{1}{|C_{i,\ell}|} \oint_{\partial C_{i,\ell}} \widehat{\boldsymbol{n} \cdot \boldsymbol{f}}(u_h) \, \mathrm{d}S = 0, \quad \forall i, \ell,
\tag{2.3}
$$

where $(\overline{u_h})_{i,\ell}$ is the cell average of u_h on $C_{i,\ell}$, and $\widehat{\boldsymbol{n} \cdot \boldsymbol{f}}(u_h)$ is taken as the monotone flux along the interface $\partial C_{i,\ell}$. Note that the trial space V_h consists of piecewise smooth polynomials on each element T_i. The number of subcells $C_{i,\ell}$ in each cell T_i is equal to the degree of freedom (DoF) of the polynomials on T_i.

For the convenience of analysis, we rewrite the SV scheme (2.3) in the framework of the discontinuous Petrov-Galerkin (DPG) method. To this end, we define the test function space as

$$
W_h := \big\{ w_h \in L^2(\Omega) : w_h|_{C_{i,\ell}} \in P^0(C_{i,\ell}), \ \forall i, \ell \big\}.
$$

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The SV scheme (2.3) is equivalent to

$$
\begin{aligned}
0 &= \int_{T_i} \partial_t u_h \cdot \chi_{C_{i,\ell}} \, \mathrm{d}\boldsymbol{x} + \oint_{\partial C_{i,\ell}} \widehat{\boldsymbol{n} \cdot \boldsymbol{f}}(u_h) \, \mathrm{d}S \\
&= \int_{T_i} \left(\partial_t u_h + \operatorname{div} \boldsymbol{f}(u_h) \right) \chi_{C_{i,\ell}} \, \mathrm{d}\boldsymbol{x} + \oint_{\partial C_{i,\ell}} \left(\widehat{\boldsymbol{n} \cdot \boldsymbol{f}}(u_h) - \boldsymbol{n} \cdot \boldsymbol{f}(u_h) \right) \mathrm{d}S,
\end{aligned}
\tag{2.4}
$$

where $\chi_{C_{i,\ell}}(\boldsymbol{x})$ is the characteristic function on the control volume $C_{i,\ell}$, and $\boldsymbol{f}(u_h)|_{\partial C_{i,\ell}}$ is the one-side limit from inside of $C_{i,\ell}$. Therefore, for any function $w_h = \sum_i (w_h)_{i,\ell} \chi_{C_{i,\ell}} \in W_h$ where $(w_h)_{i,\ell}$ are constants, then from (2.4) we have

$$
\begin{aligned}
&\int_{T_i} \left(\partial_t u_h + \operatorname{div} \boldsymbol{f}(u_h) \right) w_h \, \mathrm{d}\boldsymbol{x} + \oint_{\partial T_i} \left(\widehat{\boldsymbol{n} \cdot \boldsymbol{f}}(u_h) - \boldsymbol{n} \cdot \boldsymbol{f}(u_h) \right) w_h \, \mathrm{d}S \\
&= \int_{T_i} \left(\partial_t u_h + \operatorname{div} \boldsymbol{f}(u_h) \right) \sum_\ell (w_h)_{i,\ell} \chi_{C_{i,\ell}} \, \mathrm{d}\boldsymbol{x} + \sum_\ell \oint_{\partial C_{i,\ell}} \left(\widehat{\boldsymbol{n} \cdot \boldsymbol{f}}(u_h) - \boldsymbol{n} \cdot \boldsymbol{f}(u_h) \right) w_h \, \mathrm{d}S \\
&= \sum_\ell (w_h)_{i,j} \left[\int_{T_i} \left(\partial_t u_h + \operatorname{div} \boldsymbol{f}(u_h) \right) \chi_{C_{i,\ell}} \, \mathrm{d}\boldsymbol{x} + \oint_{\partial C_{i,\ell}} \left(\widehat{\boldsymbol{n} \cdot \boldsymbol{f}}(u_h) - \boldsymbol{n} \cdot \boldsymbol{f}(u_h) \right) \mathrm{d}S \right] = 0.
\end{aligned}
\tag{2.5}
$$

Here we use the fact $\widehat{\boldsymbol{n} \cdot \boldsymbol{f}}(u_h) = \boldsymbol{n} \cdot \boldsymbol{f}(u_h)$ at the subcell interfaces $\cup_\ell \partial C_{i,\ell} \backslash \partial T_i$ in the first equality of (2.5) due to the continuity of V_h inside T_i. Now we define the SV scheme (2.3) in the DPG form: Seek $u_h(\cdot, t) \in V_h$ such that

$$
\int_{T_i} \left(\partial_t u_h + \operatorname{div} \boldsymbol{f}(u_h) \right) w_h \, \mathrm{d}\boldsymbol{x} + \oint_{\partial T_i} \left(\widehat{\boldsymbol{n} \cdot \boldsymbol{f}}(u_h) - \boldsymbol{n} \cdot \boldsymbol{f}(u_h) \right) w_h \, \mathrm{d}S = 0, \quad \forall \, w_h \in W_h.
\tag{2.6}
$$

Note that the dimension of the trial solution space V_h should be matched with that of the test function space W_h so that the scheme (2.6) is well-defined.

From [6], we know the SV methods are quite different if we choose different subdivisions $T_i = \cup_\ell C_{i,\ell}$. For the illustration purpose, we present some examples of SV methods in one and two dimensions in the following.

2.1 Spectral Volume Scheme for 1D Hyperbolic Conservation Laws

Consider the hyperbolic conservation law (2.1) in one dimension $(d = 1)$

$$
u_t + f(u)_x = 0, \qquad (x, t) \in \Omega \times (0, T],
\tag{2.7}
$$

Take partition of $\Omega = (a, b)$ into N cells and we have

$$
\begin{aligned}
a &= x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = b, \quad h = \max_i h_i, \\
I_i &= \left(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}} \right), \quad h_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}, \quad x_i = \frac{1}{2} \left(x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}} \right).
\end{aligned}
\tag{2.8}
$$

And each cell I_i is divided into $k + 1$ CVs that

$$
\begin{aligned}
x_{i-\frac{1}{2}} &= x_{i,0} < x_{i,1} < \cdots < x_{i,k+1} = x_{i+\frac{1}{2}}, \\
I_{i,\ell} &= \left(x_{i,\ell}, x_{i,\ell+1} \right), \quad h_{i,\ell} = x_{i,\ell+1} - x_{i,\ell}, \quad 0 \le \ell \le k.
\end{aligned}
\tag{2.9}
$$

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In the SV scheme (2.6), we take the trial solution space V_h as follows:

$$V_h = V_h^k := \{v_h \in L^2(a,b) : v_h|_{I_i} \in P^k(I_i), \ i = 1, \dots, N\}, \tag{2.10}$$

where $P^k(I_i)$ denotes the set of all polynomials of degree at most k on I_i. And the test function space W_h is given as follows:

$$W_h = W_h^k := \{w \in L^2(a,b) : w|_{I_{i,\ell}} \in P^0(I_{i,\ell}), \ \forall i, \ell\}, \tag{2.11}$$

i.e., the test function space is the collection of piecewise constant functions. The semi-discrete SV scheme (2.6) becomes: Seek $u_h(\cdot, t) \in V_h^k$ such that

$$\int_{I_i} \big(\partial_t u_h + \partial_x f(u_h)\big) w_h \, dx + \Big(\widehat{f}_{i+\frac{1}{2}} - f\big((u_h)^-_{i+\frac{1}{2}}\big)\Big)(w_h)^-_{i+\frac{1}{2}}$$
$$- \Big(\widehat{f}_{i-\frac{1}{2}} - f\big((u_h)^+_{i-\frac{1}{2}}\big)\Big)(w_h)^+_{i-\frac{1}{2}} = 0, \quad \forall\, w_h \in W_h^k, \tag{2.12}$$

where $\widehat{f}_{i+\frac{1}{2}} = \widehat{f}\big((u_h)^-_{i+\frac{1}{2}}, (u_h)^+_{i+\frac{1}{2}}\big)$ is the monotone flux and $(u_h)^\pm_{i+\frac{1}{2}} = u_h\big(x^\pm_{i+\frac{1}{2}}, t\big)$ are the left and right limits of u_h at $x = x_{i+\frac{1}{2}}$. The SV scheme (2.12) can also be simplified into the following form:

$$\int_{I_{i,\ell}} (\partial_t u_h) w_h \, dx + \widehat{f}_{i,\ell+1}(w_h)^-_{i,\ell+1} - \widehat{f}_{i,\ell}(w_h)^+_{i,\ell} = 0, \quad \forall\, w_h \in W_h^k, \forall\, 1 \le i \le N, 0 \le \ell \le k, \tag{2.13}$$

where $(w_h)^-_{i,\ell}$ (resp. $(w_h)^+_{i,\ell}$) is the left (resp. right) limit of the discontinuous function w_h at the subdivision point $x = x_{i,\ell}$, i.e. $(w_h)^\pm_{i,\ell} = w_h\big(x^\pm_{i,\ell}\big)$, and the numerical flux $\widehat{f}_{i,\ell} = \widehat{f}\big((u_h)^-_{i,\ell}, (u_h)^+_{i,\ell}\big)$ with $\widehat{f}_{i+\frac{1}{2}} = \widehat{f}_{i,k+1} = \widehat{f}_{i+1,0}$. Since u_h stays smooth inside I_i, then by the consistency of the monotone flux, we have $\widehat{f}$ is continuous at the subcell interfaces inside the cell I_i, i.e. $\widehat{f}_{i,\ell} = \widehat{f}\big((u_h)^-_{i,\ell}, (u_h)^+_{i,\ell}\big) = f(u_h(x_{i,\ell}, t))$ for $\ell = 1, \dots, k$.

2.2 Spectral Volume Scheme for 2D Hyperbolic Conservation Laws

Now we consider two-dimensional conservation law (2.1) ($d = 2$) in the following:

$$u_t + f(u)_x + g(u)_y = 0, \qquad (x, y, t) \in \Omega \times (0, T], \tag{2.14}$$

on the squared domain $\Omega = (a, b) \times (c, d)$. For simplicity, we take the Cartesian grid as the partition of Ω, i.e.

$$\Omega = \cup_{i,j} K_{i,j}, \quad K_{i,j} = I_i \times J_j, \quad h = \max(h_i^x, h_j^y),$$
$$I_i = \big(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\big), \quad h_i^x = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}, \quad x_i = \frac{1}{2}\big(x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}\big),$$
$$J_j = \big(y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}\big), \quad h_j^y = y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}} \quad y_j = \frac{1}{2}\big(y_{j-\frac{1}{2}} + y_{j+\frac{1}{2}}\big). \tag{2.15}$$

We then continue to take the Cartesian grids inside the element $K_{i,j}$, with $k + 2$ subdivision points in the x-direction and y-direction respectively, then we have $(k+1)^2$ subcells on each element $K_{i,j}$, denoted as $K_{i,j,\ell,m}$ in the following.

$$K_{i,j,\ell,m} = I_{i,\ell} \times J_{j,m}, \quad I_{i,\ell} = \big(x_{i,\ell}, x_{i,\ell+1}\big), \quad J_{j,m} = \big(y_{j,m}, y_{j,m+1}\big), \quad 0 \le \ell, m \le k,$$
$$x_{i-\frac{1}{2}} = x_{i,0} < x_{i,1} < \cdots < x_{i,k+1} = x_{i+\frac{1}{2}}, \quad y_{j-\frac{1}{2}} = y_{j,0} < y_{j,1} < \cdots < y_{j,k+1} = y_{j+\frac{1}{2}}. \tag{2.16}$$

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We consider the trial solution space as the tensor product of V_h defined in (2.10), as well as the test function space. The SV scheme (2.6) for (2.14) is presented as follows: Seek $u_h(\cdot, t) \in V_h^k \times V_h^k$ such that $\forall i, j$ and $\forall w_h \in W_h^k \times W_h^k$, it holds that

$$\int_{K_{i,j}} \big(\partial_t u_h + \partial_x f(u_h) + \partial_y g(u_h)\big) w_h \, \mathrm{d}x \mathrm{d}y$$
$$+ \int_{J_j} \Big(\widehat{f} - f(u_h)\Big) w_h \bigg|_{\left(x_{i-\frac{1}{2}}^+, y\right)}^{\left(x_{i+\frac{1}{2}}^-, y\right)} \mathrm{d}y + \int_{I_i} \Big(\widehat{g} - g(u_h)\Big) w_h \bigg|_{\left(x, y_{j-\frac{1}{2}}^+\right)}^{\left(x, y_{j+\frac{1}{2}}^-\right)} \mathrm{d}x = 0, \quad (2.17)$$

where $\widehat{f}$ and $\widehat{g}$ are the monotone numerical fluxes defined on the element interfaces. Similarly, the SV scheme (2.17) can be rewritten as follows: $\forall i, j$ and $\forall w_h \in W_h^k \times W_h^k$, seek $u_h(\cdot, t) \in V_h^k \times V_h^k$ such that

$$\int_{K_{i,j,\ell,m}} (\partial_t u_h) w_h \, \mathrm{d}x \mathrm{d}y + \int_{J_{j,m}} \widehat{f} \, w_h \bigg|_{\left(x_{i,\ell}^+, y\right)}^{\left(x_{i,\ell+1}^-, y\right)} \mathrm{d}y + \int_{I_{i,\ell}} \widehat{g} \, w_h \bigg|_{\left(x, y_{j,m}^+\right)}^{\left(x, y_{j,m+1}^-\right)} \mathrm{d}x = 0, \quad \forall\, 0 \leq \ell, m \leq k, \quad (2.18)$$

with the internal fluxes are given as

$$\widehat{f}\bigg|_{\left(x_{i,\ell}^-, y\right)} = \widehat{f}\bigg|_{\left(x_{i,\ell}^+, y\right)} = f\big(u_h(x_{i,\ell}, y, t)\big), \quad 1 \leq \ell \leq k,$$
$$\widehat{g}\bigg|_{\left(x, y_{j,m}^-\right)} = \widehat{g}\bigg|_{\left(x, y_{j,m}^+\right)} = g\big(u_h(x, y_{j,m}, t)\big), \quad 1 \leq m \leq k.$$

3 Analysis of the Spectral Volume Methods

In this section, we will study a class of the SV schemes (2.6) for linear scalar conservation law, i.e. $\boldsymbol{f}(u) = \boldsymbol{a} \cdot \nabla u$ in (2.1) with a constant vector $\boldsymbol{a} = (a_1, \cdots, a_d)^T \in \mathbb{R}^d$. Without loss of generality, we take $\boldsymbol{a} = (1, \cdots, 1)^T$. Throughout the paper, we take the notation $A \lesssim B$ which means that there exists a constant $c_0 > 0$ independent of h such that $A \leq c_0 B$. With a specific setting of the subdivision points, we are able to derive the L^2 boundedness and a priori error estimates of the numerical solution in the semi-discrete analysis. In particular, we adopt the so-called *correction function* technique in [5, 6] to deduce the superconvergence of the numerical solution. Before we proceed, let us first define a class of subdivision points which lays the foundation for everything that follows.

