

In this attachment we give detailed proofs and formulas for Proposition 2.1 and Lemma 3.2.

1 Proof of Proposition 2.1

Here we give the detail proof of $\|v_I - x^{k+1} - u_I(x - \frac{h}{2}) + (x - \frac{h}{2})^{k+1}\|_{L^2(x_j, x_{j+\frac{1}{2}})} \leq Ch^{2k+5}$ with $k = 0, 1, \dots, 8$.

1. $k = 0$, $u = x$, by the definition,

$$\begin{aligned} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_I dx &= \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} x dx \\ \int_{x_j}^{x_{j+1}} v_I dx &= \int_{x_j}^{x_{j+1}} x dx \end{aligned} \quad (1.1)$$

then we have

$$\begin{aligned} u_I &= x_j, \quad \forall x \in I_j, \\ v_I &= x_{j+\frac{1}{2}}, \quad \forall x \in I_{j+\frac{1}{2}}. \end{aligned} \quad (1.2)$$

Hence, for $x \in (x_j, x_{j+\frac{1}{2}})$,

$$v_I - x^1 - u_I(x - \frac{h}{2}) + (x - \frac{h}{2})^1 = x_{j+\frac{1}{2}} - x_j - \frac{h}{2} = 0 \quad (1.3)$$

2. $k = 1$, $u = x^2$, by the definition,

$$\begin{aligned} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_I dx &= \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} x^2 dx \\ \tilde{P}_h(u_I; x; f, u)_j &= \tilde{P}_h(x^2; x; f, u)_j \\ \int_{x_j}^{x_{j+1}} v_I dx &= \int_{x_j}^{x_{j+1}} x^2 dx \\ \tilde{Q}_h(v_I; x; f, u)_{j+\frac{1}{2}} &= \tilde{Q}_h(x^2; x; f, u)_{j+\frac{1}{2}} \end{aligned} \quad (1.4)$$

then we have

$$\begin{aligned} u_I &= (2x_j - \frac{2}{3}a_j\tau_{max})x + \frac{1}{12}h^2 + \frac{2}{3}a_jx_j\tau_{max} - x_j^2, \quad \forall x \in I_j, \\ v_I &= (2x_{j+\frac{1}{2}} - \frac{2}{3}a_{j+\frac{1}{2}}\tau_{max})x + \frac{1}{12}h^2 + \frac{2}{3}a_{j+\frac{1}{2}}x_{j+\frac{1}{2}}\tau_{max} - x_{j+\frac{1}{2}}^2, \quad \forall x \in I_{j+\frac{1}{2}}. \end{aligned} \quad (1.5)$$

Hence,

$$\begin{aligned}
& \int_{x_j}^{x_{j+\frac{1}{2}}} (v_I - x^2 - u_I(x - \frac{h}{2}) + (x - \frac{h}{2})^2)^2 \\
&= \frac{1}{54} h^3 \tau_{max}^2 (a_j - a_{j+\frac{1}{2}})^2 \\
&= O(h^7)
\end{aligned} \tag{1.6}$$

3. $k = 2$, $u = x^3$, by the definition

$$\begin{aligned}
& \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_I dx = \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} x^3 dx \\
& \tilde{P}_h(u_I; x; f, u)_j = \tilde{P}_h(x^3; x; f, u)_j \\
& \tilde{P}_h(u_I; x^2; f, u)_j = \tilde{P}_h(x^3; x^2; f, u)_j \\
& \int_{x_j}^{x_{j+1}} v_I dx = \int_{x_j}^{x_{j+1}} x^3 dx \\
& \tilde{Q}_h(v_I; x; f, u)_{j+\frac{1}{2}} = \tilde{Q}_h(x^3; x; f, u)_{j+\frac{1}{2}} \\
& \tilde{Q}_h(v_I; x^2; f, u)_{j+\frac{1}{2}} = \tilde{Q}_h(x^3; x^2; f, u)_{j+\frac{1}{2}}
\end{aligned} \tag{1.7}$$

then we have

$$\begin{aligned}
& u_I = \alpha_2 x^2 + \alpha_1 x + \alpha_0, \quad \forall x \in I_j. \\
& v_I = \beta_2 x^2 + \beta_1 x + \beta_0, \quad \forall x \in I_{j+\frac{1}{2}}.
\end{aligned} \tag{1.8}$$

where

$$\begin{aligned}
\alpha_2 &= (3(80a_j^2 \tau_{max}^2 x_j - 2a_j h^2 \tau_{max} + 15h^2 x_j)) / (5(16a_j^2 \tau_{max}^2 + 3h^2)) \\
\alpha_1 &= (3(16a_j^2 h^2 \tau_{max}^2 - 640a_j^2 \tau_{max}^2 x_j^2 + 32a_j h^2 \tau_{max} x_j + 5h^4 - 120h^2 x_j^2)) \\
& \quad / (40(16a_j^2 \tau_{max}^2 + 3h^2)) \\
\alpha_0 &= - (48a_j^2 h^2 \tau_{max}^2 x_j - 640a_j^2 \tau_{max}^2 x_j^3 - 4a_j h^4 \tau_{max} \\
& \quad + 48a_j h^2 \tau_{max} x_j^2 + 15h^4 x_j - 120h^2 x_j^3) \\
& \quad / (40(16a_j^2 \tau_{max}^2 + 3h^2)) \\
\beta_2 &= (3(80a_{j+\frac{1}{2}}^2 \tau_{max}^2 x_{j+\frac{1}{2}} - 2a_{j+\frac{1}{2}} h^2 \tau_{max} + 15h^2 x_{j+\frac{1}{2}})) / (5(16a_{j+\frac{1}{2}}^2 \tau_{max}^2 + 3h^2)) \\
\beta_1 &= (3(16a_{j+\frac{1}{2}}^2 h^2 \tau_{max}^2 - 640a_{j+\frac{1}{2}}^2 \tau_{max}^2 x_{j+\frac{1}{2}}^2 + 32a_{j+\frac{1}{2}} h^2 \tau_{max} x_{j+\frac{1}{2}} + 5h^4 - 120h^2 x_{j+\frac{1}{2}}^2)) \\
& \quad / (40(16a_{j+\frac{1}{2}}^2 \tau_{max}^2 + 3h^2))
\end{aligned}$$

$$\begin{aligned}
\beta_0 = & - (48a_{j+\frac{1}{2}}^2 h^2 \tau_{max} x_{j+\frac{1}{2}} - 640a_{j+\frac{1}{2}}^2 \tau_{max} x_{j+\frac{1}{2}}^3 - 4a_{j+\frac{1}{2}} h^4 \tau_{max} \\
& + 48a_{j+\frac{1}{2}} h^2 \tau_{max} x_{j+\frac{1}{2}}^2 + 15h^4 x_{j+\frac{1}{2}} - 120h^2 x_{j+\frac{1}{2}}^3) \\
& / (40(16a_{j+\frac{1}{2}}^2 \tau_{max}^2 + 3h^2))
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
& \int_{x_j}^{x_{j+\frac{1}{2}}} (v_I - x^3 - u_I(x - \frac{h}{2}) + (x - \frac{h}{2})^3)^2 \\
& = (h^9 \tau_{max}^2 (a_j - a_{j+\frac{1}{2}})^2 (24h^2 \tau_{max}^2 (5a_j^2 + 2a_j a_{j+\frac{1}{2}} + 5a_{j+\frac{1}{2}}^2) + 512a_j^2 a_{j+\frac{1}{2}}^2 \tau_{max}^4 \\
& \quad - 45h^3 \tau_{max} (a_j + a_{j+\frac{1}{2}}) + 240a_j a_{j+\frac{1}{2}} h \tau_{max}^3 (a_j + a_{j+\frac{1}{2}}) + 18h^4)) \\
& \quad / (500(16a_j^2 \tau_{max}^2 + 3h^2)^2 (16a_{j+\frac{1}{2}}^2 \tau_{max}^2 + 3h^2)^2) \\
& = O(h^9)
\end{aligned}$$

4. $k = 3$, $u = x^4$, by the definition

$$\begin{aligned}
& \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_I dx = \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} x^4 dx \\
& \tilde{P}_h(u_I; x; f, u)_j = \tilde{P}_h(x^4; x; f, u)_j \\
& \tilde{P}_h(u_I; x^2; f, u)_j = \tilde{P}_h(x^4; x^2; f, u)_j \\
& \tilde{P}_h(u_I; x^3; f, u)_j = \tilde{P}_h(x^4; x^3; f, u)_j \\
& \int_{x_j}^{x_{j+1}} v_I dx = \int_{x_j}^{x_{j+1}} x^4 dx \\
& \tilde{Q}_h(v_I; x; f, u)_{j+\frac{1}{2}} = \tilde{Q}_h(x^4; x; f, u)_{j+\frac{1}{2}} \\
& \tilde{Q}_h(v_I; x^2; f, u)_{j+\frac{1}{2}} = \tilde{Q}_h(x^4; x^2; f, u)_{j+\frac{1}{2}} \\
& \tilde{Q}_h(v_I; x^3; f, u)_{j+\frac{1}{2}} = \tilde{Q}_h(x^4; x^3; f, u)_{j+\frac{1}{2}}
\end{aligned} \tag{1.9}$$

then we have

$$\begin{aligned}
u_I & = \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0, \quad \forall x \in I_j. \\
v_I & = \beta_3 x^3 + \beta_2 x^2 + \beta_1 x + \beta_0, \quad \forall x \in I_{j+\frac{1}{2}}.
\end{aligned} \tag{1.10}$$

where

$$\alpha_3 = (4(176a_j^3 \tau_{max}^3 + 1008a_j^2 \tau_{max}^2 x_j + 19a_j h^2 \tau_{max} + 105h^2 x_j)) / (21(48a_j^2 \tau_{max}^2 + 5h^2))$$

$$\alpha_2 = - (704a_j^3\tau_{max}^3x_j - 68a_j^2h^2\tau_{max}^2 + 2016a_j^2\tau_{max}^2x_j^2 + 76a_jh^2\tau_{max}x_j - 7h^4 + 210h^2x_j^2) / (7(48a_j^2\tau_{max}^2 + 5h^2))$$

$$\alpha_1 = - (2(44a_j^3h^2\tau_{max}^3 - 1760a_j^3\tau_{max}^3x_j^2 + 340a_j^2h^2\tau_{max}^2x_j - 3360a_j^2\tau_{max}^2x_j^3 + 5a_jh^4\tau_{max} - 190a_jh^2\tau_{max}x_j^2 + 35h^4x_j - 350h^2x_j^3)) / (35(48a_j^2\tau_{max}^2 + 5h^2))$$

$$\alpha_0 = - (-4224a_j^3h^2\tau_{max}^3x_j + 56320a_j^3\tau_{max}^3x_j^3 + 352a_j^2h^4\tau_{max}^2 - 16320a_j^2h^2\tau_{max}^2x_j^2 + 80640a_j^2\tau_{max}^2x_j^4 - 480a_jh^4\tau_{max}x_j + 6080a_jh^2\tau_{max}x_j^3 + 35h^6 - 1680h^4x_j^2 + 8400h^2x_j^4) / (1680(48a_j^2\tau_{max}^2 + 5h^2))$$

$$\beta_3 = (4(176a_{j+\frac{1}{2}}^3\tau_{max}^3 + 1008a_{j+\frac{1}{2}}^2\tau_{max}^2x_{j+\frac{1}{2}} + 19a_{j+\frac{1}{2}}h^2\tau_{max} + 105h^2x_{j+\frac{1}{2}})) / (21(48a_{j+\frac{1}{2}}^2\tau_{max}^2 + 5h^2))$$

$$\beta_2 = - (704a_{j+\frac{1}{2}}^3\tau_{max}^3x_{j+\frac{1}{2}} - 68a_{j+\frac{1}{2}}^2h^2\tau_{max}^2 + 2016a_{j+\frac{1}{2}}^2\tau_{max}^2x_{j+\frac{1}{2}}^2 + 76a_{j+\frac{1}{2}}h^2\tau_{max}x_{j+\frac{1}{2}} - 7h^4 + 210h^2x_{j+\frac{1}{2}}^2) / (7(48a_{j+\frac{1}{2}}^2\tau_{max}^2 + 5h^2))$$