3.1 Preliminaries

Let us consider the distribution of the subdivision points $\{x_{i,\ell}\}_{\ell=0}^{k+1}$ defined in (2.9) on the element I_i in one dimension. By a linear transformation $\alpha_\ell = 2(x_{i,\ell} - x_i)/h_i$, we have transformed the subdivision points to $\{\alpha_\ell\}_{\ell=0}^{k+1}$ on the reference element $[-1, 1]$ with $-1 = \alpha_0 < \alpha_1 < \ldots < \alpha_{k+1} = 1$.

Now we define the space of piecewise constant functions on $[-1, 1]$ that

$$W := \{w \in L^2([-1,1]) \,\big|\, w|_{(\alpha_\ell, \alpha_{\ell+1})} \in P^0, \ell = 0, \ldots, k\}. \quad (3.1)$$

And then we define a projection $\mathbb{P} : P^k([-1,1]) \to W$ such that for any given function $p(x) \in P^k([-1,1])$, we have $\mathbb{P}p \in W$ which satisfies

$$\begin{cases} \displaystyle\int_{-1}^{1} (p - \mathbb{P}p)q \, \mathrm{d}x = 0, & \forall q \in P^{k-1}([-1,1]), \\ \mathbb{P}p(-1) = p(-1), \end{cases} \tag{3.2}$$

where P^m is the set of polynomials whose degree is at most m. Now we take a series of piecewise constant functions $Q_m \in W$ that

$$Q_m(x) = \mathbb{P}L_m(x), \quad m = 0, \ldots, k, \tag{3.3}$$

and $L_m(x)$ is the Legendre polynomial of degree m on $[-1,1]$. Some properties and formulas of the Legendre polynomials can be found in Appendix A.1. With the definition of the projection $\mathbb{P}$, we immediately have $Q_m(-1) = L_m(-1) = (-1)^m$. Moreover, we have the following lemma.

Lemma 3.1. *The following relations*

$$\begin{cases} \displaystyle\int_{-1}^{1} L_k(x)Q_m(x) \, \mathrm{d}x = 0, & m = 0, \ldots, k-1, \\ \displaystyle\int_{-1}^{1} L_k(x)Q_k(x) \, \mathrm{d}x = 2C_k, & C_k > 0. \end{cases} \tag{3.4}$$

hold if and only if the subdivision points $\{\alpha_\ell\}_{\ell=1}^{k}$ are the zeros of the polynomial

$$R_k(x) = L_k(x) + c(x+1)L_k'(x), \quad c > -\frac{1}{k(k+1)} \tag{3.5}$$

with $C_k = \dfrac{1}{(1+k)(1+ck)}$.

To prove this lemma, we first need to make sure the zeros of the polynomial $R_k(x)$ are within $(-1,1)$, which is ensured by the next lemma. The proof of Lemma 3.1 is given in Appendix A.2.

Lemma 3.2. *Under the condition $c > -\dfrac{1}{k(k+1)}$, the given polynomial $R_k(x)$ (3.5) has k distinct zeros, and all of them are located within $(-1,1)$.*

The proof of Lemma 3.2 is given in Appendix A.3.

In Figure 1, we plot the function $R_k(x)$ with $k = 1, 2, 3, 4$ for various values of c. We can see that when $c > -\frac{1}{k(k+1)}$, the function $R_k(x)$ does have k distinct solutions in $(-1,1)$, coinciding with our analysis.

Combining (3.2) and (3.4) together, we can see the Legendre polynomials $L_m(x)$ and the piecewise constant functions $Q_m(x) = \mathbb{P}L_m(x)$, $m = 0, \ldots, k$ satisfy the following conditions:

$$\begin{cases} \displaystyle\int_{-1}^{1} L_\ell(x)Q_m(x) \, \mathrm{d}x = \delta_{\ell m}\frac{2}{2\ell+1}, & 0 \le \ell, m \le k, \ (\ell, m) \neq (k, k), \\ \displaystyle\int_{-1}^{1} L_k(x)Q_k(x) \, \mathrm{d}x = \frac{2}{(1+k)(1+ck)}, \\ Q_\ell(-1) = L_\ell(-1) = (-1)^\ell, & \ell = 0, \ldots, k. \end{cases} \tag{3.6}$$

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(a) $k = 1$.

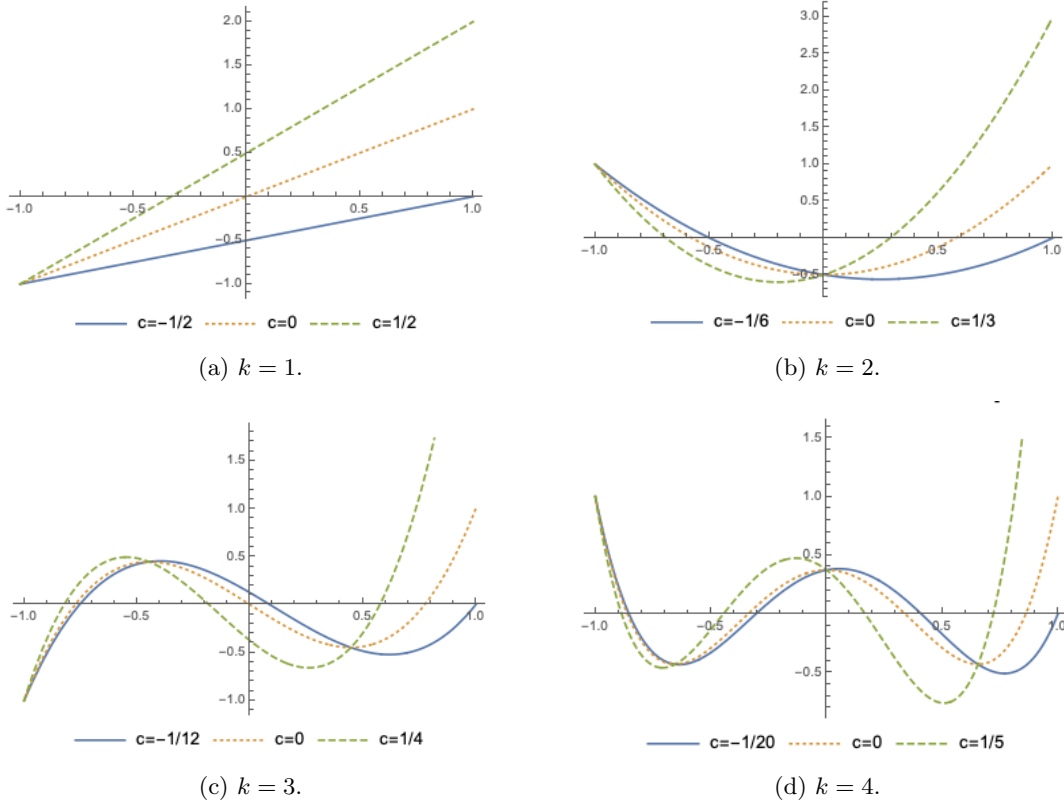(b) $k = 2$.

(c) $k = 3$.

(d) $k = 4$.

Figure 1: Plots of the function $R_k(x)$ for various values of c. Solid line, $c = -\frac{1}{k(k+1)}$; dotted line, $c = 0$; dashed line, $c = \frac{1}{k+1}$.

As we shall see later, these conditions play a crucial role in the deduction of the energy-boundedness of the SV scheme (2.6).

Now let us give the definition of the subdivision points as follows.

Definition 3.3. *The subdivision is called an admissible subdivision if the transformed subdivision points $\{\alpha_\ell\}_{\ell=1}^{k}$ are taken as the zeros of the polynomial $R_k(x)$ defined in (3.5).*

3.2 Spectral Volume Method for 1D Linear Scalar Conservation Law

In this subsection, we will consider the spectral volume (SV) method with an admissible subdivision for the one-dimensional linear scalar hyperbolic conservation law. We then study the properties of the numerical scheme, including energy-boundedness, optimal error estimates and superconvergence result.

The SV scheme (2.12) for the linear scalar hyperbolic conservation law $u_t + u_x = 0$ is

$$\int_{I_i} (\partial_t u_h) w_h \, \mathrm{d}x + B_i(u_h, w_h) = 0, \quad \forall \, w_h \in W_h^k, \, i = 1, \dots, N, \tag{3.7}$$

with the upwind numerical flux $(\widehat{u_h})_{i+\frac{1}{2}} = (u_h)_{i+\frac{1}{2}}^-$ in (2.12) and

$$B_i(u_h, w_h) = \int_{I_i} (\partial_x u_h) w_h \, \mathrm{d}x + [\![u_h]\!]_{i-\frac{1}{2}} (w_h)_{i-\frac{1}{2}}^+, \tag{3.8}$$

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and $[\![v]\!]_{i-\frac{1}{2}} = v\big(x^+_{i-\frac{1}{2}}\big) - v\big(x^-_{i-\frac{1}{2}}\big)$ denotes the jump of v at the element interface $x = x_{i-\frac{1}{2}}$.

Take $w_h = 1$ in (3.7) and sum it over i, we immediately obtain the conservation of u_h as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_a^b u_h(x,t)\,\mathrm{d}x = 0. \tag{3.9}$$

3.2.1 Energy-boundedness

Now let us proceed to obtain the stability of the SV scheme (3.7). Define a projection $\mathbb{P}_h : V_h \to W_h$ on each cell I_i, such that for a given function $v_h \in V_h$, $\mathbb{P}_h v_h \in W_h$ satisfies

$$\begin{cases} \displaystyle\int_{I_i} (v_h - \mathbb{P}_h v_h) r_h \,\mathrm{d}x = 0, \quad \forall\, r_h \in P^{k-1}(I_i), \\[2mm] \big(v_h - \mathbb{P}_h v_h\big)^+_{i-\frac{1}{2}} = 0, \end{cases} \tag{3.10}$$

which is similar to the projection $\mathbb{P}$ defined in (3.2), thus the results in the previous subsection can be applied directly. Now we take the test function $w_h = \widetilde{u_h} := \mathbb{P}_h u_h$ in (3.7) and sum it over i, we then obtain

$$\sum_i \int_{I_i} (\partial_t u_h)\widetilde{u_h}\,\mathrm{d}x + \sum_i B_i(u_h, \widetilde{u_h}) = 0. \tag{3.11}$$

By using the definition of the projection $\mathbb{P}_h$ in (3.10) and the periodic or compactly supported boundary conditions, we can obtain

$$\begin{aligned} \sum_i B_i(u_h, \widetilde{u_h}) &= \sum_i B_i(u_h, \widetilde{u_h} - u_h) + \sum_i B_i(u_h, u_h) \\ &= \sum_i \left(\int_{I_i} (\partial_x u_h) u_h \,\mathrm{d}x + [\![u_h]\!]_{i-\frac{1}{2}}(u_h)^+_{i-\frac{1}{2}} \right) \quad \text{(by (3.8) and (3.10))} \\ &= \sum_i \frac{1}{2}\left(\big((u_h)^-_{i+\frac{1}{2}}\big)^2 - \big((u_h)^+_{i-\frac{1}{2}}\big)^2 + 2[\![u_h]\!]_{i-\frac{1}{2}}(u_h)^+_{i-\frac{1}{2}} \right) \\ &= \sum_i \frac{1}{2}\left(\big((u_h)^-_{i+\frac{1}{2}}\big)^2 + \big((u_h)^+_{i+\frac{1}{2}}\big)^2 - 2(u_h)^-_{i+\frac{1}{2}}(u_h)^+_{i+\frac{1}{2}} \right) \\ &= \sum_i \frac{1}{2}[\![u_h]\!]^2_{i+\frac{1}{2}} \geq 0. \end{aligned} \tag{3.12}$$

We take the Legendre polynomials $\{\phi_{i,\ell}\}_{\ell=0}^k$ as the basis functions of V_h on I_i:

$$\phi_{i,\ell}(x) = L_\ell\left(\frac{2(x - x_i)}{h_i}\right), \quad \ell = 0, \dots, k, \ x \in I_i, \tag{3.13}$$

where $L_\ell(x)$ is the Legendre polynomials defined on $[-1, 1]$ of degree ℓ. And we assume the numerical solution u_h has the following form

$$u_h(x,t) = \sum_i \sum_{\ell=0}^k u_{i,\ell}(t)\, \phi_{i,\ell}(x). \tag{3.14}$$

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We also take

$$\varphi_{i,\ell}(x) = \mathbb{P}_h \phi_{i,\ell}(x) \in W_h^k, \quad \ell = 0, \dots, k. \tag{3.15}$$

With (3.6) and (3.10), we have that

$$
\begin{cases}
\displaystyle\int_{I_i} \phi_{i,\ell}(x)\varphi_{i,m}(x)\,\mathrm{d}x = \delta_{\ell m}\frac{h_i}{2\ell+1}, & 0 \le \ell, m \le k,\ (\ell, m) \ne (k, k), \\[3mm]
\displaystyle\int_{I_i} \phi_{i,k}(x)\varphi_{i,k}(x)\,\mathrm{d}x = \frac{h_i}{(1+k)(1+ck)}, & \\[3mm]
\varphi_{i,\ell}(x_{i-1/2}) = (-1)^\ell, & \ell = 0, \dots, k.
\end{cases} \tag{3.16}
$$

On the other hand, from the definition of $\widetilde{u_h}$, we can obtain

$$\widetilde{u_h} = \mathbb{P}_h u_h = \sum_i \sum_{\ell=0}^k u_{i,\ell}\varphi_{i,\ell}(x). \tag{3.17}$$

Therefore,

$$
\begin{aligned}
\int_{I_i} (\partial_t u_h)\widetilde{u_h}\,\mathrm{d}x &= \int_{I_i}\left(\sum_{\ell=0}^k (u_{i,\ell})_t\,\phi_{i,\ell}(x)\right)\left(\sum_{m=0}^k u_{i,m}\,\varphi_{i,m}(x)\right)\mathrm{d}x \\
&= \sum_{\ell=0}^k \sum_{m=0}^k (u_{i,\ell})_t u_{i,m} \int_{I_i} \phi_{i,\ell}(x)\varphi_{i,m}(x)\,\mathrm{d}x \\
&= \sum_{\ell=0}^k (u_{i,\ell})_t u_{i,\ell} \int_{I_i} \phi_{i,\ell}(x)\varphi_{i,\ell}(x)\,\mathrm{d}x \\
&= \frac{h_i}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{\ell=0}^{k-1} \frac{1}{2\ell+1}(u_{i,\ell})^2 + \frac{1}{(1+k)(1+ck)}(u_{i,k})^2\right).
\end{aligned} \tag{3.18}
$$

Define the energy function $E(u_h)$ as

$$E(u_h) = \sum_i h_i\left(\sum_{\ell=0}^{k-1}\frac{1}{2\ell+1}(u_{i,\ell})^2 + \frac{1}{(1+k)(1+ck)}(u_{i,k})^2\right). \tag{3.19}$$

Then, plugging (3.18) and (3.12) into (3.11), we can obtain the energy-boundedness of u_h, i.e.