$$\beta_1 = - (2(44a_{j+\frac{1}{2}}^3h^2\tau_{max}^3 - 1760a_{j+\frac{1}{2}}^3\tau_{max}^3x_{j+\frac{1}{2}}^2 + 340a_{j+\frac{1}{2}}^2h^2\tau_{max}^2x_{j+\frac{1}{2}} - 3360a_{j+\frac{1}{2}}^2\tau_{max}^2x_{j+\frac{1}{2}}^3 + 5a_{j+\frac{1}{2}}h^4\tau_{max} - 190a_{j+\frac{1}{2}}h^2\tau_{max}x_{j+\frac{1}{2}}^2 + 35h^4x_{j+\frac{1}{2}} - 350h^2x_{j+\frac{1}{2}}^3)) / (35(48a_{j+\frac{1}{2}}^2\tau_{max}^2 + 5h^2))$$

$$\beta_0 = - (-4224a_{j+\frac{1}{2}}^3h^2\tau_{max}^3x_{j+\frac{1}{2}} + 56320a_{j+\frac{1}{2}}^3\tau_{max}^3x_{j+\frac{1}{2}}^3 + 352a_{j+\frac{1}{2}}^2h^4\tau_{max}^2 - 16320a_{j+\frac{1}{2}}^2h^2\tau_{max}^2x_{j+\frac{1}{2}}^2 + 80640a_{j+\frac{1}{2}}^2\tau_{max}^2x_{j+\frac{1}{2}}^4 - 480a_{j+\frac{1}{2}}h^4\tau_{max}x_{j+\frac{1}{2}} + 6080a_{j+\frac{1}{2}}h^2\tau_{max}x_{j+\frac{1}{2}}^3 + 35h^6 - 1680h^4x_{j+\frac{1}{2}}^2 + 8400h^2x_{j+\frac{1}{2}}^4) / (1680(48a_{j+\frac{1}{2}}^2\tau_{max}^2 + 5h^2))$$

Hence, we have

$$\begin{aligned}
& \int_{x_j}^{x_{j+\frac{1}{2}}} (v_I - x^4 - u_I(x - \frac{h}{2}) + (x - \frac{h}{2})^4)^2 \\
&= (h^7 \tau_{max}^2 (a_j - a_{j+\frac{1}{2}})^2 (660160512 a_j^4 a_{j+\frac{1}{2}}^4 \tau_{max}^8 + 80 h^6 \tau_{max}^2 (18762 a_j^2 - 101 a_j a_{j+\frac{1}{2}} \\
&\quad + 18762 a_{j+\frac{1}{2}}^2) - 1400 h^5 \tau_{max}^3 (a_j + a_{j+\frac{1}{2}}) (77 a_j^2 - a_j a_{j+\frac{1}{2}} + 77 a_{j+\frac{1}{2}}^2) \\
&\quad - 1034880 a_j^2 a_{j+\frac{1}{2}}^2 h^3 \tau_{max}^5 (a_j + a_{j+\frac{1}{2}}) + 743424 a_j^2 a_{j+\frac{1}{2}}^2 h^2 \tau_{max}^6 (185 a_j^2 - a_j a_{j+\frac{1}{2}} \\
&\quad + 185 a_{j+\frac{1}{2}}^2) + 320 h^4 \tau_{max}^4 (22385 a_j^4 - 242 a_j^3 a_{j+\frac{1}{2}} + 89799 a_j^2 a_{j+\frac{1}{2}}^2 - 242 a_j a_{j+\frac{1}{2}}^3 \\
&\quad + 22385 a_{j+\frac{1}{2}}^4) - 11375 h^7 \tau_{max} (a_j + a_{j+\frac{1}{2}}) + 78625 h^8)) \\
&\quad / (617400 (48 a_j^2 \tau_{max}^2 + 5 h^2)^2 (48 a_{j+\frac{1}{2}}^2 \tau_{max}^2 + 5 h^2)^2) \\
&= O(h^{11})
\end{aligned}$$

5. $k = 4$, $u = x^5$, by the definition

$$\begin{aligned}
& \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_I dx = \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} x^5 dx \\
& \tilde{P}_h(u_I; x; f, u)_j = \tilde{P}_h(x^5; x; f, u)_j \\
& \tilde{P}_h(u_I; x^2; f, u)_j = \tilde{P}_h(x^5; x^2; f, u)_j \\
& \tilde{P}_h(u_I; x^3; f, u)_j = \tilde{P}_h(x^5; x^3; f, u)_j \\
& \tilde{P}_h(u_I; x^4; f, u)_j = \tilde{P}_h(x^5; x^4; f, u)_j \\
& \int_{x_j}^{x_{j+1}} v_I dx = \int_{x_j}^{x_{j+1}} x^5 dx \\
& \tilde{Q}_h(v_I; x; f, u)_{j+\frac{1}{2}} = \tilde{Q}_h(x^5; x; f, u)_{j+\frac{1}{2}} \\
& \tilde{Q}_h(v_I; x^2; f, u)_{j+\frac{1}{2}} = \tilde{Q}_h(x^5; x^2; f, u)_{j+\frac{1}{2}} \\
& \tilde{Q}_h(v_I; x^3; f, u)_{j+\frac{1}{2}} = \tilde{Q}_h(x^5; x^3; f, u)_{j+\frac{1}{2}} \\
& \tilde{Q}_h(v_I; x^4; f, u)_{j+\frac{1}{2}} = \tilde{Q}_h(x^5; x^4; f, u)_{j+\frac{1}{2}}
\end{aligned} \tag{1.11}$$

then we have

$$\begin{aligned}
u_I &= \alpha_4 x^4 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0, \quad \forall x \in I_j. \\
v_I &= \beta_4 x^4 + \beta_3 x^3 + \beta_2 x^2 + \beta_1 x + \beta_0, \quad \forall x \in I_{j+\frac{1}{2}}.
\end{aligned} \tag{1.12}$$

where

$$\alpha_4 = (464640 a_j^4 \tau_{max}^4 x_j + 80 a_j^2 h^2 \tau_{max}^2 (178 a_j \tau_{max} + 795 x_j) + 75 h^4 (20 a_j \tau_{max} + 21 x_j))$$

$$\begin{aligned}
& / (92928a_j^4\tau_{max}^4 + 12720a_j^2h^2\tau_{max}^2 + 315h^4) \\
\alpha_3 = & (5(-2230272a_j^4\tau_{max}^4x_j^2 - 24h^4(-262a_j^2\tau_{max}^2 + 600a_j\tau_{max}x_j + 315x_j^2) \\
& + 128a_j^2h^2\tau_{max}^2(286a_j^2\tau_{max}^2 - 1068a_j\tau_{max}x_j - 2385x_j^2) + 225h^6)) \\
& / (36(30976a_j^4\tau_{max}^4 + 4240a_j^2h^2\tau_{max}^2 + 105h^4)) \\
\alpha_2 = & - (5(-5203968a_j^4\tau_{max}^4x_j^3 + 896a_j^2h^2\tau_{max}^2x_j(286a_j^2\tau_{max}^2 - 534a_j\tau_{max}x_j - 795x_j^2) \\
& - 8h^4(-1856a_j^3\tau_{max}^3 - 5502a_j^2\tau_{max}^2x_j + 6300a_j\tau_{max}x_j^2 + 2205x_j^3) \\
& + 63h^6(26a_j\tau_{max} + 25x_j))) \\
& / (84(30976a_j^4\tau_{max}^4 + 4240a_j^2h^2\tau_{max}^2 + 105h^4)) \\
\alpha_1 = & - (5(20815872a_j^4\tau_{max}^4x_j^4 - 56h^6(-86a_j^2\tau_{max}^2 + 468a_j\tau_{max}x_j + 225x_j^2) \\
& - 3584a_j^2h^2\tau_{max}^2x_j^2(572a_j^2\tau_{max}^2 - 712a_j\tau_{max}x_j - 795x_j^2) \\
& + 32h^4(440a_j^4\tau_{max}^4 - 7424a_j^3\tau_{max}^3x_j - 11004a_j^2\tau_{max}^2x_j^2 \\
& + 8400a_j\tau_{max}x_j^3 + 2205x_j^4) + 231h^8)) \\
& / (672(30976a_j^4\tau_{max}^4 + 4240a_j^2h^2\tau_{max}^2 + 105h^4)) \\
\alpha_0 = & (62447616a_j^4\tau_{max}^4x_j^5 - 17920a_j^2h^2\tau_{max}^2x_j^3(572a_j^2\tau_{max}^2 - 534a_j\tau_{max}x_j - 477x_j^2) \\
& - 8h^6(-3608a_j^3\tau_{max}^3 - 9030a_j^2\tau_{max}^2x_j + 24570a_j\tau_{max}x_j^2 + 7875x_j^3) \\
& + 480h^4x_j(440a_j^4\tau_{max}^4 - 3712a_j^3\tau_{max}^3x_j - 3668a_j^2\tau_{max}^2x_j^2 \\
& + 2100a_j\tau_{max}x_j^3 + 441x_j^4) + 315h^8(12a_j\tau_{max} + 11x_j)) \\
& / (2016(30976a_j^4\tau_{max}^4 + 4240a_j^2h^2\tau_{max}^2 + 105h^4)) \\
\beta_4 = & (464640a_{j+\frac{1}{2}}^4\tau_{max}^4x_{j+\frac{1}{2}} + 80a_{j+\frac{1}{2}}^2h^2\tau_{max}^2(178a_{j+\frac{1}{2}}\tau_{max} + 795x_{j+\frac{1}{2}}) \\
& + 75h^4(20a_{j+\frac{1}{2}}\tau_{max} + 21x_{j+\frac{1}{2}})) \\
& / (92928a_{j+\frac{1}{2}}^4\tau_{max}^4 + 12720a_{j+\frac{1}{2}}^2h^2\tau_{max}^2 + 315h^4) \\
\beta_3 = & (5(-2230272a_{j+\frac{1}{2}}^4\tau_{max}^4x_{j+\frac{1}{2}}^2 - 24h^4(-262a_{j+\frac{1}{2}}^2\tau_{max}^2 + 600a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}} + 315x_{j+\frac{1}{2}}^2) \\
& + 128a_{j+\frac{1}{2}}^2h^2\tau_{max}^2(286a_{j+\frac{1}{2}}^2\tau_{max}^2 - 1068a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}} - 2385x_{j+\frac{1}{2}}^2) + 225h^6)) \\
& / (36(30976a_{j+\frac{1}{2}}^4\tau_{max}^4 + 4240a_{j+\frac{1}{2}}^2h^2\tau_{max}^2 + 105h^4))
\end{aligned}$$