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E(u_h) = -\sum_i \frac{1}{2}[\![u_h]\!]_{i+\frac{1}{2}}^2 \le 0.$$

Remark 3.4. *Due to the fact that the coefficient of x^k in the Legendre polynomial L_k is $\dfrac{(2k-1)!!}{k!} = \dfrac{(2k)!}{2^k(k!)^2}$, the energy $E(u_h)$ can be written as*

$$E(u_h) = \sum_i \int_{I_i}\left((u_h)^2 + \beta\left(\partial_x^k u_h\right)^2\right)\mathrm{d}x$$

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where β is given as

$$\beta = \left(\frac{h_i}{2}\right)^{2k}\left(\frac{1}{(2k-1)!!}\right)^2\left(\frac{1}{(1+k)(1+ck)} - \frac{1}{2k+1}\right). \qquad (3.20)$$

This form of energy can also be found in the VCJH scheme [38]. In fact, the VCJH scheme is similar to the scheme proposed here although they appear in different forms. The decay of the energy function $E(\cdot)$ with respect to time shows the SV scheme (3.7) is energy stable. In fact, the SV scheme is stable as well as in the L^2 norm because of the equivalence between $\sqrt{E(\cdot)}$ and the L^2 norm $\|\cdot\|$ in V_h.

Remark 3.5. *With the suitable choices of c in (3.5), there are two special cases of the polynomial $R_k(x)$, which have been studied in [6].*

1. *When $c = 0$, $R_k(x) = L_k(x)$ and $\{\alpha_m\}_{m=1}^k$ are the Gauss-Legendre quadrature nodes. In this case,*

$$E(u_h) = \sum_i h_i\left(\sum_{\ell=0}^{k-1}\frac{1}{2\ell+1}(u_{i,\ell})^2 + \frac{1}{k+1}(u_{i,k})^2\right). \qquad (3.21)$$

2. *When $c = \dfrac{1}{k+1}$, we claim that $\{\alpha_m\}_{m=1}^k$ are the right Gauss-Radau quadrature nodes. To verify this claim, we need to show the relation between $R_k(x)$ and $L_{k+1}(x) - L_k(x)$. By the properties of Legendre polynomials in Appendix A.1, we have*

$$\begin{aligned}
R_k(x) &= L_k(x) + \frac{1}{k+1}(x+1)L_k'(x)\\
&= \frac{1}{k+1}\left(L_{k+1}'(x) + L_k'(x)\right)\\
&= \frac{1}{k+1}\left((2k+1)L_k(x) + (2(k-1)+1)L_{k-1}(x) + \cdots + L_0(x)\right).
\end{aligned}$$

Given the property (A.9), we have

$$(2m+1)(x-1)L_m(x) = (m+1)\left(L_{m+1}(x) - L_m(x)\right) - m\left(L_m(x) - L_{m-1}(x)\right).$$

Thus, by this recurrence relation we have

$$\begin{aligned}
R_k(x) &= \frac{1}{k+1}\frac{1}{x-1}\Big((k+1)\left(L_{k+1}(x) - L_k(x)\right) - k\left(L_k(x) - L_{k-1}(x)\right)\\
&\quad + k\left(L_k(x) - L_{k-1}(x)\right) - (k-1)\left(L_{k-1}(x) - L_{k-2}(x)\right) + \cdots\\
&\quad + 2\left(L_2(x) - L_1(x)\right) - \left(L_1(x) - L_0(x)\right) + (x-1)L_0(x)\Big)\\
&= \frac{1}{k+1}\frac{1}{x-1}(k+1)\left(L_{k+1}(x) - L_k(x)\right) = \frac{1}{x-1}\left(L_{k+1}(x) - L_k(x)\right).
\end{aligned}$$

In this case,

$$E(u_h) = \sum_i h_i\left(\sum_{\ell=0}^{k}\frac{1}{2\ell+1}(u_{i,\ell})^2\right) = \|u_h\|^2 \qquad (3.22)$$

where $\|\cdot\|$ is the L^2 norm on (a,b). Moreover, the scheme (3.7) is equivalent to the standard DG scheme, see e.g. [6] for more details.

3.2.2   An a priori error estimate

We present the optimal error estimate of the semi-discrete SV scheme (3.7), stated in the following theorem.

Theorem 3.6. *Assume the SV scheme (3.7) has the admissible subdivision defined in Definition 3.3. If the exact solution $u \in W^{1,\infty}\big(0,T;W^{k+1,\infty}(\Omega)\big)$, then we have the following optimal error estimate*

$$\|u(\cdot,t) - u_h(\cdot,t)\| \lesssim (t+1)h^{k+1}\,, \tag{3.23}$$

with an appropriate initialization.

Proof. Since the exact solution u satisfies the SV scheme (3.7), then we can obtain the error equation as follows.

$$\int_{I_i} \big(\partial_t e\big) w_h \, \mathrm{d}x + B_i(e, w_h) = 0, \quad \forall\, w_h \in W_h^k\,, \tag{3.24}$$

where $e = u_h - u$ and $B_i(\cdot,\cdot)$ is given in (3.8). Define an interpolation operator $\mathbb{I}_h : W^{k+1,\infty}(\Omega) \to V_h^k$ such that for any $w \in W^{k+1,\infty}(\Omega)$

$$\mathbb{I}_h w\big(x_{i,\ell}^-\big) = w\big(x_{i,\ell}\big), \quad 1 \le \ell \le k+1, \,\forall\, i\,, \tag{3.25}$$

where $\{x_{i,\ell}\}_{\ell=0}^{k+1}$ are the subdivision points given in (2.9). Therefore, by the standard approximation theory [8], we have $\|w - \mathbb{I}_h w\| \lesssim h^{k+1}$. Now we denote

$$e = e_h - \varepsilon_h, \quad e_h = u_h - \mathbb{I}_h u, \quad \varepsilon_h = u - \mathbb{I}_h u\,. \tag{3.26}$$

Then the error equation (3.24) can be rewritten as follows.

$$\int_{I_i} \big(\partial_t e_h\big) w_h \, \mathrm{d}x + B_i(e_h, w_h) = \int_{I_i} \big(\partial_t \varepsilon_h\big) w_h \, \mathrm{d}x + B_i(\varepsilon_h, w_h), \quad \forall\, w_h \in W_h^k\,. \tag{3.27}$$

Assume $e_h = \sum_i \sum_{\ell=0}^k e_{i,\ell}\, \phi_{i,\ell}(x)$, and we define $\widetilde{e}_h := \mathbb{P}_h e_h = \sum_j \sum_{\ell=0}^k e_{i,\ell}\, \varphi_{i,\ell}(x) \in W_h^k$, where the projection $\mathbb{P}_h$ is defined in (3.10). By taking $w_h = \widetilde{e}_h$ in (3.27) and summing it over i, then the LHS of (3.27) becomes

$$\sum_i \left(\int_{I_i} \big(\partial_t e_h\big)\widetilde{e}_h + B_i(e_h, \widetilde{e}_h) \right) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} E(e_h) + \frac{1}{2}\sum_i [\![e_h]\!]_{i+\frac{1}{2}}^2\,. \tag{3.28}$$

And the terms on the right-hand side of (3.27) become

$$\sum_i \int_{I_i} \big(\partial_t \varepsilon_h\big)\widetilde{e}_h \, \mathrm{d}x = \|\partial_t \varepsilon_h\|\, \|\widetilde{e}_h\| \lesssim h^{k+1}\sqrt{E(e_h)}\,, \tag{3.29}$$

$$
\begin{aligned}
\sum_i B_i(\varepsilon_h, \widetilde{e}_h) &= \sum_i \left(\int_{I_i} (\varepsilon_h)_x\, \widetilde{e}_h \, \mathrm{d}x + [\![\varepsilon_h]\!]_{i-\frac{1}{2}} \big(\widetilde{e}_h\big)_{i-\frac{1}{2}}^+ \right) \\
&= \sum_i \left(\sum_{\ell=0}^k e_{i,\ell} \int_{I_i} (\varepsilon_h)_x\, \varphi_{i,\ell} \, \mathrm{d}x + [\![\varepsilon_h]\!]_{i-\frac{1}{2}} (e_h)_{i-\frac{1}{2}}^+ \right) \\
&= \sum_i \left(\sum_{\ell=0}^k e_{i,\ell} \sum_{s=0}^k \varphi_{i,\ell}\big|_{I_{i,s}} \int_{I_{i,s}} (\varepsilon_h)_x \, \mathrm{d}x + [\![\varepsilon_h]\!]_{i-\frac{1}{2}} (e_h)_{i-\frac{1}{2}}^+ \right) \\
&= \sum_i \left((\varepsilon_h)_{i+\frac{1}{2}}^- (e_h)_{i+\frac{1}{2}}^- - (\varepsilon_h)_{i-\frac{1}{2}}^+ (e_h)_{i-\frac{1}{2}}^+ + [\![\varepsilon_h]\!]_{i-\frac{1}{2}} (e_h)_{i-\frac{1}{2}}^+ \right) = 0\,.
\end{aligned}
\tag{3.30}
$$

The inequality in (3.29) is based on the equivalence relation between $\sqrt{E(\cdot)}$ and the L^2 norm in V_h. Finally, plugging (3.28) – (3.30) into (3.27) we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E(e_h(\cdot,t)) \lesssim h^{k+1}\sqrt{E(e_h(\cdot,t))}. \tag{3.31}$$

By Gronwall's inequality we can obtain $\sqrt{E(e_h(\cdot,t))} \lesssim \sqrt{E(e_h(\cdot,0))} + t\, h^{k+1}$. Now we take the initialization $u_h(x,0) = \mathbb{I}_h u_0$, and since $E(\cdot)$ is equivalent to the L^2 norm, then we obtain $\|e_h(\cdot,t)\| \lesssim t\, h^{k+1}$. With the property of the projection and triangle inequality, we obtain the desired result. $\quad\square$

3.2.3 Superconvergence

In this subsection, we consider the superconvergence property of the SV scheme (3.7). Denote $\sigma_{h,0} := \varepsilon_h = u - \mathbb{I}_h u$ defined in (3.26). Then we can define a series of correction functions $\{\sigma_{h,m}\}$ on I_i as follows: $\forall\, m \geq 1$, $\sigma_{h,m} \in V_h^k$ satisfies

$$\begin{cases} \displaystyle\sum_{s=1}^{k} \sigma_{h,m}(x_{i,s})\left(w_h\big|_{I_{i,s}} - w_h\big|_{I_{i,s-1}}\right) = \int_{I_i}\left(\partial_t\sigma_{h,m-1} - \overline{(\partial_t\sigma_{h,m-1})}_i\right)w_h\,\mathrm{d}x, \quad \forall\, w_h \in W_h^k, \\[2mm] (\sigma_{h,m})_{i+\frac{1}{2}}^- = 0, \end{cases} \tag{3.32}$$

where $(\overline{w})_i$ denotes the cell average of w on I_i. Note that the DoF of $\sigma_{h,m}\big|_{I_i}$ is $k+1$, while the constraints (3.32) has $k+2$ conditions. Fortunately, the first equation always holds when w_h equals to the same constant in each subcell $I_{i,\ell}$. Hence, the definition (3.32) is well-defined. The main result of superconvergence is stated in the following theorem.

Theorem 3.7. *Assume the SV scheme (3.7) has the admissible subdivision defined in Definition 3.3. Suppose the exact solution of (2.7) $u \in W^{m_0+2,\infty}\big(0,T;W^{k+1,\infty}(\Omega)\big) \cap W^{m_0+1,\infty}\big(0,T; W^{2k+1,\infty}(\Omega)\big)$, and u_h is the numerical solution of the SV scheme (3.7). The initial data is chosen as $u_h(x,0) = \mathbb{I}_h u_0 + \sum_{\ell=1}^{m_0+1}\sigma_{h,\ell}$, and we then have*

$$\left\| u_h - \mathbb{I}_h u + \sum_{\ell=1}^{m_0}\sigma_{h,\ell} \right\| \lesssim t\, h^{\min\{k+2+m_0,\, k_0+1\}}, \tag{3.33}$$

where k_0 is the degree of exactness of the quadrature rule with the quadrature nodes $\{\alpha_\ell\}_{\ell=1}^{k+1}$.

Before the proof of this theorem, we need some estimation for the correction functions, shown as in the following two lemmas:

Lemma 3.8. *The correction function $\sigma_{h,m}$ has the following estimation:*

$$\left\|\sigma_{h,m}\right\|_{L^2(I_i)} \lesssim h\left\|\partial_t\sigma_{h,m-1}\right\|_{L^2(I_i)}, \quad \left\|\partial_t\sigma_{h,m}\right\|_{L^2(I_i)} \lesssim h\left\|\partial_t^2\sigma_{h,m-1}\right\|_{L^2(I_i)}, \quad m \geq 1. \tag{3.34}$$

Lemma 3.9. *The estimation of the cell averages of the correction functions are given as follows:*

$$\left(\overline{\partial_t\sigma_{h,\ell}}\right)_i \lesssim h^{k_0+1}, \quad \ell = 0,\ldots,m-1,$$

where k_0 is the degree of exactness of the quadrature rule with the quadrature nodes $\{\alpha_\ell\}_{\ell=1}^{k+1}$ defined in Section 3.1, and $k_0 = 2k$ when $c = \dfrac{1}{k+1}$, otherwise $k_0 = 2k-1$.