$$\begin{aligned}
\beta_2 = & - (5(-5203968a_{j+\frac{1}{2}}^4\tau_{max}^4x_{j+\frac{1}{2}}^3 + 896a_{j+\frac{1}{2}}^2h^2\tau_{max}^2x_{j+\frac{1}{2}}(286a_{j+\frac{1}{2}}^2\tau_{max}^2 \\
& - 534a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}} - 795x_{j+\frac{1}{2}}^2) - 8h^4(-1856a_{j+\frac{1}{2}}^3\tau_{max}^3 - 5502a_{j+\frac{1}{2}}^2\tau_{max}^2x_{j+\frac{1}{2}} \\
& + 6300a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}^2 + 2205x_{j+\frac{1}{2}}^3) + 63h^6(26a_{j+\frac{1}{2}}\tau_{max} + 25x_{j+\frac{1}{2}}))) \\
& / (84(30976a_{j+\frac{1}{2}}^4\tau_{max}^4 + 4240a_{j+\frac{1}{2}}^2h^2\tau_{max}^2 + 105h^4)) \\
\beta_1 = & - (5(20815872a_{j+\frac{1}{2}}^4\tau_{max}^4x_{j+\frac{1}{2}}^4 - 56h^6(-86a_{j+\frac{1}{2}}^2\tau_{max}^2 + 468a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}} + 225x_{j+\frac{1}{2}}^2) \\
& - 3584a_{j+\frac{1}{2}}^2h^2\tau_{max}^2x_{j+\frac{1}{2}}^2(572a_{j+\frac{1}{2}}^2\tau_{max}^2 - 712a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}} - 795x_{j+\frac{1}{2}}^2) \\
& + 32h^4(440a_{j+\frac{1}{2}}^4\tau_{max}^4 - 7424a_{j+\frac{1}{2}}^3\tau_{max}^3x_{j+\frac{1}{2}} - 11004a_{j+\frac{1}{2}}^2\tau_{max}^2x_{j+\frac{1}{2}}^2 \\
& + 8400a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}^3 + 2205x_{j+\frac{1}{2}}^4) + 231h^8)) \\
& / (672(30976a_{j+\frac{1}{2}}^4\tau_{max}^4 + 4240a_{j+\frac{1}{2}}^2h^2\tau_{max}^2 + 105h^4)) \\
\beta_0 = & (62447616a_{j+\frac{1}{2}}^4\tau_{max}^4x_{j+\frac{1}{2}}^5 - 17920a_{j+\frac{1}{2}}^2h^2\tau_{max}^2x_{j+\frac{1}{2}}^3(572a_{j+\frac{1}{2}}^2\tau_{max}^2 \\
& - 534a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}} - 477x_{j+\frac{1}{2}}^2) - 8h^6(-3608a_{j+\frac{1}{2}}^3\tau_{max}^3 - 9030a_{j+\frac{1}{2}}^2\tau_{max}^2x_{j+\frac{1}{2}} \\
& + 24570a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}^2 + 7875x_{j+\frac{1}{2}}^3) \\
& + 480h^4x_{j+\frac{1}{2}}(440a_{j+\frac{1}{2}}^4\tau_{max}^4 - 3712a_{j+\frac{1}{2}}^3\tau_{max}^3x_{j+\frac{1}{2}} - 3668a_{j+\frac{1}{2}}^2\tau_{max}^2x_{j+\frac{1}{2}}^2 \\
& + 2100a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}^3 + 441x_{j+\frac{1}{2}}^4) + 315h^8(12a_{j+\frac{1}{2}}\tau_{max} + 11x_{j+\frac{1}{2}})) \\
& / (2016(30976a_{j+\frac{1}{2}}^4\tau_{max}^4 + 4240a_{j+\frac{1}{2}}^2h^2\tau_{max}^2 + 105h^4))
\end{aligned}$$

Hence, we have

$$\begin{aligned}
& \int_{x_j}^{x_{j+\frac{1}{2}}} (v_I - x^5 - u_I(x - \frac{h}{2}) + (x - \frac{h}{2})^5)^2 \\
= & (h^{13}\tau_{max}^2(a_j - a_{j+\frac{1}{2}})^2(74297776099295232a_j^6a_{j+\frac{1}{2}}^6\tau_{max}^{12} \\
& - 17160990584995840a_j^5a_{j+\frac{1}{2}}^5h\tau_{max}^{11}(a_j + a_{j+\frac{1}{2}}) \\
& + 189000h^{10}\tau_{max}^2(1679899a_j^2 - 1244578a_ja_{j+\frac{1}{2}} + 1679899a_{j+\frac{1}{2}}^2) \\
& + 7938000h^9\tau_{max}^3(a_j + a_{j+\frac{1}{2}})(110956a_j^2 - 146113a_ja_{j+\frac{1}{2}} + 110956a_{j+\frac{1}{2}}^2) \\
& + 317194240a_j^4a_{j+\frac{1}{2}}^4h^2\tau_{max}^{10}(51995327a_j^2 - 6974930a_ja_{j+\frac{1}{2}} + 51995327a_{j+\frac{1}{2}}^2) \\
& - 7208960a_j^3a_{j+\frac{1}{2}}^3h^3\tau_{max}^9(a_j + a_{j+\frac{1}{2}})(411149046a_j^2 - 110513705a_ja_{j+\frac{1}{2}} \\
& + 411149046a_{j+\frac{1}{2}}^2) + 806400h^8\tau_{max}^4(4609881a_j^4 - 5666392a_j^3a_{j+\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
& + 21455275a_j^2a_{j+\frac{1}{2}}^2 - 5666392a_ja_{j+\frac{1}{2}}^3 + 4609881a_{j+\frac{1}{2}}^4) \\
& + 403200h^7\tau_{max}^5(a_j + a_{j+\frac{1}{2}})(11867779a_j^4 - 59803436a_j^3a_{j+\frac{1}{2}} \\
& + 53324708a_j^2a_{j+\frac{1}{2}}^2 - 59803436a_ja_{j+\frac{1}{2}}^3 + 11867779a_{j+\frac{1}{2}}^4) \\
& - 47308800a_ja_{j+\frac{1}{2}}h^5\tau_{max}^7(a_j + a_{j+\frac{1}{2}})(2637558a_j^4 - 2676409a_j^3a_{j+\frac{1}{2}} \\
& + 11391631a_j^2a_{j+\frac{1}{2}}^2 - 2676409a_ja_{j+\frac{1}{2}}^3 + 2637558a_{j+\frac{1}{2}}^4) \\
& + 1966080a_j^2a_{j+\frac{1}{2}}^2h^4\tau_{max}^8(532478408a_j^4 - 225653868a_j^3a_{j+\frac{1}{2}} \\
& + 1900203439a_j^2a_{j+\frac{1}{2}}^2 - 225653868a_ja_{j+\frac{1}{2}}^3 + 532478408a_{j+\frac{1}{2}}^4) \\
& + 307200h^6\tau_{max}^6(45865050a_j^6 - 72215220a_j^5a_{j+\frac{1}{2}} + 800405235a_j^4a_{j+\frac{1}{2}}^2 \\
& - 293286178a_j^3a_{j+\frac{1}{2}}^3 + 800405235a_j^2a_{j+\frac{1}{2}}^4 - 72215220a_ja_{j+\frac{1}{2}}^5 + 45865050a_{j+\frac{1}{2}}^6) \\
& + 40059613125h^{11}\tau_{max}(a_j + a_{j+\frac{1}{2}}) + 8661350250h^{12})) \\
& /((254016(30976a_j^4\tau_{max}^4 + 4240a_j^2h^2\tau_{max}^2 + 105h^4)^2(30976a_{j+\frac{1}{2}}^4\tau_{max}^4 \\
& + 4240a_{j+\frac{1}{2}}^2h^2\tau_{max}^2 + 105h^4)^2) \\
& =O(h^{13})
\end{aligned}$$

6. $k = 5$, $u = x^6$, by the definition

$$\begin{aligned}
\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_I dx &= \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} x^6 dx \\
\tilde{P}_h(u_I; x; f, u)_j &= \tilde{P}_h(x^6; x; f, u)_j \\
\tilde{P}_h(u_I; x^2; f, u)_j &= \tilde{P}_h(x^6; x^2; f, u)_j \\
\tilde{P}_h(u_I; x^3; f, u)_j &= \tilde{P}_h(x^6; x^3; f, u)_j \\
\tilde{P}_h(u_I; x^4; f, u)_j &= \tilde{P}_h(x^6; x^4; f, u)_j \\
\tilde{P}_h(u_I; x^5; f, u)_j &= \tilde{P}_h(x^6; x^5; f, u)_j \\
\int_{x_j}^{x_{j+1}} v_I dx &= \int_{x_j}^{x_{j+1}} x^6 dx \\
\tilde{Q}_h(v_I; x; f, u)_{j+\frac{1}{2}} &= \tilde{Q}_h(x^6; x; f, u)_{j+\frac{1}{2}} \\
\tilde{Q}_h(v_I; x^2; f, u)_{j+\frac{1}{2}} &= \tilde{Q}_h(x^6; x^2; f, u)_{j+\frac{1}{2}} \\
\tilde{Q}_h(v_I; x^3; f, u)_{j+\frac{1}{2}} &= \tilde{Q}_h(x^6; x^3; f, u)_{j+\frac{1}{2}} \\
\tilde{Q}_h(v_I; x^4; f, u)_{j+\frac{1}{2}} &= \tilde{Q}_h(x^6; x^4; f, u)_{j+\frac{1}{2}} \\
\tilde{Q}_h(v_I; x^5; f, u)_{j+\frac{1}{2}} &= \tilde{Q}_h(x^6; x^5; f, u)_{j+\frac{1}{2}}
\end{aligned} \tag{1.13}$$

then we have

$$\begin{aligned}
u_I &= \alpha_5 x^5 + \alpha_4 x^4 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0, \quad \forall x \in I_j. \\
v_I &= \beta_5 x^5 + \beta_4 x^4 + \beta_3 x^3 + \beta_2 x^2 + \beta_1 x + \beta_0, \quad \forall x \in I_{j+\frac{1}{2}}.
\end{aligned} \tag{1.14}$$

where

$$\begin{aligned}
\alpha_5 &= (630(33x_j + 145a_j\tau_{max})h^4 + 288a_j^2\tau_{max}^2(70455x_j + 481a_j\tau_{max})h^2 \\
&\quad + 5632a_j^4\tau_{max}^4(35193x_j - 1331a_j\tau_{max})) \\
&\quad /((11(315h^4 + 307440a_j^2\tau_{max}^2h^2 + 3003136a_j^4\tau_{max}^4))) \\
\alpha_4 &= (5(3465h^6 - 12(10395x_j^2 + 91350a_j\tau_{max}x_j - 191168a_j^2\tau_{max}^2)h^4 \\
&\quad + 64a_j^2\tau_{max}^2(-1902285x_j^2 - 25974a_j\tau_{max}x_j \\
&\quad + 348082a_j^2\tau_{max}^2)h^2 + 33792a_j^4x_j\tau_{max}^4(2662a_j\tau_{max} - 35193x_j))) \\
&\quad /((132(315h^4 + 307440a_j^2\tau_{max}^2h^2 + 3003136a_j^4\tau_{max}^4)))
\end{aligned}$$

$$\begin{aligned}
\alpha_3 = & (5(-9(1155x_j + 5531a_j\tau_{max})h^6 + 12(10395x_j^3 + 137025a_j\tau_{max}x_j^2 \\
& - 573504a_j^2\tau_{max}^2x_j - 21856a_j^3\tau_{max}^3)h^4 + 64a_j^2\tau_{max}^2(1902285x_j^3 \\
& + 38961a_j\tau_{max}x_j^2 - 1044246a_j^2\tau_{max}^2x_j + 34606a_j^3\tau_{max}^3)h^2 \\
& - 101376a_j^4x_j^2\tau_{max}^4(1331a_j\tau_{max} - 11731x_j)) \\
& /((99(315h^4 + 307440a_j^2\tau_{max}^2h^2 + 3003136a_j^4\tau_{max}^4))
\end{aligned}$$

$$\begin{aligned}
\alpha_2 = & -(5(2871h^8 - 48(3465x_j^2 + 33186a_j\tau_{max}x_j - 23006a_j^2\tau_{max}^2)h^6 \\
& + 32(31185x_j^4 + 548100a_j\tau_{max}x_j^3 - 3441024a_j^2\tau_{max}^2x_j^2 - 262272a_j^3\tau_{max}^3x_j \\
& + 329432a_j^4\tau_{max}^4)h^4 + 512a_j^2x_j\tau_{max}^2(1902285x_j^3 + 51948a_j\tau_{max}x_j^2 \\
& - 2088492a_j^2\tau_{max}^2x_j + 138424a_j^3\tau_{max}^3)h^2 \\
& - 270336a_j^4x_j^3\tau_{max}^4(5324a_j\tau_{max} - 35193x_j)) \\
& /((1056(315h^4 + 307440a_j^2\tau_{max}^2h^2 + 3003136a_j^4\tau_{max}^4))
\end{aligned}$$