The proof of Lemma 3.8 and Lemma 3.9 are given in Appendix A.4 and Appendix A.5, respectively. Next, we start to prove Theorem 3.7.

Proof. First, we add the correction functions to both sides of the error equation (3.27) and obtain

$$\int_{I_i} \partial_t \Big(e_h + \sum_{\ell=1}^{m} \sigma_{h,\ell} \Big) w_h \, \mathrm{d}x + B_i(e_h, w_h) = \int_{I_i} \partial_t \Big(\sum_{\ell=0}^{m} \sigma_{h,\ell} \Big) w_h \, \mathrm{d}x + B_i(\varepsilon_h, w_h), \ \forall \, w_h \in W_h^k.$$

(3.35)

From the definition of the correction functions in (3.32), we have

$$\int_{I_i} \Big(\partial_t \sigma_{h,m-1} - \big(\overline{\partial_t \sigma_{h,m-1}} \big)_i \Big) w_h \, \mathrm{d}x$$

$$= \sum_{s=1}^{k} \sigma_{h,m}(x_{i,s}) \Big(w_h \big|_{I_{i,s}} - w_h \big|_{I_{i,s-1}} \Big) - (\sigma_{h,m})_{i+\frac{1}{2}}^{-}(w_h)_{i+\frac{1}{2}}^{-} + (\sigma_{h,m})_{i-\frac{1}{2}}^{-}(w_h)_{i-\frac{1}{2}}^{+}$$

$$= -\sum_{s=0}^{k} w_h \big|_{I_{i,s}} \Big(\sigma_{h,m}(x_{i,s+1}) - \sigma_{h,m}(x_{i,s}) \Big) - [\![\sigma_{h,m}]\!]_{i-\frac{1}{2}}(w_h)_{i-\frac{1}{2}}^{+}$$

$$= -\int_{I_i} w_h \big(\partial_x \sigma_{h,m} \big) \mathrm{d}x - [\![\sigma_{h,m}]\!]_{i-\frac{1}{2}}(w_h)_{i-\frac{1}{2}}^{+} = -B_i(\sigma_{h,m}, w_h), \quad \forall \, w_h \in W_h^k.$$

(3.36)

With $B_i(\varepsilon_h, w_h) = 0$ and (3.36), (3.35) can be transformed into the following form:

$$\int_{I_i} \partial_t \Big(e_h + \sum_{\ell=1}^{m} \sigma_{h,\ell} \Big) w_h \, \mathrm{d}x + B_i \Big(e_h + \sum_{\ell=1}^{m} \sigma_{h,\ell}, w_h \Big) = \int_{I_i} \Big(\partial_t \sigma_{h,m} + \sum_{\ell=0}^{m-1} \big(\overline{\partial_t \sigma_{h,\ell}} \big)_i \Big) w_h \, \mathrm{d}x. \quad (3.37)$$

To have the higher order estimates, we need Lemmas 3.8 and 3.9 for the estimation of the right-hand side of (3.37), i.e. $\partial_t \sigma_{h,m}$ and $\big(\overline{\partial_t \sigma_{h,\ell}} \big)_i$, $\ell = 0, \ldots, m-1$. These two lemmas are crucial in obtaining the superconvergence result. With Lemma 3.8, we have

$$\big\| \partial_t \sigma_{h,m} \big\|_{L^2(I_i)} \lesssim h \big\| \partial_t^2 \sigma_{h,m-1} \big\|_{L^2(I_i)} \lesssim \cdots \lesssim h^m \big\| \partial_t^{m+1} \sigma_{h,0} \big\|_{L^2(I_i)}.$$

This indicates

$$\big\| \partial_t \sigma_{h,m} \big\| \le h^m \big\| \partial_t^{m+1} \varepsilon_h \big\| \lesssim h^{k+1+m}. \tag{3.38}$$

From Lemma 3.9, we have the estimates of the cell averages of the correction functions that

$$\big(\overline{\partial_t \sigma_{h,\ell}} \big)_i \lesssim h^{k_0+1}, \quad \ell = 0, \ldots, m-1. \tag{3.39}$$

Now we take $w_h = \mathbb{P}_h \big(e_h + \sum_{\ell=1}^{m} \sigma_{h,\ell} \big) = \widetilde{e_h} + \sum_{\ell=1}^{m} \widetilde{\sigma_{h,\ell}}$ in (3.37) and summing it over i, with (3.38) and (3.39) we can obtain

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} E \Big(e_h + \sum_{\ell=1}^{m} \sigma_{h,\ell} \Big) + \frac{1}{2} \sum_i \Big[\!\!\Big[e_h + \sum_{\ell=1}^{m} \sigma_{h,\ell} \Big]\!\!\Big]_{i+\frac{1}{2}}^2 \lesssim h^{\min\{k+1+m, k_0+1\}} \sqrt{E \Big(e_h + \sum_{\ell=1}^{m} \sigma_{h,\ell} \Big)}.$$

By the Gronwall's inequality, we have

$$\sqrt{E \Big(e_h + \sum_{\ell=1}^{m} \sigma_{h,\ell} \Big)(t)} \lesssim \sqrt{E \Big(e_h + \sum_{\ell=1}^{m} \sigma_{h,\ell} \Big)(0)} + t \, h^{\min\{k+1+m, k_0+1\}}.$$

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Now we take the initialization that $u_h(x,0) = \mathbb{I}_h u_0(x) - \sum_{\ell=1}^m \sigma_{h,\ell}(x,0)$, we can obtain

$$\sqrt{E\left(e_h + \sum_{\ell=1}^m \sigma_{h,\ell}\right)(t)} \lesssim t\,h^{\min\{k+1+m,k_0+1\}}\,.$$

This indicates $\|e_h + \sum_{\ell=1}^m \sigma_{h,\ell}\| \lesssim t\,h^{\min\{k+1+m,k_0+1\}}$ due to the equivalence of the $\sqrt{E(\cdot)}$ and the L^2 norm $\|\cdot\|$ in V_h. From the Lemma 3.8, we have $\|\sigma_{h,m}\|_{L^2(I_i)} \lesssim h\,\|\partial_t\sigma_{h,m-1}\|_{L^2(I_i)} \lesssim h^m \|\partial_t^m \varepsilon_h\|_{L^2(I_i)}$, then we get $\|\sigma_{h,m}\| \lesssim h^{k+1+m}$. By the triangle inequality, we have

$$\left\|e_h + \sum_{\ell=1}^{m-1} \sigma_{h,\ell}\right\| \leq \left\|e_h + \sum_{\ell=1}^m \sigma_{h,\ell}\right\| + \|\sigma_{h,m}\| \lesssim t\,h^{\min\{k+2+m,k_0+1\}}. \tag{3.40}$$

Set $m = m_0 + 1$ in (3.40), we obtain the desired result. $\qquad\square$

Therefore, if we take $m_0 \geq k_0 - k - 1$, then we can obtain

$$\left\|u_h - \mathbb{I}_h u + \sum_{\ell=1}^{m_0} \sigma_{h,\ell}\right\| \lesssim t\,h^{k_0+1} = \begin{cases} t\,h^{2k+1}, & c = \dfrac{1}{k+1}, \\ t\,h^{2k}, & \text{otherwise.} \end{cases} \tag{3.41}$$

Corollary 3.10. *Under the conditions of Theorem 3.7, we have*

$$e_{u,c} \lesssim (1+t)h^{\min\{k+2+m_0,k_0+1\}}, \quad e_{u,p} \lesssim t\,h^{k+\frac{3}{2}}, \tag{3.42}$$

where $e_{u,c}$ and $e_{u,p}$ are given as

$$e_{u,c} = \left(\frac{1}{N}\sum_i \left(\frac{1}{h_i}\int_{I_i}(u - u_h)\,\mathrm{d}x\right)^2\right)^{\frac{1}{2}}, \quad e_{u,p} = \max_{i,s}\left|(u - u_h)(x_{i,s})\right|.$$

3.3 Spectral Volume Method for Multi-dimensional Linear Scalar Conservation Laws

In this subsection, we proceed to extend the previous 1D results to multi-dimensional linear scalar conservation law. For simplicity, we consider the two-dimensional case, and the results can be extended to the higher dimensions without any difficulties.

The SV scheme (2.17) can be rewritten as follows: $\forall w_h \in W_h^k \times W_h^k$, seek $u_h(\cdot,t) \in V_h^k \times V_h^k$ such that

$$\int_{K_{i,j}}(\partial_t u_h)w_h\,\mathrm{d}x\mathrm{d}y + B_{i,j}^x(u_h,w_h) + B_{i,j}^y(u_h,w_h) = 0\,, \tag{3.43}$$

where $B_{i,j}^x(u_h,w_h)$ and $B_{i,j}^y(u_h,w_h)$ are defined as

$$
\begin{aligned}
B_{i,j}^x(u_h,w_h) &= \int_{K_{i,j}}(\partial_x u_h)w_h\,\mathrm{d}x\mathrm{d}y + \int_{J_j}\llbracket u_h\rrbracket\left(x_{i-\frac{1}{2}},y\right)w_h\left(x_{i-\frac{1}{2}}^+,y\right)\mathrm{d}y\,, \\
B_{i,j}^y(u_h,w_h) &= \int_{K_{i,j}}(\partial_y u_h)w_h\,\mathrm{d}x\mathrm{d}y + \int_{I_i}\llbracket u_h\rrbracket\left(x,y_{j-\frac{1}{2}}\right)w_h\left(x,y_{j-\frac{1}{2}}^+\right)\mathrm{d}x\,,
\end{aligned}
\tag{3.44}
$$

with $[\![u_h]\!](x_{i-\frac{1}{2}}, y) = u_h(x_{i-\frac{1}{2}}^+, y, t) - u_h(x_{i-\frac{1}{2}}^-, y, t)$ and $[\![u_h]\!](x, y_{j-\frac{1}{2}}) = u_h(x, y_{j-\frac{1}{2}}^+, t) - u_h(x, y_{j-\frac{1}{2}}^-, t)$.
Now let us consider the case that $\{(x_{i,\ell},\, y_{j,m})\}$, $1 \le \ell, m \le k$ are roots of the equation $R_{i,k}^x(x)\, R_{j,k}^y(y) = 0$ where $R_{i,k}^x(x)$ and $R_{j,k}^y(y)$ are defined as:

$$R_{i,k}^x(x) = R_k\left(\frac{2(x - x_i)}{h_i^x}\right), \quad R_{j,k}^y(y) = R_k\left(\frac{2(y - y_j)}{h_j^y}\right), \tag{3.45}$$

and $R_k(x)$ is the polynomial given in (3.5) with possibly different parameters c_x and c_y.

By taking $w_h = 1$ in (3.43) and summing it over i, j, we can obtain the conservation of u_h as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_h(x, y, t)\,\mathrm{d}x\mathrm{d}y = 0. \tag{3.46}$$

3.3.1 Energy-boundedness

We take the basis functions $\{\phi_{i,\ell}(x)\phi_{j,m}(y)\}$ on $K_{i,j}$ and assume the numerical solution u_h has the following form:

$$u_h(x, y, t) = \sum_{i,j}\sum_{0\le\ell,m\le k} u_{i,j,\ell,m}(t)\,\phi_{i,\ell}(x)\phi_{j,m}(y). \tag{3.47}$$

Denote the function $\widetilde{u_h} = \mathbb{Q}_h u_h$ as

$$\mathbb{Q}_h u_h(x, y, t) = \mathbb{P}_h^x\,\mathbb{P}_h^y\,u_h(x, y, t) = \sum_{i,j}\sum_{0\le\ell,m\le k} u_{i,j,\ell,m}(t)\,\varphi_{i,\ell}(x)\varphi_{j,m}(y). \tag{3.48}$$

Here, $\varphi_{i,\ell}(x) = \mathbb{P}_h^x\phi_{i,\ell}(x)$ and $\varphi_{j,m}(y) = \mathbb{P}_h^y\phi_{j,m}(y)$ are the functions obtained by the 1D projection (3.2) on x- and y-direction, respectively. Plugging $w_h = \widetilde{u_h}$ into (3.43), we then obtain

$$\int_{K_{i,j}} (\partial_t u_h)\widetilde{u_h}\,\mathrm{d}x\mathrm{d}y + B_{i,j}^x(u_h, \widetilde{u_h}) + B_{i,j}^y(u_h, \widetilde{u_h}) = 0. \tag{3.49}$$

Let us consider $B_{i,j}^x(u_h, \widetilde{u_h})$ at first. We separate $B_{i,j}^x(u_h, \widetilde{u_h})$ into two parts:

$$B_{i,j}^x(u_h, \widetilde{u_h}) = B_{i,j}^x(u_h, u_h) + B_{i,j}^x(u_h, \widetilde{u_h} - u_h). \tag{3.50}$$

For the convenience of analysis, we introduce the following notations:

$$u_h = \sum_{i,j}\left(u_{h,*}^y(x,y) + u_{h,k}^y(x)\phi_{j,k}(y)\right), \quad u_{h,*}^y(x,y) = \sum_{m=0}^{k-1} u_{h,m}^y(x)\phi_{j,m}(y), \quad u_{h,m}^y(x) = \sum_{\ell=0}^{k} u_{i,j,\ell,m}\,\phi_{i,\ell}(x),$$

$$\widetilde{u_h} = \sum_{i,j}\left(\widetilde{u_{h,*}^y}(x,y) + \widetilde{u_{h,k}^y}(x)\varphi_{j,k}(y)\right), \quad \widetilde{u_{h,*}^y}(x,y) = \sum_{m=0}^{k-1} \widetilde{u_{h,m}^y}(x)\varphi_{j,m}(y), \quad \widetilde{u_{h,m}^y}(x) = \sum_{\ell=0}^{k} u_{i,j,\ell,m}\,\varphi_{i,\ell}(x).$$