$$\begin{aligned}
\alpha_1 = & (3465(29x_j + 149a_j\tau_{max})h^8 - 336(5775x_j^3 + 82965a_j\tau_{max}x_j^2 \\
& - 115030a_j^2\tau_{max}^2x_j - 12178a_j^3\tau_{max}^3)h^6 + 160(43659x_j^5 + 959175a_j\tau_{max}x_j^4 \\
& - 8029056a_j^2\tau_{max}^2x_j^3 - 917952a_j^3\tau_{max}^3x_j^2 + 2306024a_j^4\tau_{max}^4x_j - 53240a_j^5\tau_{max}^5)h^4 \\
& + 17920a_j^2x_j^2\tau_{max}^2(380457x_j^3 + 12987a_j\tau_{max}x_j^2 - 696164a_j^2\tau_{max}^2x_j \\
& + 69212a_j^3\tau_{max}^3)h^2 - 1892352a_j^4x_j^4\tau_{max}^4(6655a_j\tau_{max} - 35193x_j)) \\
& /((3696(315h^4 + 307440a_j^2\tau_{max}^2h^2 + 3003136a_j^4\tau_{max}^4))
\end{aligned}$$

$$\begin{aligned}
\alpha_0 = & (17325h^{10} - 84(14355x_j^2 + 147510a_j\tau_{max}x_j - 37792a_j^2\tau_{max}^2)h^8 + 32(363825x_j^4 \\
& + 6969060a_j\tau_{max}x_j^3 - 14493780a_j^2\tau_{max}^2x_j^2 - 3068856a_j^3\tau_{max}^3x_j + 873136a_j^4\tau_{max}^4)h^6 \\
& - 1920x_j(14553x_j^5 + 383670a_j\tau_{max}x_j^4 - 4014528a_j^2\tau_{max}^2x_j^3 - 611968a_j^3\tau_{max}^3x_j^2 \\
& + 2306024a_j^4\tau_{max}^4x_j - 106480a_j^5\tau_{max}^5)h^4 - 14336a_j^2x_j^3\tau_{max}^2(1902285x_j^3 \\
& + 77922a_j\tau_{max}x_j^2 - 5221230a_j^2\tau_{max}^2x_j + 692120a_j^3\tau_{max}^3)h^2 \\
& + 22708224a_j^4x_j^5\tau_{max}^4(2662a_j\tau_{max} - 11731x_j)) \\
& /((88704(315h^4 + 307440a_j^2\tau_{max}^2h^2 + 3003136a_j^4\tau_{max}^4))
\end{aligned}$$

$$\beta_5 = (630(33x_{j+\frac{1}{2}} + 145a_{j+\frac{1}{2}}\tau_{max})h^4 + 288a_{j+\frac{1}{2}}^2\tau_{max}^2(70455x_{j+\frac{1}{2}} + 481a_{j+\frac{1}{2}}\tau_{max})h^2 + 5632a_{j+\frac{1}{2}}^4\tau_{max}^4(35193x_{j+\frac{1}{2}} - 1331a_{j+\frac{1}{2}}\tau_{max}))$$

$$/(11(315h^4 + 307440a_{j+\frac{1}{2}}^2\tau_{max}^2h^2 + 3003136a_{j+\frac{1}{2}}^4\tau_{max}^4))$$

$$\beta_4 = (5(3465h^6 - 12(10395x_{j+\frac{1}{2}}^2 + 91350a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}} - 191168a_{j+\frac{1}{2}}^2\tau_{max}^2)h^4$$

$$+ 64a_{j+\frac{1}{2}}^2\tau_{max}^2(-1902285x_{j+\frac{1}{2}}^2 - 25974a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}$$

$$+ 348082a_{j+\frac{1}{2}}^2\tau_{max}^2)h^2 + 33792a_{j+\frac{1}{2}}^4x_{j+\frac{1}{2}}\tau_{max}^4(2662a_{j+\frac{1}{2}}\tau_{max} - 35193x_{j+\frac{1}{2}})))$$

$$/(132(315h^4 + 307440a_{j+\frac{1}{2}}^2\tau_{max}^2h^2 + 3003136a_{j+\frac{1}{2}}^4\tau_{max}^4))$$

$$\beta_3 = (5(-9(1155x_{j+\frac{1}{2}} + 5531a_{j+\frac{1}{2}}\tau_{max})h^6 + 12(10395x_{j+\frac{1}{2}}^3 + 137025a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}^2$$

$$- 573504a_{j+\frac{1}{2}}^2\tau_{max}^2x_{j+\frac{1}{2}} - 21856a_{j+\frac{1}{2}}^3\tau_{max}^3)h^4 + 64a_{j+\frac{1}{2}}^2\tau_{max}^2(1902285x_{j+\frac{1}{2}}^3$$

$$+ 38961a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}^2 - 1044246a_{j+\frac{1}{2}}^2\tau_{max}^2x_{j+\frac{1}{2}} + 34606a_{j+\frac{1}{2}}^3\tau_{max}^3)h^2$$

$$- 101376a_{j+\frac{1}{2}}^4x_{j+\frac{1}{2}}^2\tau_{max}^4(1331a_{j+\frac{1}{2}}\tau_{max} - 11731x_{j+\frac{1}{2}})))$$

$$/(99(315h^4 + 307440a_{j+\frac{1}{2}}^2\tau_{max}^2h^2 + 3003136a_{j+\frac{1}{2}}^4\tau_{max}^4))$$

$$\beta_2 = -(5(2871h^8 - 48(3465x_{j+\frac{1}{2}}^2 + 33186a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}} - 23006a_{j+\frac{1}{2}}^2\tau_{max}^2)h^6$$

$$+ 32(31185x_{j+\frac{1}{2}}^4 + 548100a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}^3 - 3441024a_{j+\frac{1}{2}}^2\tau_{max}^2x_{j+\frac{1}{2}}^2$$

$$- 262272a_{j+\frac{1}{2}}^3\tau_{max}^3x_{j+\frac{1}{2}} + 329432a_{j+\frac{1}{2}}^4\tau_{max}^4)h^4 + 512a_{j+\frac{1}{2}}^2x_{j+\frac{1}{2}}\tau_{max}^2(1902285x_{j+\frac{1}{2}}^3$$

$$+ 51948a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}^2 - 2088492a_{j+\frac{1}{2}}^2\tau_{max}^2x_{j+\frac{1}{2}} + 138424a_{j+\frac{1}{2}}^3\tau_{max}^3)h^2$$

$$- 270336a_{j+\frac{1}{2}}^4x_{j+\frac{1}{2}}^3\tau_{max}^4(5324a_{j+\frac{1}{2}}\tau_{max} - 35193x_{j+\frac{1}{2}})))$$

$$/(1056(315h^4 + 307440a_{j+\frac{1}{2}}^2\tau_{max}^2h^2 + 3003136a_{j+\frac{1}{2}}^4\tau_{max}^4))$$

$$\beta_1 = (3465(29x_{j+\frac{1}{2}} + 149a_{j+\frac{1}{2}}\tau_{max})h^8 - 336(5775x_{j+\frac{1}{2}}^3 + 82965a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}^2$$

$$- 115030a_{j+\frac{1}{2}}^2\tau_{max}^2x_{j+\frac{1}{2}} - 12178a_{j+\frac{1}{2}}^3\tau_{max}^3)h^6 + 160(43659x_{j+\frac{1}{2}}^5$$

$$+ 959175a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}^4 - 8029056a_{j+\frac{1}{2}}^2\tau_{max}^2x_{j+\frac{1}{2}}^3 - 917952a_{j+\frac{1}{2}}^3\tau_{max}^3x_{j+\frac{1}{2}}^2$$

$$+ 2306024a_{j+\frac{1}{2}}^4\tau_{max}^4x_{j+\frac{1}{2}} - 53240a_{j+\frac{1}{2}}^5\tau_{max}^5)h^4$$

$$+ 17920a_{j+\frac{1}{2}}^2x_{j+\frac{1}{2}}^2\tau_{max}^2(380457x_{j+\frac{1}{2}}^3 + 12987a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}^2 - 696164a_{j+\frac{1}{2}}^2\tau_{max}^2x_{j+\frac{1}{2}}$$

$$+ 69212a_{j+\frac{1}{2}}^3\tau_{max}^3)h^2 - 1892352a_{j+\frac{1}{2}}^4x_{j+\frac{1}{2}}^4\tau_{max}^4(6655a_{j+\frac{1}{2}}\tau_{max} - 35193x_{j+\frac{1}{2}}))$$

$$\begin{aligned}
& /((3696(315h^4 + 307440a_{j+\frac{1}{2}}^2\tau_{max}^2h^2 + 3003136a_{j+\frac{1}{2}}^4\tau_{max}^4)) \\
\beta_0 = & (17325h^{10} - 84(14355x_{j+\frac{1}{2}}^2 + 147510a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}} - 37792a_{j+\frac{1}{2}}^2\tau_{max}^2)x_{j+\frac{1}{2}}^8 \\
& + 32(363825x_{j+\frac{1}{2}}^4 + 6969060a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}^3 - 14493780a_{j+\frac{1}{2}}^2\tau_{max}^2x_{j+\frac{1}{2}}^2 \\
& - 3068856a_{j+\frac{1}{2}}^3\tau_{max}^3x_{j+\frac{1}{2}} + 873136a_{j+\frac{1}{2}}^4\tau_{max}^4)x_{j+\frac{1}{2}}^6 \\
& - 1920x_{j+\frac{1}{2}}(14553x_{j+\frac{1}{2}}^5 + 383670a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}^4 - 4014528a_{j+\frac{1}{2}}^2\tau_{max}^2x_{j+\frac{1}{2}}^3 \\
& - 611968a_{j+\frac{1}{2}}^3\tau_{max}^3x_{j+\frac{1}{2}}^2 + 2306024a_{j+\frac{1}{2}}^4\tau_{max}^4x_{j+\frac{1}{2}} - 106480a_{j+\frac{1}{2}}^5\tau_{max}^5)x_{j+\frac{1}{2}}^4 \\
& - 14336a_{j+\frac{1}{2}}^2x_{j+\frac{1}{2}}^3\tau_{max}^2(1902285x_{j+\frac{1}{2}}^3 + 77922a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}^2 \\
& - 5221230a_{j+\frac{1}{2}}^2\tau_{max}^2x_{j+\frac{1}{2}} + 692120a_{j+\frac{1}{2}}^3\tau_{max}^3)x_{j+\frac{1}{2}}^2 \\
& + 22708224a_{j+\frac{1}{2}}^4x_{j+\frac{1}{2}}^5\tau_{max}^4(2662a_{j+\frac{1}{2}}\tau_{max} - 11731x_{j+\frac{1}{2}})) \\
& /((88704(315h^4 + 307440a_{j+\frac{1}{2}}^2\tau_{max}^2h^2 + 3003136a_{j+\frac{1}{2}}^4\tau_{max}^4))
\end{aligned}$$

Hence, we have

$$\begin{aligned}
& \int_{x_j}^{x_{j+\frac{1}{2}}} (v_I - x^6 - u_I(x - \frac{h}{2}) + (x - \frac{h}{2})^6)^2 \\
= & (h^{11}\tau_{max}^2(a_j - a_{j+\frac{1}{2}}))^2(195012000131713231589585453056a_j^8a_{j+\frac{1}{2}}^8\tau_{max}^{16} \\
& - 201891950852331333916557312a_j^6a_{j+\frac{1}{2}}^6h^3\tau_{max}^{13}(a_j + a_{j+\frac{1}{2}}) \\
& + 1800338400h^{14}\tau_{max}^2(13438284110a_j^2 - 51320885931a_ja_{j+\frac{1}{2}} \\
& + 13438284110a_{j+\frac{1}{2}}^2) + 100416141472907132928a_j^6a_{j+\frac{1}{2}}^6h^2\tau_{max}^{14}(397624535a_j^2 \\
& - 18450573a_ja_{j+\frac{1}{2}} + 397624535a_{j+\frac{1}{2}}^2) - 2200413600h^{13}\tau_{max}^3(10662687321a_j^3 \\
& + 292187561882a_j^2a_{j+\frac{1}{2}} + 292187561882a_ja_{j+\frac{1}{2}}^2 + 10662687321a_{j+\frac{1}{2}}^3) \\
& + 228614400h^{12}\tau_{max}^4(2094049467654a_j^4 - 8864817048804a_j^3a_{j+\frac{1}{2}} \\
& + 295334595970901a_j^2a_{j+\frac{1}{2}}^2 - 8864817048804a_ja_{j+\frac{1}{2}}^3 + 2094049467654a_{j+\frac{1}{2}}^4) \\
& - 23253221376a_j^4a_{j+\frac{1}{2}}^4h^5\tau_{max}^{11}(2316625977228695a_j^3 \\
& + 3103819172036596a_j^2a_{j+\frac{1}{2}} + 3103819172036596a_ja_{j+\frac{1}{2}}^2 \\
& + 2316625977228695a_{j+\frac{1}{2}}^3) + 1056964608a_j^4a_{j+\frac{1}{2}}^4h^4\tau_{max}^{12}(1972327189988974885a_j^4 \\
& - 357721171163321350a_j^3a_{j+\frac{1}{2}} + 8331528893607301417a_j^2a_{j+\frac{1}{2}}^2
\end{aligned}$$

$$\begin{aligned}
& - 357721171163321350a_j a_{j+\frac{1}{2}}^3 + 1972327189988974885a_{j+\frac{1}{2}}^4) \\
& - 11735539200h^{11}\tau_{max}^5(54815097606a_j^5 + 1107739544932a_j^4a_{j+\frac{1}{2}} + 4065675623387a_j^3a_{j+\frac{1}{2}}^2 \\
& + 4065675623387a_j^2a_{j+\frac{1}{2}}^3 + 1107739544932a_ja_{j+\frac{1}{2}}^4 + 54815097606a_{j+\frac{1}{2}}^5) \\
& - 19074908160a_j^2a_{j+\frac{1}{2}}^2h^7\tau_{max}^9(179294959298488a_j^5 + 308410865284400a_j^4a_{j+\frac{1}{2}} \\
& + 770619223416953a_j^3a_{j+\frac{1}{2}}^2 + 770619223416953a_j^2a_{j+\frac{1}{2}}^3 + 308410865284400a_ja_{j+\frac{1}{2}}^4 \\
& + 179294959298488a_{j+\frac{1}{2}}^5) + 487710720h^{10}\tau_{max}^6(5441989209404a_j^6 \\
& - 26664212415965a_j^5a_{j+\frac{1}{2}} + 2805981567667107a_j^4a_{j+\frac{1}{2}}^2 - 492603505952389a_j^3a_{j+\frac{1}{2}}^3 \\
& + 2805981567667107a_j^2a_{j+\frac{1}{2}}^4 - 26664212415965a_ja_{j+\frac{1}{2}}^5 + 5441989209404a_{j+\frac{1}{2}}^6) \\
& + 1189085184a_j^2a_{j+\frac{1}{2}}^2h^6\tau_{max}^{10}(3522080594245675a_j^6 - 16385982633633415a_j^5a_{j+\frac{1}{2}} \\
& + 466662656914525505a_j^4a_{j+\frac{1}{2}}^2 - 65580553513354431a_j^3a_{j+\frac{1}{2}}^3 + 466662656914525505a_j^2a_{j+\frac{1}{2}}^4 \\
& - 16385982633633415a_ja_{j+\frac{1}{2}}^5 + 3522080594245675a_{j+\frac{1}{2}}^6) \\
& - 1788272640h^9\tau_{max}^7(1947374673365a_j^7 + 36401168398920a_j^6a_{j+\frac{1}{2}} \\
& + 422084292605924a_j^5a_{j+\frac{1}{2}}^2 + 595382564504327a_j^4a_{j+\frac{1}{2}}^3 + 595382564504327a_j^3a_{j+\frac{1}{2}}^4 \\
& + 422084292605924a_j^2a_{j+\frac{1}{2}}^5 + 36401168398920a_ja_{j+\frac{1}{2}}^6 + 1947374673365a_{j+\frac{1}{2}}^7) \\
& + 185794560h^8\tau_{max}^8(11547805227035a_j^8 - 106377383972860a_j^7a_{j+\frac{1}{2}} \\
& + 39707200376096864a_j^6a_{j+\frac{1}{2}}^2 - 22325282290601846a_j^5a_{j+\frac{1}{2}}^3 \\
& + 259867644987323578a_j^4a_{j+\frac{1}{2}}^4 - 22325282290601846a_j^3a_{j+\frac{1}{2}}^5 \\
& + 39707200376096864a_j^2a_{j+\frac{1}{2}}^6 - 106377383972860a_ja_{j+\frac{1}{2}}^7 + 11547805227035a_{j+\frac{1}{2}}^8) \\
& + 604417549765837500h^{15}\tau_{max}(a_j + a_{j+\frac{1}{2}}) + 71270165866433625h^{16})) \\
& /((10819049472(3003136a_j^4\tau_{max}^4 + 307440a_j^2h^2\tau_{max}^2 + 315h^4)^2(3003136a_{j+\frac{1}{2}}^4\tau_{max}^4 \\
& + 307440a_{j+\frac{1}{2}}^2h^2\tau_{max}^2 + 315h^4)^2) \\
& =O(h^{15})
\end{aligned}$$

7. $k = 6$, $u = x^7$, by the definition

$$\begin{aligned}
\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_I dx &= \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} x^7 dx \\
\tilde{P}_h(u_I; x; f, u)_j &= \tilde{P}_h(x^7; x; f, u)_j \\
\tilde{P}_h(u_I; x^2; f, u)_j &= \tilde{P}_h(x^7; x^2; f, u)_j \\
\tilde{P}_h(u_I; x^3; f, u)_j &= \tilde{P}_h(x^7; x^3; f, u)_j \\
\tilde{P}_h(u_I; x^4; f, u)_j &= \tilde{P}_h(x^7; x^4; f, u)_j \\
\tilde{P}_h(u_I; x^5; f, u)_j &= \tilde{P}_h(x^7; x^5; f, u)_j \\
\tilde{P}_h(u_I; x^6; f, u)_j &= \tilde{P}_h(x^7; x^6; f, u)_j \\
\int_{x_j}^{x_{j+1}} v_I dx &= \int_{x_j}^{x_{j+1}} x^7 dx \\
\tilde{Q}_h(v_I; x; f, u)_{j+\frac{1}{2}} &= \tilde{Q}_h(x^7; x; f, u)_{j+\frac{1}{2}} \\
\tilde{Q}_h(v_I; x^2; f, u)_{j+\frac{1}{2}} &= \tilde{Q}_h(x^7; x^2; f, u)_{j+\frac{1}{2}} \\
\tilde{Q}_h(v_I; x^3; f, u)_{j+\frac{1}{2}} &= \tilde{Q}_h(x^7; x^3; f, u)_{j+\frac{1}{2}} \\
\tilde{Q}_h(v_I; x^4; f, u)_{j+\frac{1}{2}} &= \tilde{Q}_h(x^7; x^4; f, u)_{j+\frac{1}{2}} \\
\tilde{Q}_h(v_I; x^5; f, u)_{j+\frac{1}{2}} &= \tilde{Q}_h(x^7; x^5; f, u)_{j+\frac{1}{2}} \\
\tilde{Q}_h(v_I; x^6; f, u)_{j+\frac{1}{2}} &= \tilde{Q}_h(x^7; x^6; f, u)_{j+\frac{1}{2}}
\end{aligned} \tag{1.15}$$

then we have

$$\begin{aligned}
u_I &= \alpha_6 x^6 + \alpha_5 x^5 + \alpha_4 x^4 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0, \quad \forall x \in I_j. \\
v_I &= \beta_6 x^6 + \beta_5 x^5 + \beta_4 x^4 + \beta_3 x^3 + \beta_2 x^2 + \beta_1 x + \beta_0, \quad \forall x \in I_{j+\frac{1}{2}}.
\end{aligned} \tag{1.16}$$

where

$$\begin{aligned}
\alpha_6 &= (7(94332080128a_j^6 \tau_{max}^6 x_j - 1280a_j^4 h^2 \tau_{max}^4 (4415546a_j \tau_{max} - 42028623x_j) \\
&\quad - 672a_j^2 h^4 \tau_{max}^2 (789340a_j \tau_{max} - 6671691x_j) + 63h^6 (67718a_j \tau_{max} + 2145x_j))) \\
&\quad / (13(7256313856a_j^6 \tau_{max}^6 + 4138202880a_j^4 h^2 \tau_{max}^4 + 344875104a_j^2 h^4 \tau_{max}^2 + 10395h^6)) \\
\alpha_5 &= (21(-754656641024a_j^6 \tau_{max}^6 x_j^2 - 504h^6 (-1271288a_j^2 \tau_{max}^2 + 135436a_j \tau_{max} x_j + 2145x_j^2) \\
&\quad + 256a_j^2 h^4 \tau_{max}^2 (28301533a_j^2 \tau_{max}^2 + 33152280a_j \tau_{max} x_j - 140105511x_j^2)
\end{aligned}$$

$$\begin{aligned}
& + 2048a_j^4 h^2 \tau_{max}^4 (4448202a_j^2 \tau_{max}^2 + 44155460a_j \tau_{max} x_j - 210143115x_j^2) + 28665h^8) \\
& /((104(7256313856a_j^6 \tau_{max}^6 + 4138202880a_j^4 h^2 \tau_{max}^4 + 344875104a_j^2 h^4 \tau_{max}^2 + 10395h^6)) \\
\alpha_4 = & - (35(-8301223051264a_j^6 \tau_{max}^6 x_j^3 + 67584a_j^4 h^2 \tau_{max}^4 x_j(4448202a_j^2 \tau_{max}^2 \\
& + 22077730a_j \tau_{max} x_j - 70047705x_j^2) - 24h^6(103797556a_j^3 \tau_{max}^3 - 881002584a_j^2 \tau_{max}^2 x_j \\
& + 46928574a_j \tau_{max} x_j^2 + 495495x_j^3) - 256a_j^2 h^4 \tau_{max}^2 (108151636a_j^3 \tau_{max}^3 \\
& - 933950589a_j^2 \tau_{max}^2 x_j - 547012620a_j \tau_{max} x_j^2 + 1541160621x_j^3) \\
& + 2079h^8(15216a_j \tau_{max} + 455x_j))) \\
& /((1144(7256313856a_j^6 \tau_{max}^6 + 4138202880a_j^4 h^2 \tau_{max}^4 + 344875104a_j^2 h^4 \tau_{max}^2 + 10395h^6)) \\
\alpha_3 = & - (35(16602446102528a_j^6 \tau_{max}^6 x_j^4 - 132h^8(-6457912a_j^2 \tau_{max}^2 + 1917216a_j \tau_{max} x_j \\
& + 28665x_j^2) - 45056a_j^4 h^2 \tau_{max}^4 x_j^2(26689212a_j^2 \tau_{max}^2 + 88310920a_j \tau_{max} x_j - 210143115x_j^2) \\
& + 16h^6(569473600a_j^4 \tau_{max}^4 + 1245570672a_j^3 \tau_{max}^3 x_j - 5286015504a_j^2 \tau_{max}^2 x_j^2 \\
& + 187714296a_j \tau_{max} x_j^3 + 1486485x_j^4) + 512a_j^2 h^4 \tau_{max}^2 (13427128a_j^4 \tau_{max}^4 \\
& + 432606544a_j^3 \tau_{max}^3 x_j - 1867901178a_j^2 \tau_{max}^2 x_j^2 - 729350160a_j \tau_{max} x_j^3 \\
& + 1541160621x_j^4) + 63063h^{10})) \\
& /((2288(7256313856a_j^6 \tau_{max}^6 + 4138202880a_j^4 h^2 \tau_{max}^4 + 344875104a_j^2 h^4 \tau_{max}^2 + 10395h^6)) \\
\alpha_2 = & (7(49807338307584a_j^6 \tau_{max}^6 x_j^5 - 675840a_j^4 h^2 \tau_{max}^4 x_j^3(8896404a_j^2 \tau_{max}^2 + 22077730a_j \tau_{max} x_j \\
& - 42028623x_j^2) - 180h^8(7154768a_j^3 \tau_{max}^3 - 71037032a_j^2 \tau_{max}^2 x_j \\
& + 10544688a_j \tau_{max} x_j^2 + 105105x_j^3) + 1536a_j^2 h^4 \tau_{max}^2 x_j(67135640a_j^4 \tau_{max}^4 \\
& + 1081516360a_j^3 \tau_{max}^3 x_j - 3113168630a_j^2 \tau_{max}^2 x_j^2 - 911687700a_j \tau_{max} x_j^3 \\
& + 1541160621x_j^4) + 16h^6(-999477488a_j^5 \tau_{max}^5 + 8542104000a_j^4 \tau_{max}^4 x_j \\
& + 9341780040a_j^3 \tau_{max}^3 x_j^2 - 26430077520a_j^2 \tau_{max}^2 x_j^3 + 703928610a_j \tau_{max} x_j^4 + 4459455x_j^5) \\
& + 99h^{10}(334172a_j \tau_{max} + 9555x_j))) \\
& /((2288(7256313856a_j^6 \tau_{max}^6 + 4138202880a_j^4 h^2 \tau_{max}^4 + 344875104a_j^2 h^4 \tau_{max}^2 + 10395h^6)) \\
\alpha_1 = & (7(-796917412921344a_j^6 \tau_{max}^6 x_j^6 - 1584h^{10}(-2907158a_j^2 \tau_{max}^2 + 2005032a_j \tau_{max} x_j
\end{aligned}$$