Then we immediately obtain

$$\begin{aligned}
\sum_{i,j} B_{i,j}^x(u_h, u_h) &= \frac{1}{2}\sum_{i,j}\int_{J_j} [\![u_h]\!]^2(x_{i+\frac{1}{2}}, y)\,\mathrm{d}y \\
&= \frac{1}{2}\sum_{i,j}\int_{J_j} [\![u_{h,*}^y]\!]^2(x_{i+\frac{1}{2}}, y)\,\mathrm{d}y + \frac{1}{2(2k+1)}\sum_{i,j} h_j^y[\![u_{h,k}^y]\!]^2_{i-\frac{1}{2}}.
\end{aligned} \tag{3.51}$$

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Different from the one-dimensional case, in two dimensions we have $B_{i,j}^x(u_h, \widetilde{u_h} - u_h) \neq 0$. In fact, by repeatedly utilizing (3.6), we have

$$
\begin{aligned}
&\sum_{i,j} B_{i,j}^x(u_h, \widetilde{u_h} - u_h) \\
&= \sum_{i,j} B_{i,j}^x\left(u_h, \widetilde{u_{h,*}^y} + \widetilde{u_{h,k}^y}(x)\varphi_{j,k}(y) - u_{h,*}^y - u_{h,k}^y(x)\phi_{j,k}(y)\right) \\
&= \sum_{i,j} B_{i,j}^x\left(u_{h,k}^y(x)\phi_{j,k}(y), \widetilde{u_{h,k}^y}(x)\varphi_{j,k}(y) - u_{h,k}^y(x)\phi_{j,k}(y)\right) \\
&= \sum_{i,j} \int_{J_j} \phi_{j,k}(y)\left(\varphi_{j,k}(y) - \phi_{j,k}(y)\right)\left(\int_{I_i} (\partial_x u_{h,k}^y)u_{h,k}^y \, \mathrm{d}x + [\![u_{h,k}^y]\!]_{i-\frac{1}{2}}\left(u_{h,k}^y\right)_{i-\frac{1}{2}}^+\right)\mathrm{d}y \\
&= \frac{k(1 - c_y(k+1))}{2(k+1)(2k+1)(c_y k + 1)} \sum_{i,j} h_j^y [\![u_{h,k}^y]\!]_{i+\frac{1}{2}}^2 \, .
\end{aligned}
\tag{3.52}
$$

Therefore, with (3.50) - (3.52) we have

$$
\sum_{i,j} B_{i,j}^x(u_h, \widetilde{u_h}) = \frac{1}{2}\sum_{i,j}\int_{J_j} [\![u_{h,*}^y]\!]^2\left(x_{i+\frac{1}{2}}, y\right)\mathrm{d}y + \frac{1}{2(k+1)(c_y k + 1)}\sum_{i,j} h_j^y [\![u_{h,k}^y]\!]_{i+\frac{1}{2}}^2 \, .
\tag{3.53}
$$

Similarly, we have

$$
\sum_{i,j} B_{i,j}^y(u_h, \widetilde{u_h}) = \frac{1}{2}\sum_{i,j}\int_{I_i} [\![u_{h,*}^x]\!]^2\left(x, y_{j+\frac{1}{2}}\right)\mathrm{d}x + \frac{1}{2(k+1)(c_x k + 1)}\sum_{i,j} h_i^x [\![u_{h,k}^x]\!]_{j+\frac{1}{2}}^2 \, ,
\tag{3.54}
$$

where $u_{h,*}^x$ and $u_{h,k}^x$ are defined as

$$
u_{h,*}^x(x,y) = \sum_{\ell=0}^{k-1} u_{h,\ell}^x(y)\phi_{i,\ell}(x), \quad u_{h,\ell}^x(y) = \sum_{m=0}^k u_{i,j,\ell,m}\,\phi_{j,m}(y)\, .
$$

Therefore, from (3.53) and (3.54) we then obtain

$$
\sum_{i,j} B_{i,j}^x(u_h, \widetilde{u_h}) + \sum_{i,j} B_{i,j}^y(u_h, \widetilde{u_h}) \geq 0
\tag{3.55}
$$

provided by $c_x, c_y > -\dfrac{1}{k(k+1)}$.

On the other hand, with the help of (3.6), we can define the energy

$$
E(u_h) := \sum_{i,j}\int_{K_{i,j}} u_h\,\widetilde{u_h}\,\mathrm{d}x\mathrm{d}y = \sum_{i,j}\sum_{0\leq \ell,m\leq k} c_{\ell,m}\,h_i^x h_j^y (u_{i,j,\ell,m})^2
\tag{3.56}
$$

where $c_{\ell,m} > 0$ are given as

$$
c_{\ell,m} = \begin{cases}
\dfrac{1}{(2\ell+1)(2m+1)}, & 0 \leq \ell, m \leq k-1, \\[2mm]
\dfrac{1}{(2\ell+1)(k+1)(c_y k + 1)}, & 0 \leq \ell \leq k-1,\, m = k, \\[2mm]
\dfrac{1}{(2m+1)(k+1)(c_x k + 1)}, & \ell = k,\, 0 \leq m \leq k-1, \\[2mm]
\dfrac{1}{(k+1)^2(c_x k + 1)(c_y k + 1)}, & \ell = m = k.
\end{cases}
$$

With (3.49), (3.55) and (3.56), we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E(u_h) = \sum_{i,j}\int_{K_{i,j}}(\partial_t u_h)\widetilde{u_h}\,\mathrm{d}x\mathrm{d}y = -\sum_{i,j}B_{i,j}^x(u_h,\widetilde{u_h}) - \sum_{i,j}B_{i,j}^y(u_h,\widetilde{u_h}) \le 0, \qquad (3.57)$$

which implies the SV scheme is energy stable.

3.3.2   An a priori error estimate

In this subsection, we consider the error estimates of the SV scheme (3.43). We adopt some notations the same as in the 1D case for convenience and hope it would not cause any ambiguities. Since the exact solution satisfies the numerical scheme (3.43), then we can have the error equation as follows:

$$\int_{K_{i,j}}(\partial_t e)w_h\,\mathrm{d}x\mathrm{d}y + B_{i,j}^x(e,w_h) + B_{i,j}^y(e,w_h) = 0, \quad \forall\, w_h \in W_h^k \times W_h^k, \qquad (3.58)$$

where $e = u_h - u$. We also define the projection $\mathbb{I}_h\colon W^{k+1,\infty}(\Omega) \to V_h^k \times V_h^k$ such that

$$\mathbb{I}_h w\big(x_{i,\ell}, y_{j,m}\big) = w\big(x_{i,\ell}, y_{j,m}\big), \quad 1 \le \ell, m \le k+1, \forall\, i,j. \qquad (3.59)$$

We still denote

$$e = e_h - \varepsilon_h, \quad e_h = u_h - \mathbb{I}_h u, \quad \varepsilon_h = u - \mathbb{I}_h u. \qquad (3.60)$$

Then the error equation (3.58) becomes

$$\begin{aligned}
\int_{K_{i,j}}(\partial_t e_h)w_h\,\mathrm{d}x\mathrm{d}y + B_{i,j}^x(e_h,w_h) + B_{i,j}^y(e_h,w_h) \\
= \int_{K_{i,j}}(\partial_t \varepsilon_h)w_h\,\mathrm{d}x\mathrm{d}y + B_{i,j}^x(\varepsilon_h,w_h) + B_{i,j}^y(\varepsilon_h,w_h), \quad \forall\, w_h \in W_h^k \times W_h^k.
\end{aligned} \qquad (3.61)$$

Let us define

$$e_h = \sum_{i,j}\sum_{0\le\ell,m\le k}e_{i,j,\ell,m}\phi_{i,\ell}(x)\phi_{j,m}(y), \quad \widetilde{e}_h = \sum_{i,j}\sum_{0\le\ell,m\le k}e_{i,j,\ell,m}\varphi_{i,\ell}(x)\varphi_{j,m}(y).$$

From the stability analysis, if we take $w_h = \widetilde{e}_h$ in the LHS of (3.61) and sum it over i,j we can obtain

$$\begin{aligned}
\sum_{i,j}&\bigg(\int_{K_{i,j}}(\partial_t e_h)\,\widetilde{e}_h\,\mathrm{d}x\mathrm{d}y + B_{i,j}^x(e_h,\widetilde{e}_h) + B_{i,j}^y(e_h,\widetilde{e}_h)\bigg) \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E(e_h) + \frac{1}{2}\sum_{i,j}\int_{J_j}\Big(\llbracket e_{h,*}^y\rrbracket\big(x_{i+\frac{1}{2}},y\big)\Big)^2\mathrm{d}y + \frac{1}{2(k+1)(c_y k+1)}\sum_{i,j}h_j^y\llbracket e_{h,k}^y\rrbracket_{i+\frac{1}{2}}^2 \\
&\quad + \frac{1}{2}\sum_{i,j}\int_{I_i}\Big(\llbracket e_{h,*}^x\rrbracket\big(x,y_{j+\frac{1}{2}}\big)\Big)^2\mathrm{d}x + \frac{1}{2(k+1)(c_x k+1)}\sum_{i,j}h_i^x\llbracket e_{h,k}^x\rrbracket_{j+\frac{1}{2}}^2 \\
&\ge \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E(e_h)
\end{aligned} \qquad (3.62)$$

where $e_{h,*}^x, e_{h,k}^x, e_{h,*}^y, e_{h,k}^y$ are given as

$$e_{h,*}^x(x,y) = \sum_{\ell=0}^{k-1} e_{h,\ell}^x(y)\phi_{i,\ell}(x), \quad e_{h,\ell}^x(y) = \sum_{m=0}^{k} e_{i,j,\ell,m}\,\phi_{j,m}(y), \quad 0 \le \ell \le k,$$

$$e_{h,*}^y(x,y) = \sum_{m=0}^{k-1} e_{h,m}^y(x)\phi_{j,m}(y), \quad e_{h,m}^y(x) = \sum_{l=0}^{k} e_{i,j,\ell,m}\,\phi_{i,\ell}(x), \quad 0 \le m \le k.$$

Also, we have

$$\sum_{i,j} \int_{K_{i,j}} (\partial_t \varepsilon_h) w_h \,\mathrm{d}x\mathrm{d}y \le \|\partial_t \varepsilon_h\|\|w_h\| \lesssim h^{k+1}\|w_h\|. \tag{3.63}$$

Therefore, it is crucial to estimate $B_{i,j}^x(\varepsilon_h, w_h) + B_{i,j}^y(\varepsilon_h, w_h)$. Our goal is to obtain

$$\sum_{i,j} \left(B_{i,j}^x(\varepsilon_h, w_h) + B_{i,j}^y(\varepsilon_h, w_h) \right) \lesssim h^{k+1}\|w_h\|. \tag{3.64}$$

Now we consider $u = Tu + Ru$ on $K_{i,j}$ for any $Tu \in P^{k+1}(K_{i,j})$ which is a $(k+2)$-th order approximation of u, i.e. $\|Ru\| = \|u - Tu\| \lesssim h^{k+2}\|u\|_{H^{k+2}}$. since $\varepsilon_h = u - \mathbb{I}_h u = Tu + Ru - \mathbb{I}_h(Tu + Ru)$, then (3.64) is equivalent to

$$\begin{aligned}
&\sum_{i,j} \left(B_{i,j}^x(Tu - \mathbb{I}_h(Tu), w_h) + B_{i,j}^y(Tu - \mathbb{I}_h(Tu), w_h) \right) \\
&+ \sum_{i,j} \left(B_{i,j}^x(Ru - \mathbb{I}_h(Ru), w_h) + B_{i,j}^y(Ru - \mathbb{I}_h(Ru), w_h) \right) \lesssim h^{k+1}\|w_h\|.
\end{aligned} \tag{3.65}$$

A direct calculation leads to

$$\sum_{i,j} \left(B_{i,j}^x(Ru - \mathbb{I}_h(Ru), w_h) + B_{i,j}^y(Ru - \mathbb{I}_h(Ru), w_h) \right) \lesssim h^{k+1}\|w_h\|.$$

Therefore, (3.65) holds true if $B_{i,j}^x(Tu - \mathbb{I}_h(Tu), w_h) + B_{i,j}^y(Tu - \mathbb{I}_h(Tu), w_h) = 0$. Since $\mathbb{I}_h$ is a polynomial preserving operator, then it suffices to show that $q_I = x^{k+1}, y^{k+1}$ satisfy

$$B_{i,j}^x(q_I - \mathbb{I}_h q_I, w_h) + B_{i,j}^y(q_I - \mathbb{I}_h q_I, w_h) = 0, \quad \forall\, w_h \in W_h^k \times W_h^k. \tag{3.66}$$

When $q_I = x^{k+1}$, by the definition of the projection $\mathbb{I}_h$ we have $(q_I - \mathbb{I}_h q_I)(x_{i,\ell}, y) = 0$, $\forall\, i$, $1 \le \ell \le k+1$, thus we have

$$B_{i,j}^x(q_I - \mathbb{I}_h q_I, w_h) = \int_{J_j} \sum_{\ell=0}^{k} \left((q_I - \mathbb{I}_h q_I)(x_{i,\ell+1}^-, y) - (q_I - \mathbb{I}_h q_I)(x_{i,\ell}^-, y) \right) w_h(x,y)\big|_{x \in I_{i,\ell}} \,\mathrm{d}y = 0. \tag{3.67}$$

And since $\partial_y(q_I - \mathbb{I}_h q_I) = 0$, $[\![q_I - \mathbb{I}_h q_I]\!](x, y_{j+\frac{1}{2}}) = 0$, then we have

$$B_{i,j}^y(q_I - \mathbb{I}_h q_I, w_h) = 0. \tag{3.68}$$

Hence (3.66) holds true for $q_I = x^{k+1}$. Similarly, we can also prove (3.66) for $q_I = y^{k+1}$. Plugging (3.62) – (3.64) into (3.61), we can obtain

$$\frac{1}{2}\frac{d}{dt}E(e_h) \lesssim h^{k+1}\|\widetilde{e}_h\| \lesssim h^{k+1}\sqrt{E(e_h)}\,. \tag{3.69}$$

By the Gronwall's inequality we can obtain $\sqrt{E(e_h)}(t) \lesssim \sqrt{E(e_h)}(0) + t\,h^{2k+2}$. By taking the initialization $u_h(x,y,0) = \mathbb{I}_h u_0$ and since $\sqrt{E(\cdot)}$ is equivalent to the L^2 norm, we then obtain $\|e_h(t)\| \lesssim t\,h^{k+1}$. With the property of the projection and triangle inequality, we can obtain the optimal error estimate for the SV scheme (3.43), stated in the following theorem.