$$\begin{aligned}
& + 28665x_j^2) + 32440320a_j^4h^2\tau_{max}^4x_j^4(4448202a_j^2\tau_{max}^2 + 8831092a_j\tau_{max}x_j \\
& - 14009541x_j^2) + 864h^8(54537656a_j^4\tau_{max}^4 + 143095360a_j^3\tau_{max}^3x_j - 710370320a_j^2\tau_{max}^2x_j^2 \\
& + 70297920a_j\tau_{max}x_j^3 + 525525x_j^4) - 24576a_j^2h^4\tau_{max}^2x_j^2(201406920a_j^4\tau_{max}^4 \\
& + 2163032720a_j^3\tau_{max}^3x_j - 4669752945a_j^2\tau_{max}^2x_j^2 - 1094025240a_j\tau_{max}x_j^3 + 1541160621x_j^4) \\
& - 256h^6(-55859408a_j^6\tau_{max}^6 - 5996864928a_j^5\tau_{max}^5x_j + 25626312000a_j^4\tau_{max}^4x_j^2 \\
& + 18683560080a_j^3\tau_{max}^3x_j^3 - 39645116280a_j^2\tau_{max}^2x_j^4 + 844714332a_j\tau_{max}x_j^5 \\
& + 4459455x_j^6) + 637065h^{12})) \\
& /((109824(7256313856a_j^6\tau_{max}^6 + 4138202880a_j^4h^2\tau_{max}^4 + 344875104a_j^2h^4\tau_{max}^2 + 10395h^6))) \\
\alpha_0 = & (796917412921344a_j^6\tau_{max}^6x_j^7 - 15138816a_j^4h^2\tau_{max}^4x_j^5(13344606a_j^2\tau_{max}^2 \\
& + 22077730a_j\tau_{max}x_j - 30020445x_j^2) + 1008h^{10}(1438204a_j^3\tau_{max}^3 - 31978738a_j^2\tau_{max}^2x_j \\
& + 11027676a_j\tau_{max}x_j^2 + 105105x_j^3) + 172032a_j^2h^4\tau_{max}^2x_j^3(67135640a_j^4\tau_{max}^4 \\
& + 540758180a_j^3\tau_{max}^3x_j - 933950589a_j^2\tau_{max}^2x_j^2 - 182337540a_j\tau_{max}x_j^3 + 220165803x_j^4) \\
& - 32h^8(-967818016a_j^5\tau_{max}^5 + 10307616984a_j^4\tau_{max}^4x_j + 13522511520a_j^3\tau_{max}^3x_j^2 \\
& - 44753330160a_j^2\tau_{max}^2x_j^3 + 3321576720a_j\tau_{max}x_j^4 + 19864845x_j^5) \\
& + 1792h^6x_j(-55859408a_j^6\tau_{max}^6 - 2998432464a_j^5\tau_{max}^5x_j + 8542104000a_j^4\tau_{max}^4x_j^2 \\
& + 4670890020a_j^3\tau_{max}^3x_j^3 - 7929023256a_j^2\tau_{max}^2x_j^4 + 140785722a_j\tau_{max}x_j^5 \\
& + 637065x_j^6) - 693h^{12}(232088a_j\tau_{max} + 6435x_j)) \\
& /((109824(7256313856a_j^6\tau_{max}^6 + 4138202880a_j^4h^2\tau_{max}^4 + 344875104a_j^2h^4\tau_{max}^2 + 10395h^6))) \\
\beta_6 = & (7(94332080128a_{j+\frac{1}{2}}^6\tau_{max}^6x_{j+\frac{1}{2}} - 1280a_{j+\frac{1}{2}}^4h^2\tau_{max}^4(4415546a_{j+\frac{1}{2}}\tau_{max} - 42028623x_{j+\frac{1}{2}}) \\
& - 672a_{j+\frac{1}{2}}^2h^4\tau_{max}^2(789340a_{j+\frac{1}{2}}\tau_{max} - 6671691x_{j+\frac{1}{2}}) + 63h^6(67718a_{j+\frac{1}{2}}\tau_{max} + 2145x_{j+\frac{1}{2}}))) \\
& /((13(7256313856a_{j+\frac{1}{2}}^6\tau_{max}^6 + 4138202880a_{j+\frac{1}{2}}^4h^2\tau_{max}^4 + 344875104a_{j+\frac{1}{2}}^2h^4\tau_{max}^2 + 10395h^6))) \\
\beta_5 = & (21(-754656641024a_{j+\frac{1}{2}}^6\tau_{max}^6x_{j+\frac{1}{2}}^2 - 504h^6(-1271288a_{j+\frac{1}{2}}^2\tau_{max}^2 \\
& + 135436a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}} + 2145x_{j+\frac{1}{2}}^2) \\
& + 256a_{j+\frac{1}{2}}^2h^4\tau_{max}^2(28301533a_{j+\frac{1}{2}}^2\tau_{max}^2 + 33152280a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}} - 140105511x_{j+\frac{1}{2}}^2)
\end{aligned}$$

$$\begin{aligned}
& + 2048a_{j+\frac{1}{2}}^4 h^2 \tau_{max}^4 (4448202a_{j+\frac{1}{2}}^2 \tau_{max}^2 + 44155460a_{j+\frac{1}{2}} \tau_{max} x_{j+\frac{1}{2}} \\
& - 210143115x_{j+\frac{1}{2}}^2) + 28665h^8)) \\
& / (104(7256313856a_{j+\frac{1}{2}}^6 \tau_{max}^6 + 4138202880a_{j+\frac{1}{2}}^4 h^2 \tau_{max}^4 + 344875104a_{j+\frac{1}{2}}^2 h^4 \tau_{max}^2 + 10395h^6)) \\
\beta_4 = & - (35(-8301223051264a_{j+\frac{1}{2}}^6 \tau_{max}^6 x_{j+\frac{1}{2}}^3 + 67584a_{j+\frac{1}{2}}^4 h^2 \tau_{max}^4 x_{j+\frac{1}{2}} (4448202a_{j+\frac{1}{2}}^2 \tau_{max}^2 \\
& + 22077730a_{j+\frac{1}{2}} \tau_{max} x_{j+\frac{1}{2}} - 70047705x_{j+\frac{1}{2}}^2) - 24h^6(103797556a_{j+\frac{1}{2}}^3 \tau_{max}^3 \\
& - 881002584a_{j+\frac{1}{2}}^2 \tau_{max}^2 x_{j+\frac{1}{2}} \\
& + 46928574a_{j+\frac{1}{2}} \tau_{max} x_{j+\frac{1}{2}}^2 + 495495x_{j+\frac{1}{2}}^3) - 256a_{j+\frac{1}{2}}^2 h^4 \tau_{max}^2 (108151636a_{j+\frac{1}{2}}^3 \tau_{max}^3 \\
& - 933950589a_{j+\frac{1}{2}}^2 \tau_{max}^2 x_{j+\frac{1}{2}} - 547012620a_{j+\frac{1}{2}} \tau_{max} x_{j+\frac{1}{2}}^2 + 1541160621x_{j+\frac{1}{2}}^3) \\
& + 2079h^8(15216a_{j+\frac{1}{2}} \tau_{max} + 455x_{j+\frac{1}{2}}))) \\
& / (1144(7256313856a_{j+\frac{1}{2}}^6 \tau_{max}^6 + 4138202880a_{j+\frac{1}{2}}^4 h^2 \tau_{max}^4 + 344875104a_{j+\frac{1}{2}}^2 h^4 \tau_{max}^2 + 10395h^6)) \\
\beta_3 = & - (35(16602446102528a_{j+\frac{1}{2}}^6 \tau_{max}^6 x_{j+\frac{1}{2}}^4 - 132h^8(-6457912a_{j+\frac{1}{2}}^2 \tau_{max}^2 + 1917216a_{j+\frac{1}{2}} \tau_{max} x_{j+\frac{1}{2}} \\
& + 28665x_{j+\frac{1}{2}}^2) - 45056a_{j+\frac{1}{2}}^4 h^2 \tau_{max}^4 x_{j+\frac{1}{2}}^2 (26689212a_{j+\frac{1}{2}}^2 \tau_{max}^2 \\
& + 88310920a_{j+\frac{1}{2}} \tau_{max} x_{j+\frac{1}{2}} - 210143115x_{j+\frac{1}{2}}^2) \\
& + 16h^6(569473600a_{j+\frac{1}{2}}^4 \tau_{max}^4 + 1245570672a_{j+\frac{1}{2}}^3 \tau_{max}^3 x_{j+\frac{1}{2}} - 5286015504a_{j+\frac{1}{2}}^2 \tau_{max}^2 x_{j+\frac{1}{2}}^2 \\
& + 187714296a_{j+\frac{1}{2}} \tau_{max} x_{j+\frac{1}{2}}^3 + 1486485x_{j+\frac{1}{2}}^4) + 512a_{j+\frac{1}{2}}^2 h^4 \tau_{max}^2 (13427128a_{j+\frac{1}{2}}^4 \tau_{max}^4 \\
& + 432606544a_{j+\frac{1}{2}}^3 \tau_{max}^3 x_{j+\frac{1}{2}} - 1867901178a_{j+\frac{1}{2}}^2 \tau_{max}^2 x_{j+\frac{1}{2}}^2 - 729350160a_{j+\frac{1}{2}} \tau_{max} x_{j+\frac{1}{2}}^3 \\
& + 1541160621x_{j+\frac{1}{2}}^4) + 63063h^{10})) \\
& / (2288(7256313856a_{j+\frac{1}{2}}^6 \tau_{max}^6 + 4138202880a_{j+\frac{1}{2}}^4 h^2 \tau_{max}^4 + 344875104a_{j+\frac{1}{2}}^2 h^4 \tau_{max}^2 + 10395h^6)) \\
\beta_2 = & (7(49807338307584a_{j+\frac{1}{2}}^6 \tau_{max}^6 x_{j+\frac{1}{2}}^5 - 675840a_{j+\frac{1}{2}}^4 h^2 \tau_{max}^4 x_{j+\frac{1}{2}}^3 (8896404a_{j+\frac{1}{2}}^2 \tau_{max}^2 \\
& + 22077730a_{j+\frac{1}{2}} \tau_{max} x_{j+\frac{1}{2}} \\
& - 42028623x_{j+\frac{1}{2}}^2) - 180h^8(7154768a_{j+\frac{1}{2}}^3 \tau_{max}^3 - 71037032a_{j+\frac{1}{2}}^2 \tau_{max}^2 x_{j+\frac{1}{2}} \\
& + 10544688a_{j+\frac{1}{2}} \tau_{max} x_{j+\frac{1}{2}}^2 + 105105x_{j+\frac{1}{2}}^3) + 1536a_{j+\frac{1}{2}}^2 h^4 \tau_{max}^2 x_{j+\frac{1}{2}} (67135640a_{j+\frac{1}{2}}^4 \tau_{max}^4 \\
& + 1081516360a_{j+\frac{1}{2}}^3 \tau_{max}^3 x_{j+\frac{1}{2}} - 3113168630a_{j+\frac{1}{2}}^2 \tau_{max}^2 x_{j+\frac{1}{2}}^2 - 911687700a_{j+\frac{1}{2}} \tau_{max} x_{j+\frac{1}{2}}^3 \\
& + 1541160621x_{j+\frac{1}{2}}^4) + 16h^6(-999477488a_{j+\frac{1}{2}}^5 \tau_{max}^5 + 8542104000a_{j+\frac{1}{2}}^4 \tau_{max}^4 x_{j+\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
& + 9341780040a_{j+\frac{1}{2}}^3\tau_{max}^3x_{j+\frac{1}{2}}^2 - 26430077520a_{j+\frac{1}{2}}^2\tau_{max}^2x_{j+\frac{1}{2}}^3 \\
& + 703928610a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}^4 + 4459455x_{j+\frac{1}{2}}^5) \\
& + 99h^{10}(334172a_{j+\frac{1}{2}}\tau_{max} + 9555x_{j+\frac{1}{2}})) \\
& /((2288(7256313856a_{j+\frac{1}{2}}^6\tau_{max}^6 + 4138202880a_{j+\frac{1}{2}}^4h^2\tau_{max}^4 \\
& + 344875104a_{j+\frac{1}{2}}^2h^4\tau_{max}^2 + 10395h^6)) \\
\beta_1 = & (7(-796917412921344a_{j+\frac{1}{2}}^6\tau_{max}^6x_{j+\frac{1}{2}}^6 - 1584h^{10}(-2907158a_{j+\frac{1}{2}}^2\tau_{max}^2 + 2005032a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}} \\
& + 28665x_{j+\frac{1}{2}}^2) + 32440320a_{j+\frac{1}{2}}^4h^2\tau_{max}^4x_{j+\frac{1}{2}}^4(4448202a_{j+\frac{1}{2}}^2\tau_{max}^2 + 8831092a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}} \\
& - 14009541x_{j+\frac{1}{2}}^2) + 864h^8(54537656a_{j+\frac{1}{2}}^4\tau_{max}^4 \\
& + 143095360a_{j+\frac{1}{2}}^3\tau_{max}^3x_{j+\frac{1}{2}}^3 - 710370320a_{j+\frac{1}{2}}^2\tau_{max}^2x_{j+\frac{1}{2}}^2 \\
& + 70297920a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}^3 + 525525x_{j+\frac{1}{2}}^4) - 24576a_{j+\frac{1}{2}}^2h^4\tau_{max}^2x_{j+\frac{1}{2}}^2(201406920a_{j+\frac{1}{2}}^4\tau_{max}^4 \\
& + 2163032720a_{j+\frac{1}{2}}^3\tau_{max}^3x_{j+\frac{1}{2}}^3 - 4669752945a_{j+\frac{1}{2}}^2\tau_{max}^2x_{j+\frac{1}{2}}^2 \\
& - 1094025240a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}^3 + 1541160621x_{j+\frac{1}{2}}^4) \\
& - 256h^6(-55859408a_{j+\frac{1}{2}}^6\tau_{max}^6 - 5996864928a_{j+\frac{1}{2}}^5\tau_{max}^5x_{j+\frac{1}{2}}^5 + 25626312000a_{j+\frac{1}{2}}^4\tau_{max}^4x_{j+\frac{1}{2}}^2 \\
& + 18683560080a_{j+\frac{1}{2}}^3\tau_{max}^3x_{j+\frac{1}{2}}^3 - 39645116280a_{j+\frac{1}{2}}^2\tau_{max}^2x_{j+\frac{1}{2}}^4 + 844714332a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}^5 \\
& + 4459455x_{j+\frac{1}{2}}^6) + 637065h^{12})) \\
& /((109824(7256313856a_{j+\frac{1}{2}}^6\tau_{max}^6 + 4138202880a_{j+\frac{1}{2}}^4h^2\tau_{max}^4 \\
& + 344875104a_{j+\frac{1}{2}}^2h^4\tau_{max}^2 + 10395h^6)) \\
\beta_0 = & (796917412921344a_{j+\frac{1}{2}}^6\tau_{max}^6x_{j+\frac{1}{2}}^7 - 15138816a_{j+\frac{1}{2}}^4h^2\tau_{max}^4x_{j+\frac{1}{2}}^5(13344606a_{j+\frac{1}{2}}^2\tau_{max}^2 \\
& + 22077730a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}} - 30020445x_{j+\frac{1}{2}}^2) \\
& + 1008h^{10}(1438204a_{j+\frac{1}{2}}^3\tau_{max}^3 - 31978738a_{j+\frac{1}{2}}^2\tau_{max}^2x_{j+\frac{1}{2}} \\
& + 11027676a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}^2 + 105105x_{j+\frac{1}{2}}^3) + 172032a_{j+\frac{1}{2}}^2h^4\tau_{max}^2x_{j+\frac{1}{2}}^3(67135640a_{j+\frac{1}{2}}^4\tau_{max}^4 \\
& + 540758180a_{j+\frac{1}{2}}^3\tau_{max}^3x_{j+\frac{1}{2}}^3 - 933950589a_{j+\frac{1}{2}}^2\tau_{max}^2x_{j+\frac{1}{2}}^2 \\
& - 182337540a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}^3 + 220165803x_{j+\frac{1}{2}}^4) \\
& - 32h^8(-967818016a_{j+\frac{1}{2}}^5\tau_{max}^5 + 10307616984a_{j+\frac{1}{2}}^4\tau_{max}^4x_{j+\frac{1}{2}}^4 + 13522511520a_{j+\frac{1}{2}}^3\tau_{max}^3x_{j+\frac{1}{2}}^2
\end{aligned}$$