Theorem 3.11. *Assume the SV scheme* (3.43) *has the admissible subdivision in both x- and y-direction, with the initialization $u_h(x,y,0) = \mathbb{I}_h u_0$ defined in* (3.59)*. If the exact solution $u \in W^{1,\infty}\big(0,T;W^{k+1,\infty}(\Omega)\big)$, then we have the following optimal error estimate:*

$$\|u(\cdot,t) - u_h(\cdot,t)\| \lesssim (t+1)h^{k+1}\,. \tag{3.70}$$

4   Numerical Tests

In this section, we conduct some numerical experiments to verify the theoretical results given in Section 3. We consider the one-dimensional problems only for illustration purpose. We adopt the Gauss-Legendre quadrature with 6 nodes to compute the error in L^2 norm. For smooth problems, as the spatial error is quite small, we use the ninth order Runge-Kutta method [19] in time discretization and take the quadruple precision in the computation. While for nonsmooth problems, we use the third order total variation diminishing Runge-Kutta (TVDRK) method [29] in time discretization. Different choices of c are taken in $R_k(x)$ to see the performance of different subdivision in the SV method.

Example 4.1. *Consider the linear scalar conservation law that $f(u) = u$ in* (2.7) *on $[0,2\pi]$ with the initial condition $u_0(x) = \cos x$ and periodic boundary condition. The final time is $T = 1.2$.*

In Table 1, we can see the L^2 error and order $\|u - u_h\|$ is $O(h^{k+1})$, while $\|u_h - \mathbb{I}_h u\|$ is one order higher than $\|u - u_h\|$ as the mesh is refined. With the correction functions, the L^2 error of $\|e_h + \sum_{\ell=1}^k \sigma_{h,\ell}\|$ is $O(h^{2k})$, and $e_{u,c}$ is $O(h^{2k})$, which coincide with the theoretical results presented in Section 3. The convergence order of $e_{u,p}$ is $k+2$, a half order higher than the theoretical prediction. The numerical results in Table 1 are similar to the one in Table 3, with $c = 0$ and 1, respectively. When $c = 1/(k+1)$, from Table 2 we can see the convergence orders of $\|u - u_h\|$, $\|u_h - \mathbb{I}_h u\|$, $e_{u,p}$ are similar to $c = 0, 1$, but it is one order higher in the L^2 errors of $\|e_h + \sum_{\ell=1}^k \sigma_{h,\ell}\|$ and $e_{u,c}$. Table 4 shows the L^2 errors and orders in the nonuniform mesh (10% random perturbation of the uniform mesh) and $c = 1/4$. Thus in Table 4, for $k = 2, 4$, the convergence orders are similar as in Table 1 and Table 3, and for $k = 3$, the convergence orders are similar as in Table 2.

Example 4.2. *Consider the linear scalar conservation law that $f(u) = u$ in* (2.7) *on $[0,2\pi]$ with the initial condition $u_0(x) = \begin{cases} \sin(2x), & 0.3\pi \le x \le 1.1\pi, \\ \cos x - 0.5, & otherwise \end{cases}$ and periodic boundary condition. The final time is $T = 10$.*

In Fig. 2, we show the energy norm $\sqrt{E(\cdot)}$ and L^2 norm $\|\cdot\|$ of the numerical solution against time, with different choices of c in $R_k(x)$ defined in (3.5). Here, no extra limiter is applied on the

Table 1: Errors and convergence orders of $\|u - u_h\|$, $\|e_h\| = \|u_h - \mathbb{I}_h u\|$, $\|e_h + \sum_{\ell=1}^{k} \sigma_{h,\ell}\|$, $e_{u,c}$, and $e_{u,p}$ in Example 4.1, with a uniform mesh $h_i = 2\pi/N$, $c = 0$.

k	N	$\|u - u_h\|$	order	$\|u_h - \mathbb{I}_h u\|$	order	$\|e_h + \sum_{\ell=1}^{k} \sigma_{h,\ell}\|$	order	$e_{u,c}$	order	$e_{u,p}$	order
2	24	2.441E-04	–	1.769E-05	–	9.249E-06	–	4.479E-06	–	1.262E-05	–
	48	3.049E-05	3.001	1.113E-06	3.990	5.782E-07	4.000	2.903E-07	3.947	7.970E-07	3.985
	72	9.033E-06	3.000	2.203E-07	3.995	1.142E-07	4.000	5.802E-08	3.972	1.578E-07	3.994
	96	3.810E-06	3.000	6.977E-08	3.997	3.614E-08	4.000	1.846E-08	3.980	4.998E-08	3.996
	120	1.951E-06	3.000	2.860E-08	3.997	1.480E-08	4.000	7.587E-09	3.985	2.048E-08	3.998
	144	1.129E-06	3.000	1.380E-08	3.998	7.138E-09	4.000	3.667E-09	3.988	9.882E-09	3.998
3	24	4.029E-06	–	1.595E-07	–	5.095E-09	–	2.474E-09	–	1.209E-07	–
	48	2.518E-07	4.000	4.982E-09	5.001	7.961E-11	6.000	4.000E-11	5.951	3.812E-09	4.987
	72	4.975E-08	4.000	6.561E-10	5.000	6.989E-12	6.000	3.551E-12	5.973	5.048E-10	4.986
	96	1.574E-08	4.000	1.557E-10	5.000	1.244E-12	6.000	6.356E-13	5.981	1.200E-10	4.995
	120	6.448E-09	4.000	5.101E-11	5.000	3.261E-13	6.000	1.672E-13	5.985	3.937E-11	4.994
	144	3.109E-09	4.000	2.050E-11	5.000	1.092E-13	6.000	5.610E-14	5.988	1.583E-11	4.996
4	24	5.306E-08	–	1.588E-09	–	1.481E-12	–	7.194E-13	–	1.141E-09	–
	48	1.659E-09	4.999	2.482E-11	6.000	5.774E-15	8.002	2.902E-15	7.954	1.782E-11	6.001
	72	2.184E-10	5.000	2.179E-12	6.000	2.253E-16	8.000	1.145E-16	7.973	1.567E-12	5.995
	96	5.184E-11	5.000	3.878E-13	6.000	2.256E-17	8.000	1.152E-17	7.981	2.789E-13	6.000
	120	1.699E-11	5.000	1.017E-13	6.000	3.784E-18	8.000	1.940E-18	7.985	7.313E-14	5.999
	144	6.827E-12	5.000	3.405E-14	6.000	8.801E-19	8.000	4.521E-19	7.988	2.449E-14	5.999

Table 2: Errors and convergence orders of $\|u - u_h\|$, $\|u_h - \mathbb{I}_h u\|$, $\|e_h + \sum_{\ell=1}^{k} \sigma_{h,\ell}\|$, $e_{u,c}$, and $e_{u,p}$ in Example 4.1, with a uniform mesh $h_i = 2\pi/N$, $c = 1/(k+1)$.

k	N	$\|u - u_h\|$	order	$\|u_h - \mathbb{I}_h u\|$	order	$\|e_h + \sum_{\ell=1}^{k} \sigma_{h,\ell}\|$	order	$e_{u,c}$	order	$e_{u,p}$	order
2	24	1.552E-04	–	7.612E-06	–	3.625E-07	–	1.901E-07	–	5.092E-06	–
	48	1.940E-05	3.000	4.752E-07	4.002	1.135E-08	4.998	5.926E-09	5.004	3.246E-07	3.971
	72	5.748E-06	3.000	9.385E-08	4.001	1.495E-09	4.999	7.793E-10	5.004	6.467E-08	3.979
	96	2.425E-06	3.000	2.969E-08	4.000	3.547E-10	5.000	1.848E-10	5.003	2.052E-08	3.989
	120	1.241E-06	3.000	1.216E-08	4.000	1.162E-10	5.000	6.052E-11	5.002	8.425E-09	3.990
	144	7.185E-07	3.000	5.865E-09	4.000	4.672E-11	5.000	2.431E-11	5.002	4.068E-09	3.993
3	24	2.496E-06	–	8.682E-08	–	1.270E-10	–	6.717E-11	–	5.925E-08	–
	48	1.561E-07	3.999	2.713E-09	5.000	9.925E-13	7.000	5.209E-13	7.011	1.848E-09	5.003
	72	3.083E-08	4.000	3.573E-10	5.000	5.809E-14	7.000	3.039E-14	7.008	2.436E-10	4.998
	96	9.756E-09	4.000	8.479E-10	5.000	7.754E-15	7.000	4.049E-15	7.006	5.871E-11	5.000
	120	3.996E-09	4.000	2.778E-11	5.000	1.626E-15	7.000	8.483E-16	7.005	1.894E-11	5.000
	144	1.927E-09	4.000	1.117E-11	5.000	4.538E-16	7.000	2.366E-16	7.004	7.613E-12	4.999
4	24	3.238E-08	–	8.803E-10	–	2.703E-14	–	1.428E-14	–	6.363E-10	–
	48	1.012E-09	4.999	1.376E-11	6.000	5.251E-17	9.008	2.764E-17	9.013	9.931E-12	6.002
	72	1.333E-10	5.000	1.208E-12	6.000	1.365E-18	9.001	7.159E-19	9.011	8.729E-13	5.997
	96	3.164E-11	5.000	2.150E-13	6.000	1.025E-19	8.999	5.363E-20	9.008	1.554E-13	6.000
	120	1.037E-11	5.000	5.636E-14	6.000	1.376E-20	9.001	7.188E-21	9.006	4.073E-14	6.000
	144	4.167E-12	5.000	1.887E-14	6.000	2.667E-21	9.000	1.392E-21	9.005	1.364E-14	5.999

SV schemes. Though we only conduct the semi-discrete analysis of the SV schemes, we can still see all of them decay as time evolves when the SV scheme (3.7) coupled with the 3rd order TVDRK time discretization. This also confirms our theoretical results.

Table 3: Errors and convergence orders of $\|u - u_h\|$, $\|u_h - \mathbb{I}_h u\|$, $\|e_h + \sum_{\ell=1}^k \sigma_{h,\ell}\|$, $e_{u,c}$, and $e_{u,p}$ in Example 4.1, with a uniform mesh $h_i = 2\pi/N$, $c = 1$.

k	N	$\|u - u_h\|$	order	$\|u_h - \mathbb{I}_h u\|$	order	$\|e_h + \sum_{\ell=1}^k \sigma_{h,\ell}\|$	order	$e_{u,c}$	order	$e_{u,p}$	order
2	24	1.380E-04	–	1.036E-05	–	6.165E-06	–	3.013E-06	–	7.781E-06	–
	48	1.724E-05	3.002	6.604E-07	3.972	3.854E-07	4.000	1.940E-07	3.957	4.930E-07	3.980
	72	5.106E-06	3.001	1.314E-07	3.983	7.614E-08	4.000	3.871E-08	3.974	9.778E-08	3.990
	96	2.154E-06	3.000	4.172E-08	3.987	2.409E-08	4.000	1.231E-08	3.982	3.099E-08	3.994
	120	1.103E-06	3.000	1.712E-08	3.990	9.868E-09	4.000	5.059E-09	3.986	1.271E-08	3.996
	144	6.381E-07	3.000	8.271E-09	3.992	4.759E-09	4.000	2.445E-09	3.988	6.132E-09	3.997
3	24	2.226E-06	–	8.953E-08	–	3.821E-10	–	1.865E-09	–	6.676E-08	–
	48	1.391E-07	4.000	2.792E-09	5.003	5.972E-13	6.000	3.004E-11	5.956	2.107E-09	4.986
	72	2.748E-08	4.000	3.677E-10	5.000	5.242E-14	6.000	2.665E-12	5.974	2.793E-10	4.984
	96	8.696E-09	4.000	8.724E-11	5.000	9.330E-15	6.000	4.768E-13	5.982	6.644E-11	4.992
	120	3.562E-09	4.000	2.859E-11	5.000	2.446E-15	6.000	1.254E-13	5.986	2.180E-11	4.993
	144	1.718E-09	4.000	1.149E-11	5.000	8.191E-16	6.000	4.208E-14	5.988	8.772E-12	4.994
4	24	2.917E-08	–	8.823E-09	–	1.186E-12	–	5.779E-13	–	7.760E-09	–
	48	9.117E-10	5.000	1.379E-11	6.000	4.623E-15	8.003	2.325E-15	7.958	1.211E-11	6.002
	72	1.201E-10	5.000	1.210E-12	6.000	1.802E-16	8.002	9.162E-17	7.975	1.064E-12	5.998
	96	2.849E-11	5.000	2.154E-13	6.000	1.805E-17	7.999	9.222E-18	7.981	1.893E-13	6.000
	120	9.336E-12	5.000	5.646E-14	6.000	3.028E-18	8.000	1.552E-18	7.986	4.962E-14	6.001
	144	3.752E-12	5.000	1.891E-14	6.000	7.041E-19	8.001	3.617E-19	7.989	1.662E-14	5.999

Table 4: Errors and convergence orders of $\|u - u_h\|$, $\|u_h - \mathbb{I}_h u\|$, $\|e_h + \sum_{\ell=1}^k \sigma_{h,\ell}\|$, $e_{u,c}$, and $e_{u,p}$ in Example 4.1, with 10% random perturbation of the uniform mesh, $c = 1/4$.

k	N	$\|u - u_h\|$	order	$\|u_h - \mathbb{I}_h u\|$	order	$\|e_h + \sum_{\ell=1}^k \sigma_{h,\ell}\|$	order	$e_{u,c}$	order	$e_{u,p}$	order
2	24	1.759E-04	–	9.381E-06	–	1.702E-06	–	8.297E-07	–	9.317E-06	–
	48	2.285E-05	3.240	6.087E-07	3.856	1.032E-07	4.409	5.132E-08	3.827	6.551E-07	3.848
	72	6.619E-06	2.968	1.188E-07	4.199	2.056E-08	3.867	1.060E-08	4.109	1.385E-07	3.770
	96	2.818E-06	2.970	3.648E-08	3.928	6.490E-09	4.293	3.309E-09	4.204	3.967E-08	4.822
	120	1.429E-06	3.247	1.535E-08	4.075	2.644E-09	4.014	1.369E-09	3.928	1.754E-08	3.281
	144	8.176E-07	2.615	7.239E-09	4.333	1.266E-09	4.088	6.592E-10	3.757	9.307E-09	4.065
3	24	5.534E-06	–	2.471E-07	–	1.442E-10	–	7.934E-11	–	8.385E-08	–
	48	2.994E-07	4.037	7.045E-09	5.323	7.601E-13	7.835	6.651E-13	7.233	3.185E-09	4.764
	72	6.497E-08	3.979	8.104E-10	5.184	4.107E-14	7.083	3.702E-14	7.134	4.747E-10	4.707
	96	2.115E-08	3.929	2.376E-10	4.211	5.494E-15	7.464	5.163E-15	7.144	1.179E-10	5.470
	120	8.840E-09	4.060	8.279E-11	4.826	1.181E-15	6.708	1.050E-15	7.340	4.093E-11	4.529
	144	4.332E-09	4.106	3.096E-11	5.118	3.214E-16	7.307	2.870E-16	7.107	1.696E-11	5.336
4	24	3.288E-08	–	1.123E-09	–	2.685E-13	–	1.050E-13	–	1.464E-09	–
	48	1.143E-09	5.379	1.578E-11	5.772	8.491E-16	8.366	4.695E-16	8.505	2.316E-11	5.878
	72	1.471E-10	4.873	1.500E-12	6.847	3.655E-17	7.831	1.805E-17	8.346	1.938E-12	6.118
	96	3.459E-11	5.205	2.623E-13	6.125	3.494E-18	7.654	1.908E-18	8.578	3.345E-13	6.083
	120	1.200E-11	4.909	6.902E-14	5.501	5.855E-19	9.097	3.383E-19	7.242	9.016E-14	5.972
	144	4.628E-12	5.053	2.206E-14	6.420	1.417E-19	7.570	7.524E-20	8.373	3.109E-14	6.288

5   Concluding Remarks

In this paper, we study a class of spectral volume (SV) methods for linear scalar conservation laws, where the subdivision points are the zeros of a specific polynomial with a parameter in it. With different choices of the parameter, the subdivision points are also changed accordingly, as well as the SV method. For some specific choices of parameters, the SV method could reduce to some existing schemes, such as the discontinuous Galerkin (DG) method, thus it is natural to
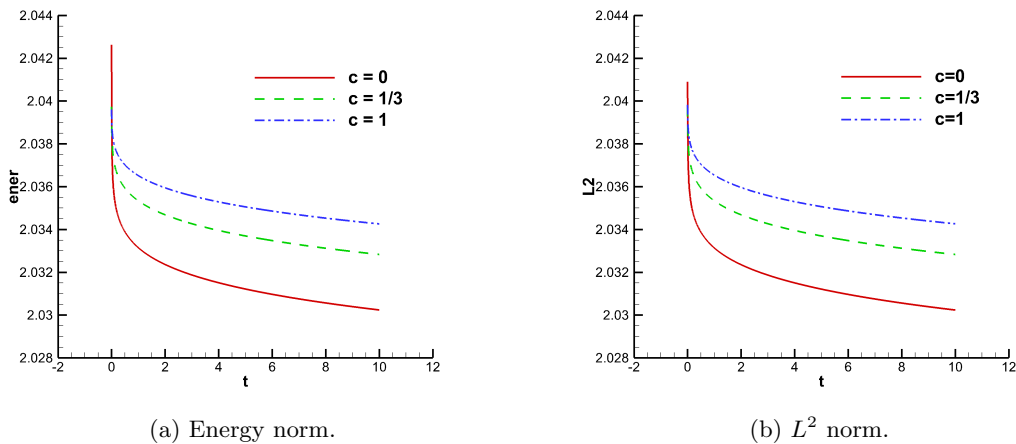
(a) Energy norm.   (b) L^2 norm.

Figure 2: The energy norm and L^2 norm of the numerical solution in Example 4.2. $k = 2$, $N = 288$. Solid line, $c = 0$; dashed line, $c = 1/3$; dash-dotted line, $c = 1$.

analyze the SV method in the Galerkin framework. By writing the SV scheme into the Petrov-Galerkin form, we mimic the standard semi-discrete analysis of the DG method, and show the energy stability and error estimates for one- and two-dimensional problems. The key ingredient of the analysis is the orthogonality property between the selected functions in the trial solution space and the test function space, which is determined by the zeros of the specific polynomial. By adopting the so-called *correction function* technique in [5], we then obtain the superconvergence results of the numerical solution. The error between the numerical solution and projection of the exact solution would be $O(h^{2k+1})$ with a specific subdivision (when the SV method is equivalent to the DG method) or it would be $O(h^{2k})$ otherwise, depending on the exactness of the quadrature with the subdivision points as the quadrature points. To verify the theoretical results, we show some numerical tests with different choices of parameters and the numerical results coincide with the theoretical results well. The technique proposed here is not directly applicable to nonlinear conservation laws, since the orthogonality property cannot be used when treating the nonlinear term. Therefore, the extension of the current analysis to nonlinear conservation laws would be one of future works.

A   Some formulas and proofs of some lemmas

In this section, we present some basic formulas and the proofs of some lemmas used in the paper.

A.1   Properties of the Legendre polynomials

Here, we give some properties of the Legendre polynomials we used in our analysis.

Prop 1.   Orthogonality:

$$\int_{-1}^{1} L_m(x)L_n(x)dx = \frac{2}{2m+1}\delta_{mn}. \tag{A.1}$$

Prop 2.   Point values:

$$L_n(-1) = (-1)^n, \quad L_n(1) = 1. \tag{A.2}$$

Prop 3.   Rodrigues' formula:

$$L_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left(x^2 - 1 \right)^n. \tag{A.3}$$

Prop 4.   Explicit representations:

$$L_n(x) = \frac{1}{2^n} \sum_{m=0}^{n} \binom{n}{m}^2 (x-1)^{n-m}(x+1)^m. \tag{A.4}$$

Prop 5.   Recurrence relations:

$$\frac{(x^2 - 1)}{n} L_n'(x) = x L_n(x) - L_{n-1}(x), \tag{A.5}$$

$$L_{n+1}'(x) = (n+1)L_n(x) + x L_n'(x), \tag{A.6}$$

$$L_{n+1}'(x) = (2n+1)L_n(x) + (2(n-2)+1)L_{n-2}(x) + (2(n-4)+1)L_{n-4}(x) + \cdots \tag{A.7}$$

$$(2n+1)L_n(x) = L_{n+1}'(x) - L_{n-1}'(x), \tag{A.8}$$

$$(n+1)L_{n+1}(x) = (2n+1)x L_n(x) - n L_{n-1}(x). \tag{A.9}$$

A.2   Proof of Lemma 3.1

Suppose $\{\alpha_\ell\}_{\ell=1}^k$ are roots of $R_k(x)$. Denote a polynomial $S_{k+1}(x) = (x-1)R_k(x)$ on $[-1, 1]$. Hence, $S_{k+1}(\alpha_1) = \cdots = S_{k+1}(\alpha_k) = S_{k+1}(1) = 0$ and $S_{k+1}(-1) = -2L_k(-1) = 2(-1)^{k+1}$. Moreover, for $Q_m(x)$ defined in (3.3), we assume

$$Q_m(x)|_{(\alpha_\ell, \alpha_{\ell+1})} = q_{m,\ell}, \ \ell = 0, \dots, k,$$

then we have

$$\int_{-1}^{1} S_{k+1}'(x) Q_m(x) \, dx = \sum_{n=0}^{k} q_{m,\ell} S_{k+1}(x)|_{\alpha_\ell}^{\alpha_{\ell+1}} = -(-1)^m 2(-1)^{k+1}$$
$$= \begin{cases} 2, & \mathrm{mod}(k+m, 2) = 0, \\ -2, & \mathrm{mod}(k+m, 2) = 1. \end{cases} \tag{A.10}$$

On the other hand, using the properties of the Legendre polynomial in Appendix A.1, we have

$$S_{k+1}(x) = (x-1)L_k(x) + c(x^2 - 1)L_k'(x)$$
$$= (x-1)L_k(x) + c\,k\big(x L_k(x) - L_{k-1}(x)\big).$$

By taking the derivative of $S_{k+1}(x)$, we can obtain

$$S_{k+1}'(x) = (1+ck)L_k(x) + (1+ck)x L_k'(x) - L_k'(x) - ck L_{k-1}'(x)$$
$$= (1+ck)L_k(x) + (1+ck)\left(L_{k+1}'(x) - (k+1)L_k(x)\right) - L_k'(x) - ck L_{k-1}'(x)$$
$$= (1+ck)L_k(x) + (1+ck)\left((2k+1)L_k(x) + L_{k-1}'(x) - (k+1)L_k(x)\right) - L_k'(x) - ck L_{k-1}'(x)$$
$$= (1+ck)(k+1)L_k(x) - L_k'(x) + L_{k-1}'(x)$$
$$= (1+ck)(k+1)L_k(x) - (2(k-1)+1)L_{k-1}(x) - (2(k-3)+1)L_{k-3}(x) - \cdots$$
$$\quad + (2(k-2)+1)L_{k-2}(x) + (2(k-4)+1)L_{k-4}(x) + \cdots$$

Hence, we have

$$\int_{-1}^{1} S'_{k+1}(x)Q_m(x)\,\mathrm{d}x = (1+ck)(k+1)\int_{-1}^{1} L_k(x)Q_m(x)\,\mathrm{d}x$$

$$+ \begin{cases} (2m+1)\int_{-1}^{1} L_m(x)Q_m(x) = 2, & \mathrm{mod}(k-m,2)=0,\ m \neq k; \\ -(2m+1)\int_{-1}^{1} L_m(x)Q_m(x) = -2, & \mathrm{mod}(k-m,2)=1; \\ 0, & m = k. \end{cases}$$

(A.11)

Comparing (A.10) with (A.11), we have

$$\int_{-1}^{1} L_k(x)Q_m(x)\,\mathrm{d}x = \begin{cases} 0, & m = 0,\ldots,k-1, \\ \dfrac{2}{(1+k)(1+ck)}, & m = k, \end{cases}$$

i.e., (3.4) holds with $C_k = \dfrac{1}{(1+k)(1+ck)}$.

Conversely, suppose (3.4) holds. Define a polynomial on $[-1,1]$:

$$\widetilde{S}_{k+1}(x) = a_k(x-\alpha_1)\cdots(x-\alpha_k)(x-1),$$

where a_k is a scaling constant such that $\widetilde{S}_{k+1}(-1) = 2(-1)^{k+1}$. Therefore,

$$\int_{-1}^{1} \widetilde{S}'_{k+1}(x)Q_m(x)\,\mathrm{d}x = -q_{m,0}\widetilde{S}_{k+1}(-1) = 2(-1)^{k+m}. \tag{A.12}$$

On the other hand, since $\{L_\ell(x)\}_{\ell=0}^{k}$ is a set of basis functions of polynomial degree at most k. Then $\widetilde{S}'_{k+1}(x)$ can be presented as

$$\widetilde{S}'_{k+1}(x) = \sum_{\ell=0}^{k} b_\ell L_\ell(x).$$

Then we have

$$\int_{-1}^{1} \widetilde{S}'_{k+1}(x)Q_m(x)\,\mathrm{d}x = \sum_{\ell=0}^{k} b_\ell \int_{-1}^{1} L_\ell(x)Q_m(x)\,\mathrm{d}x$$

$$= \begin{cases} \dfrac{2}{2m+1}b_m, & m = 0,\ldots,k-1, \\ 2C_k b_k, & m = k. \end{cases}$$

(A.13)

Comparing (A.12) with (A.13), we obtain

$$b_m = \begin{cases} \dfrac{2m+1}{2}2(-1)^{k+m} = (-1)^{k+m}(2m+1), & m = 0,\ldots,k-1, \\ \dfrac{1}{2C_k}2(-1)^{2k} = \dfrac{1}{C_k}, & m = k. \end{cases}$$

Therefore, by the properties of the Legendre polynomials given in Appendix A.1, we have

$$
\begin{aligned}
\widetilde{S}'_{k+1}(x) &= \frac{1}{C_k}L_k(x) - (2(k-1)-1)L_{k-1}(x) - (2(k-3)-1)L_{k-3}(x) - \cdots \\
&\quad + (2(k-2)+1)L_{k-2}(x) + (2(k-4)+1)L_{k-4}(x) + \cdots \\
&= \frac{1}{C_k}L_k(x) - L'_k(x) + L'_{k-1}(x) \\
&= \frac{1}{C_k}L_k(x) - L'_k(x) - ckL'_{k-1}(x) + (1+ck)(L'_{k+1}(x) - (2k+1)L_k(x)) \\
&= \frac{1}{C_k}L_k(x) - L'_k(x) - ckL'_{k-1}(x) + (1+ck)\big((k+1)xL_k(x) + xL'_k(x) - (2k+1)L_k(x)\big) \\
&= \Big(\frac{1}{C_k} - (1+ck)(k+1)\Big)L_k(x) + \big((1+ck)xL_k(x) - L_k(x) - ckL_{k-1}(x)\big)'.
\end{aligned}
$$

By taking $c = \dfrac{1}{k(k+1)C_k} - \dfrac{1}{k}$ so that $\dfrac{1}{C_k} - (1+ck)(k+1) = 0$, then we have

$$
\begin{aligned}
\widetilde{S}_{k+1}(x) &= (1+ck)xL_k(x) - L_k(x) - ckL_{k-1}(x) + \widetilde{C} \\
&= (x-1)\big(L_k(x) + c(x+1)L'_k(x)\big) + \widetilde{C},
\end{aligned}
$$

where $\widetilde{C}$ is a constant. Since $\widetilde{S}_{k+1}(1) = 0$ implies $\widetilde{C} = 0$, then we have

$$
a_k(x-\alpha_1)\cdots(x-\alpha_k) = L_k(x) + c(x+1)L'_k(x) = R_k(x),
$$

which indicates $\{\alpha_\ell\}_{\ell=1}^k$ are the roots of $R_k(x)$.