$$\begin{aligned}
& - 44753330160a_{j+\frac{1}{2}}^2\tau_{max}^2x_{j+\frac{1}{2}}^3 + 3321576720a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}^4 + 19864845x_{j+\frac{1}{2}}^5) \\
& + 1792h^6x_{j+\frac{1}{2}}(-55859408a_{j+\frac{1}{2}}^6\tau_{max}^6 - 2998432464a_{j+\frac{1}{2}}^5\tau_{max}^5x_{j+\frac{1}{2}} + 8542104000a_{j+\frac{1}{2}}^4\tau_{max}^4x_{j+\frac{1}{2}}^2 \\
& + 4670890020a_{j+\frac{1}{2}}^3\tau_{max}^3x_{j+\frac{1}{2}}^3 - 7929023256a_{j+\frac{1}{2}}^2\tau_{max}^2x_{j+\frac{1}{2}}^4 + 140785722a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}^5 \\
& + 637065x_{j+\frac{1}{2}}^6) - 693h^{12}(232088a_{j+\frac{1}{2}}\tau_{max} + 6435x_{j+\frac{1}{2}})) \\
& /((109824(7256313856a_{j+\frac{1}{2}}^6\tau_{max}^6 + 4138202880a_{j+\frac{1}{2}}^4h^2\tau_{max}^4 \\
& + 344875104a_{j+\frac{1}{2}}^2h^4\tau_{max}^2 + 10395h^6))
\end{aligned}$$

Hence, we have

$$\begin{aligned}
& \int_{x_j}^{x_{j+\frac{1}{2}}} (v_I - x^7 - u_I(x - \frac{h}{2}) + (x - \frac{h}{2})^7)^2 \\
& = ((a_j - a_{j+\frac{1}{2}})^2 h^{17} \tau_{max}^2 (437403046994236795781925h^{20} \\
& + 8859873555495770497489800(a_j + a_{j+\frac{1}{2}})\tau_{max}h^{19} \\
& + 1485279180(1699073037709623145a_j^2 - 16158953949797393574a_{j+\frac{1}{2}}a_j \\
& + 1699073037709623145a_{j+\frac{1}{2}}^2)\tau_{max}^2h^{18} - 2973528918360(1414536046558650a_j^3 \\
& + 101689335736344761a_{j+\frac{1}{2}}a_j^2 + 101689335736344761a_{j+\frac{1}{2}}^2a_j \\
& + 1414536046558650a_{j+\frac{1}{2}}^3)\tau_{max}^3h^{17} + 3168595584(20961537958383611375a_j^4 \\
& - 322425487440101544600a_{j+\frac{1}{2}}a_j^3 + 151880015288861843693378a_{j+\frac{1}{2}}^2a_j^2 \\
& - 322425487440101544600a_{j+\frac{1}{2}}^3a_j + 20961537958383611375a_{j+\frac{1}{2}}^4)\tau_{max}^4h^{16} \\
& - 257448391200(420292950978969849a_j^5 + 17072570927322274705a_{j+\frac{1}{2}}a_j^4 \\
& + 586768804488348852256a_{j+\frac{1}{2}}^2a_j^3 + 586768804488348852256a_{j+\frac{1}{2}}^3a_j^2 \\
& + 17072570927322274705a_{j+\frac{1}{2}}^4a_j + 420292950978969849a_{j+\frac{1}{2}}^5)\tau_{max}^5h^{15} \\
& + 3840721920(156595436000740513590a_j^6 - 3699346840438621615628a_{j+\frac{1}{2}}a_j^5 \\
& + 3109924558495450458531461a_{j+\frac{1}{2}}^2a_j^4 - 3855737499325029678375014a_{j+\frac{1}{2}}^3a_j^3 \\
& + 3109924558495450458531461a_{j+\frac{1}{2}}^4a_j^2 - 3699346840438621615628a_{j+\frac{1}{2}}^5a_j \\
& + 156595436000740513590a_{j+\frac{1}{2}}^6)\tau_{max}^6h^{14} - 549223234560(1593068096417356625a_j^7 \\
& - 11325611182537259267a_{j+\frac{1}{2}}a_j^6 + 7442033876711042429922a_{j+\frac{1}{2}}^2a_j^5
\end{aligned}$$

$$\begin{aligned}
& - 165818214087284083403612a_{j+\frac{1}{2}}^3a_j^4 - 165818214087284083403612a_{j+\frac{1}{2}}^4a_j^3 \\
& + 7442033876711042429922a_{j+\frac{1}{2}}^5a_j^2 - 11325611182537259267a_{j+\frac{1}{2}}^6a_j \\
& + 1593068096417356625a_{j+\frac{1}{2}}^7\tau_{max}^7h^{13} + 585252864(3667960856961456465300a_j^8 \\
& - 126786200646401196602900a_{j+\frac{1}{2}}^7a_j^7 + 167960141291628617113597954a_{j+\frac{1}{2}}^2a_j^6 \\
& - 552419651770920469669003840a_{j+\frac{1}{2}}^3a_j^5 + 11457736972910639970259979445a_{j+\frac{1}{2}}^4a_j^4 \\
& - 552419651770920469669003840a_{j+\frac{1}{2}}^5a_j^3 + 167960141291628617113597954a_{j+\frac{1}{2}}^6a_j^2 \\
& - 126786200646401196602900a_{j+\frac{1}{2}}^7a_j + 3667960856961456465300a_{j+\frac{1}{2}}^8\tau_{max}^8h^{12} \\
& - 19020718080(116883680195507193925a_j^9 - 12913068248935915017815a_{j+\frac{1}{2}}^8a_j^8 \\
& + 1986166975590076367754380a_{j+\frac{1}{2}}^2a_j^7 - 107237984339516532769057736a_{j+\frac{1}{2}}^3a_j^6 \\
& + 676505105051936160968017607a_{j+\frac{1}{2}}^4a_j^5 + 676505105051936160968017607a_{j+\frac{1}{2}}^5a_j^4 \\
& - 107237984339516532769057736a_{j+\frac{1}{2}}^6a_j^3 + 1986166975590076367754380a_{j+\frac{1}{2}}^7a_j^2 \\
& - 12913068248935915017815a_{j+\frac{1}{2}}^8a_j + 116883680195507193925a_{j+\frac{1}{2}}^9\tau_{max}^9h^{11} \\
& + 3901685760(653854776598314061485a_j^{10} - 26347537693490147722742a_{j+\frac{1}{2}}^9a_j^9 \\
& + 77009004765064962422325163a_{j+\frac{1}{2}}^2a_j^8 - 527477199135104510286360772a_{j+\frac{1}{2}}^3a_j^7 \\
& + 37103073127910502944858616329a_{j+\frac{1}{2}}^4a_j^6 - 713564523544966917683070426a_{j+\frac{1}{2}}^5a_j^5 \\
& + 37103073127910502944858616329a_{j+\frac{1}{2}}^6a_j^4 - 527477199135104510286360772a_{j+\frac{1}{2}}^7a_j^3 \\
& + 77009004765064962422325163a_{j+\frac{1}{2}}^8a_j^2 - 26347537693490147722742a_{j+\frac{1}{2}}^9a_j \\
& + 653854776598314061485a_{j+\frac{1}{2}}^{10}\tau_{max}^{10}h^{10} + 101443829760a_ja_{j+\frac{1}{2}}(4559721353965663034369a_j^9 \\
& - 1154079108387189138422257a_{j+\frac{1}{2}}^8a_j^8 + 129370543069772118288464708a_{j+\frac{1}{2}}^2a_j^7 \\
& - 2767084241034372123203306878a_{j+\frac{1}{2}}^3a_j^6 - 2142066506103509583429763149a_{j+\frac{1}{2}}^4a_j^5 \\
& - 2142066506103509583429763149a_{j+\frac{1}{2}}^5a_j^4 - 2767084241034372123203306878a_{j+\frac{1}{2}}^6a_j^3 \\
& + 129370543069772118288464708a_{j+\frac{1}{2}}^7a_j^2 - 1154079108387189138422257a_{j+\frac{1}{2}}^8a_j \\
& + 4559721353965663034369a_{j+\frac{1}{2}}^9\tau_{max}^{11}h^9 \\
& + 16647192576a_{j+\frac{1}{2}}^2a_j^2(18461283971311741166192542a_j^8
\end{aligned}$$