A.3 Proof of Lemma 3.2

First, we prove that $R_k(x)$ given in (3.5) has k distinct zeros in $(-1,1)$. Since

$$
\begin{aligned}
(x+1)&\frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}(x^2-1)^k \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left((x+1)\frac{\mathrm{d}^k}{\mathrm{d}x^k}(x^2-1)^k\right) - \frac{\mathrm{d}^k}{\mathrm{d}x^k}(x^2-1)^k \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}\left((x+1)\frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}}(x^2-1)^k - \frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}}(x^2-1)^k\right)\right) - \frac{\mathrm{d}^k}{\mathrm{d}x^k}(x^2-1)^k \\
&\ \ \vdots \\
&= \frac{\mathrm{d}^k}{\mathrm{d}x^k}\big(2kx(x+1)(x^2-1)^{k-1} - k(x^2-1)^k\big).
\end{aligned}
$$

Therefore, combining with the property (A.3), we can have

$$
\begin{aligned}
2^k k!R_k(x) &= \frac{\mathrm{d}^k}{\mathrm{d}x^k}\left((x^2-1)^k + 2c\,k\,x(x+1)(x^2-1)^{k-1} - c\,k(x^2-1)^k\right) \\
&= \frac{\mathrm{d}^k}{\mathrm{d}x^k}\left[(x-1)^{k-1}(x+1)^k\big((1+c\,k)x - (1-c\,k)\big)\right].
\end{aligned}
$$

Note that $x = -1$ and $x = 1$ are the roots of the polynomial $G_{2k}(x) = (x-1)^{k-1}(x+1)^k\big((1+c\,k)x - (1-c\,k)\big)$ with multiplicity k and $k-1$, respectively. Hence, by repeatedly using the Rolle's

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theorem, we have $R_k(x)$ has at least $(k-1)$ different roots between -1 and 1. Given the expression (A.4), we know that

$$L'_k(x) = \frac{1}{2^k} \sum_{m=1}^{k-1} \binom{k}{m}^2 \left((k-m)(x-1)^{k-m-1}(x+1)^m + m(x-1)^{k-m}(x+1)^{m-1} \right)$$
$$+ \frac{1}{2^k} k(x+1)^{k-1} + \frac{1}{2^k} k(x-1)^{k-1}.$$

This tell us that $L'_k(1) = \frac{1}{2} k(k+1)$, and $R_k(1) = L_k(1) + 2cL'_k(1) = 1 + ck(k+1) > 0$ when $c > -\frac{1}{k(k+1)}$. On the other hand, we know that $R_k(-1) = L_k(-1) = (-1)^k$. Therefore, if k is odd, then we have $R_k(-1) = (-1)^k = -1 < 0$ and $R_k(1) > 0$, and it indicates $R_k(x)$ has odd zero points in $(-1, 1)$. From above deduction, we know $R_k(x)$ has at least $(k-1)$ different roots between -1 and 1 and has at most k roots, thus $R_k(x)$ has k distinct roots within $(-1, 1)$. Similarly, if k is even, then $R_k(-1) = (-1)^k = 1 > 0$ and $R_k(1) > 0$, $R_k(x)$ has even zero points. Consequently, $R_k(x)$ has k distinct zeros in $(-1, 1)$.

A.4 Proof of Lemma 3.8

Let us estimate $\sigma_{h,m}$ first. Assume $\sigma_{h,m} = \sum_{\ell=0}^{k} (\sigma_{h,m})_{i,\ell} \phi_{i,\ell}(x)$ on I_i, and $\{\phi_{i,\ell}\}_{\ell=0}^{k}$ are defined in (3.13). It suffices to estimate the coefficients $(\sigma_{h,m})_{i,\ell}$, $0 \le \ell \le k$. From (3.36), we obtain

$$\left| B_i(\sigma_{h,m}, w_h) \right| \le \|\partial_t \sigma_{h,m-1}\|_{L^2(I_i)} \|w_h\|_{L^2(I_i)}. \tag{A.14}$$

Now let us take $w_h = \varphi_{i,k} + \varphi_{i,k-1}$ in $B_i(\sigma_{h,m}, w_h)$ where $\{\varphi_{i,\ell}\}_{\ell=0}^{k}$ are given in (3.15), then we have

$$B_i(\sigma_{h,m}, \varphi_{i,k} + \varphi_{i,k-1}) = (\sigma_{h,m})_{i,k} \int_{I_i} \phi'_{i,k} \varphi_{i,k-1} \, \mathrm{d}x. \tag{A.15}$$

Since $\int_{I_i} \phi'_{i,k} \varphi_{i,k-1} \, \mathrm{d}x = 2$, then by (A.14) and (A.15) we have

$$\left| (\sigma_{h,m})_{i,k} \right| \lesssim h^{\frac{1}{2}} \|\partial_t \sigma_{h,m-1}\|_{L^2(I_i)}. \tag{A.16}$$

Now taking $w_h = \varphi_{i,k-1} + \varphi_{i,k-2}$ in $B_i(\sigma_{h,m}, w_h)$, we have

$$B_i(\sigma_{h,m}, \varphi_{i,k-1} + \varphi_{i,k-2})$$
$$= \int_{I_i} \left((\sigma_{h,m})_{i,k} \phi'_{i,k} + (\sigma_{h,m})_{i,k-1} \phi'_{i,k-1} \right) \left(\varphi_{i,k-1} + \varphi_{i,k-2} \right) \mathrm{d}x \tag{A.17}$$
$$= 2 (\sigma_{h,m})_{i,k} + 2 (\sigma_{h,m})_{i,k-1}.$$

Thus, from (A.14), (A.16) and (A.17) we can obtain

$$\left| (\sigma_{h,m})_{i,k-1} \right| \lesssim h^{\frac{1}{2}} \|\partial_t \sigma_{h,m-1}\|_{L^2(I_i)}.$$

Similarly, we have the estimates for the coefficients $(\sigma_{h,m})_{i,\ell}$, $1 \le \ell \le k-2$:

$$\left| (\sigma_{h,m})_{i,\ell} \right| \lesssim h^{\frac{1}{2}} \|\partial_t \sigma_{h,m-1}\|_{L^2(I_i)}, \quad 1 \le \ell \le k-2.$$

By the constraint $(\sigma_{h,m})_{i-\frac{1}{2}} = 0$, then we can also have $\left| (\sigma_{h,m})_{i,0} \right| \lesssim h^{\frac{1}{2}} \|\partial_t \sigma_{h,m-1}\|_{L^2(I_i)}$. Finally, with the estimation of the coefficients $(\sigma_{h,m})_{i,\ell}$, we can obtain $\|\sigma_{h,m}\|_{L^2(I_i)} \lesssim h \|\partial_t \sigma_{h,m-1}\|_{L^2(I_i)}$. By taking the time derivative of (3.32) and follow the same lines above, we can also obtain $\|\partial_t \sigma_{h,m}\|_{L^2(I_i)} \lesssim h \|\partial_t^2 \sigma_h^{m-1}\|_{L^2(I_i)}$.

A.5 Proof of Lemma 3.9

First let us define an interpolation

$$\mathbb{I} : C^0([-1,1]) \to P^k([-1,1]), \quad \mathbb{I}w(\alpha_\ell) = w(\alpha_\ell), \, \ell = 1, \dots, k+1,$$

where $\{\alpha_m\}_{\ell=1}^{k+1}$ are the zeros of the polynomial $(x-1)R_k(x)$. Then we have

$$\int_{-1}^{1} \left(p - \mathbb{I}p \right) \mathrm{d}x = 0, \quad \forall p \in P^{k_0}([-1,1]), \tag{A.18}$$

where the degree of exactness $k_0 = 2k$ when $c = \dfrac{1}{k+1}$, otherwise $k_0 = 2k - 1$.

To obtain the degree of exactness of the quadrature rule, we can check the representation of $(x - 1)R_k(x)$ by the Legendre polynomials (see e.g. [4, Theorem 6.1.1]). The quadrature rule with nodes $\{\alpha_\ell\}_{\ell=1}^{k+1}$ is exact for $P^{k+1+s}([-1,1])$ if there exist the numbers $\mu_{s+1}, \dots, \mu_{k+1}$ such that

$$(x-1)R_k(x) = \sum_{\nu=s+1}^{k+1} \mu_\nu L_\nu(x). \tag{A.19}$$

By the properties of the Legendre polynomials in Appendix A.1, we have

$$
\begin{aligned}
(x-1)R_k(x) &= (x-1)L_k(x) + c(x^2 - 1)L_k'(x) \\
&= xL_k(x) - L_k(x) + ck\big(xL_k(x) - L_{k-1}(x)\big) \\
&= -ckL_{k-1}(x) - L_k(x) + \frac{1+ck}{2k+1}\big((k+1)L_{k+1}(x) + kL_{k-1}(x)\big) \\
&= \frac{k(1 - c(k+1))}{2k+1}L_{k-1}(x) - L_k(x) + \frac{(1+ck)(k+1)}{2k+1}L_{k+1}(x).
\end{aligned}
\tag{A.20}
$$

Therefore, for $c = \dfrac{1}{k+1}$, we have $s = k - 1$ in (A.19), otherwise $s = k - 2$. As $k_0 = k + 1 + s$, then we have obtained the degree of exactness of the quadrature rule.

Now we proceed to prove Lemma 3.9. From the definition of $\mathbb{I}_h$ in (3.25), we immediately have

$$\left(\overline{\partial_t \sigma_{h,0}} \right)_i = \frac{1}{h_i} \int_{I_i} \left(\partial_t u - \mathbb{I}_h \partial_t u \right) \mathrm{d}x \lesssim h^{k_0+1}.$$

For $1 \le \ell \le m - 1$, we define a class of polynomials

$$\Phi_{i,0} = \frac{2}{h_i} \partial_x^{-1} \phi_{i,0}, \quad \Phi_{i,\ell} = \partial_x^{-1}\big(\Phi_{i,\ell-1} - (\overline{\Phi_{i,\ell-1}})_i\big), \, \ell \ge 1,$$

where $\partial_x^{-1} w(x) = \int_{x_{i-\frac{1}{2}}}^{x} w(y)\,\mathrm{d}y$ is an antiderivative of w, and clearly $\Phi_{i,\ell}$ is a polynomial of degree $\ell + 1$. Assume $\sigma_{h,\ell} = \sum_{s=0}^{k} (\sigma_{h,\ell})_{i,s}\phi_{i,s}(x)$ on I_i, then we have

$$
\begin{aligned}
2\left(\overline{\partial_t \sigma_{h,\ell}} \right)_i &= 2(\partial_t \sigma_{h,\ell})_{i,0} = \int_{I_i} \partial_t \sigma_{h,\ell}\, \partial_x \Phi_{i,0}\,\mathrm{d}x = -B_i(\partial_t \sigma_{h,\ell}, \partial_x \Phi_{i,0}) \quad \text{(Similar to (3.36))} \\
&= \int_{I_i} \big(\partial_t^2 \sigma_{i,\ell-1} - (\overline{\partial_t^2 \sigma_{i,\ell-1}})_i\big)\Phi_{i,0}\,\mathrm{d}x = \int_{I_i} \partial_t^2 \sigma_{i,\ell-1}\big(\Phi_{i,0} - (\overline{\Phi_{i,0}})_i\big)\,\mathrm{d}x \\
&= \int_{I_i} \partial_t^2 \sigma_{h,\ell-1}\, \partial_x \Phi_{i,1}\,\mathrm{d}x = -B_i(\partial_t^2 \sigma_{h,\ell-1}, \partial_x \Phi_{i,1}) \quad \text{(Similar to (3.36))}
\end{aligned}
$$

$$= \cdots = \int_{I_i} \partial_t^3 \sigma_{h,\ell-2} \, \partial_x \Phi_{i,2} \, \mathrm{d}x = \cdots = \int_{I_i} \partial_t^{\ell+1} \sigma_{h,0} \, \partial_x \Phi_{i,\ell} \, \mathrm{d}x.$$

We also have the remainder for the interpolation $\mathbb{I}_h$ given as

$$\partial_t^{\ell+1} \sigma_{h,0} = \partial_t^{\ell+1} u - \mathbb{I}_h \partial_t^{\ell+1} u = \frac{\partial_x^{k+1} \partial_t^{\ell+1} u(\xi(x))}{(k+1)!} \prod_{s=1}^{k+1} (x - x_{i,s}).$$

From (A.20), we know that $\prod_{s=1}^{k+1} (x - x_{i,s}) \perp P^{k_0-k-1}(I_i)$. Since $\partial_x \Phi_{i,\ell} \in P^\ell(I_i)$, then for any $q \in P^{k_0-k-1-\ell}(I_i)$ we have

$$
\begin{aligned}
\int_{I_i} \partial_t^{\ell+1} \sigma_{h,0} \, \partial_x \Phi_{i,\ell} \, \mathrm{d}x &= \int_{I_i} \frac{\partial_x^{k+1} \partial_t^{\ell+1} u(\xi(x))}{(k+1)!} \prod_{s=1}^{k+1} (x - x_{i,s}) \, \partial_x \Phi_{i,\ell}(x) \, \mathrm{d}x \\
&= \int_{I_i} \left(\frac{\partial_x^{k+1} \partial_t^{\ell+1} u(\xi(x))}{(k+1)!} - q(x) \right) \prod_{s=1}^{k+1} (x - x_{i,s}) \, \partial_x \Phi_{i,\ell}(x) \, \mathrm{d}x,
\end{aligned}
\tag{A.21}
$$

For $x \in I_i$, we have

$$\min_{q \in P^{k_0-k-1-\ell}(I_i)} \left| \frac{\partial_x^{k+1} \partial_t^{\ell+1} u(\xi(x))}{(k+1)!} - q(x) \right| \lesssim h^{k_0-k-\ell}, \quad \left| \prod_{s=1}^{k+1} (x - x_{i,s}) \right| \leq h^{k+1}, \quad \left| \partial_x \Phi_{i,\ell}(x) \right| \lesssim h^{\ell-1}.$$

This indicates $\left| \int_{I_i} \partial_t^{\ell+1} \sigma_{h,0} \, \partial_x \Phi_{i,\ell} \, \mathrm{d}x \right| \lesssim h^{k_0+1}$ by plugging the above estimates into (A.21), and it results in $\left| \left(\overline{\partial_t \sigma_{h,\ell}} \right)_i \right| \lesssim h^{k_0+1}$, thus completes the proof.

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