$$\begin{aligned}
& - 183011002492177671027896240a_{j+\frac{1}{2}}a_j^7 + 56697160731182582899942819925a_{j+\frac{1}{2}}^2a_j^6 \\
& + 2372628401241137086503337520a_{j+\frac{1}{2}}^3a_j^5 + 190469561341043092690908870388a_{j+\frac{1}{2}}^4a_j^4 \\
& + 2372628401241137086503337520a_{j+\frac{1}{2}}^5a_j^3 + 56697160731182582899942819925a_{j+\frac{1}{2}}^6a_j^2 \\
& - 183011002492177671027896240a_{j+\frac{1}{2}}^7a_j + 18461283971311741166192542a_{j+\frac{1}{2}}^8)\tau_{max}^{12}h^8 \\
& + 1082067517440a_j^3a_{j+\frac{1}{2}}^3(18321852318992025713799428a_j^7 \\
& - 1321003108505324002163493396a_{j+\frac{1}{2}}^6 \\
& - 498594225191753361165098406a_{j+\frac{1}{2}}^2a_j^5 - 4512798124101130032009521037a_{j+\frac{1}{2}}^3a_j^4 \\
& - 4512798124101130032009521037a_{j+\frac{1}{2}}^4a_j^3 - 498594225191753361165098406a_{j+\frac{1}{2}}^5a_j^2 \\
& - 1321003108505324002163493396a_{j+\frac{1}{2}}^6a_j + 18321852318992025713799428a_{j+\frac{1}{2}}^7)\tau_{max}^{13}h^7 \\
& + 31708938240a_j^4a_{j+\frac{1}{2}}^4(48802141800690392120765929181a_j^6 \\
& + 11048478168216937331149943798a_{j+\frac{1}{2}}a_j^5 + 651595638176670808303246681584a_{j+\frac{1}{2}}^2a_j^4 \\
& + 43557999647137173043199301698a_{j+\frac{1}{2}}^3a_j^3 + 651595638176670808303246681584a_{j+\frac{1}{2}}^4a_j^2 \\
& + 11048478168216937331149943798a_{j+\frac{1}{2}}^5a_j + 48802141800690392120765929181a_{j+\frac{1}{2}}^6)\tau_{max}^{14}h^6 \\
& + 1097319516733440a_j^5a_{j+\frac{1}{2}}^5(3198843901481228518023451a_j^5 \\
& - 22843767273592440900178964a_{j+\frac{1}{2}}a_j^4 - 13165800443519527815052054a_{j+\frac{1}{2}}^2a_j^3 \\
& - 13165800443519527815052054a_{j+\frac{1}{2}}^3a_j^2 - 22843767273592440900178964a_{j+\frac{1}{2}}^4a_j \\
& + 3198843901481228518023451a_{j+\frac{1}{2}}^5)\tau_{max}^{15}h^5 \\
& + 67645734912a_j^6a_{j+\frac{1}{2}}^6(492258158772799267571707352258a_j^4 \\
& + 115400788701858795739154476200a_{j+\frac{1}{2}}a_j^3 + 1968592966632694779138376243595a_{j+\frac{1}{2}}^2a_j^2 \\
& + 115400788701858795739154476200a_{j+\frac{1}{2}}^3a_j + 492258158772799267571707352258a_{j+\frac{1}{2}}^4)\tau_{max}^{16}h^4 \\
& + 5852370755911680a_j^7a_{j+\frac{1}{2}}^7(10862152607395926487179716a_j^3 \\
& - 13228624002089287227191317a_{j+\frac{1}{2}}a_j^2 - 13228624002089287227191317a_{j+\frac{1}{2}}^2a_j \\
& + 10862152607395926487179716a_{j+\frac{1}{2}}^3)\tau_{max}^{17}h^3 \\
& + 798923638576250880a_j^8a_{j+\frac{1}{2}}^8(258161172004769893077187a_j^2
\end{aligned}$$

$$\begin{aligned}
& + 50057250128499149598270a_{j+\frac{1}{2}}a_j + 258161172004769893077187a_{j+\frac{1}{2}}^2\tau_{max}^{18}h^2 \\
& + 304539049496485812590397344184454726287360a_j^9a_{j+\frac{1}{2}}^9(a_j + a_{j+\frac{1}{2}})\tau_{max}^{19}h \\
& + 276774410375026167841490291916617329672192a_j^{10}a_{j+\frac{1}{2}}^{10}\tau_{max}^{20})) \\
& /((6124884480(10395h^6 + 344875104a_j^2\tau_{max}^2h^4 + 4138202880a_j^4\tau_{max}^4h^2 \\
& + 7256313856a_j^6\tau_{max}^6)^2(10395h^6 + 344875104a_{j+\frac{1}{2}}^2\tau_{max}^2h^4 + 4138202880a_{j+\frac{1}{2}}^4\tau_{max}^4h^2 \\
& + 7256313856a_{j+\frac{1}{2}}^6\tau_{max}^6)^2) \\
& =O(h^{17})
\end{aligned}$$

8. $k = 7, u = x^8$, by the definition

$$\begin{aligned}
\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_I dx &= \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} x^8 dx \\
\tilde{P}_h(u_I; x^m; f, u)_j &= \tilde{P}_h(x^8; x^m; f, u)_j, \quad m = 1, \dots, 7 \\
\int_{x_j}^{x_{j+1}} v_I dx &= \int_{x_j}^{x_{j+1}} x^8 dx \\
\tilde{Q}_h(v_I; x^m; f, u)_{j+\frac{1}{2}} &= \tilde{Q}_h(x^8; x^m; f, u)_{j+\frac{1}{2}}, \quad m = 1, \dots, 7
\end{aligned} \tag{1.17}$$

then we have

$$\begin{aligned}
u_I &= \sum_{m=0}^7 \alpha_m x^m, \quad \forall x \in I_j. \\
v_I &= \sum_{m=0}^7 \beta_m x^m, \quad \forall x \in I_{j+\frac{1}{2}}.
\end{aligned} \tag{1.18}$$

where

$$\begin{aligned}
\alpha_7 &= (8(4096a_j^6\tau_{max}^6(2354497739a_j\tau_{max} + 239935274463x_j) \\
& - 6912a_j^4h^2\tau_{max}^4(879791707a_j\tau_{max} - 21360056375x_j) \\
& - 54432a_j^2h^4\tau_{max}^2(13002767a_j\tau_{max} - 89468841x_j) \\
& + 51975h^6(14369a_j\tau_{max} + 39x_j))) \\
& /((3(327591628066816a_j^6\tau_{max}^6 + 49213569888000a_j^4h^2\tau_{max}^4 \\
& + 1623322651104a_j^2h^4\tau_{max}^2 + 675675h^6)) \\
\alpha_6 &= (28(-20480a_j^6\tau_{max}^6x_j(4708995478a_j\tau_{max} + 239935274463x_j)
\end{aligned}$$

$$\begin{aligned}
& - 945h^6(-407585262a_j^2\tau_{max}^2 + 7902950a_j\tau_{max}x_j + 10725x_j^2) \\
& + 288a_j^2h^4\tau_{max}^2(36101338556a_j^2\tau_{max}^2 + 24575229630a_j\tau_{max}x_j - 84548054745x_j^2) \\
& + 1280a_j^4h^2\tau_{max}^4(50856691162a_j^2\tau_{max}^2 + 47508752178a_j\tau_{max}x_j \\
& - 576721522125x_j^2) + 259875h^8)) \\
& /((15(327591628066816a_j^6\tau_{max}^6 + 49213569888000a_j^4h^2\tau_{max}^4 \\
& + 1623322651104a_j^2h^4\tau_{max}^2 + 675675h^6)) \\
\alpha_5 = & - (56(-266240a_j^6\tau_{max}^6x_j^2(2354497739a_j\tau_{max} + 79978424821x_j) \\
& - 9h^6(95613443942a_j^3\tau_{max}^3 - 556353882630a_j^2\tau_{max}^2x_j + 5393763375a_j\tau_{max}x_j^2 \\
& + 4879875x_j^3) - 288a_j^2h^4\tau_{max}^2(27266771676a_j^3\tau_{max}^3 - 469317401228a_j^2\tau_{max}^2x_j \\
& - 159738992595a_j\tau_{max}x_j^2 + 366374903895x_j^3) + 1280a_j^4h^2\tau_{max}^4(5911894398a_j^3\tau_{max}^3 \\
& + 661136985106a_j^2\tau_{max}^2x_j + 308806889157a_j\tau_{max}x_j^2 - 2499126595875x_j^3) \\
& + 61425h^8(21053a_j\tau_{max} + 55x_j))) \\
& /((65(327591628066816a_j^6\tau_{max}^6 + 49213569888000a_j^4h^2\tau_{max}^4 \\
& + 1623322651104a_j^2h^4\tau_{max}^2 + 675675h^6)) \\
\alpha_4 = & - ((7(2641275h^{10} + 2129920a_j^6x_j^3\tau_{max}^6(239935274463x_j + 9417990956a_j\tau_{max}) \\
& - 5616h^8(28875x_j^2 + 22105650a_jx_j\tau_{max} - 383499197a_j^2\tau_{max}^2) \\
& - 30720a_j^4h^2x_j\tau_{max}^4(-2499126595875x_j^3 + 411742518876a_jx_j^2\tau_{max} \\
& + 1322273970212a_j^2x_j\tau_{max}^2 + 23647577592a_j^3\tau_{max}^3) \\
& + 72h^6(14639625x_j^4 + 21575053500a_jx_j^3\tau_{max} - 3338123295780a_j^2x_j^2\tau_{max}^2 \\
& + 1147361327304a_j^3x_j\tau_{max}^3 + 687203235848a_j^4\tau_{max}^4) + 256a_j^2h^4\tau_{max}^2(9892122405165x_j^4 \\
& - 5750603733420a_jx_j^3\tau_{max} - 25343139666312a_j^2x_j^2\tau_{max}^2 + 2944811341008a_j^3x_j\tau_{max}^3 \\
& + 1087949803400a_j^4\tau_{max}^4))) \\
& /((156(675675h^6 + 1623322651104a_j^2h^4\tau_{max}^2 \\
& + 49213569888000a_j^4h^2\tau_{max}^4 + 327591628066816a_j^6\tau_{max}^6)))
\end{aligned}$$

$$\begin{aligned}
\alpha_3 = & (7(96525h^{10}(301x_j + 118347a_j\tau_{max}) + 4685824a_j^6x_j^4\tau_{max}^6(239935274463x_j \\
& + 11772488695a_j\tau_{max}) - 4752h^8(125125x_j^3 + 143686725a_jx_j^2\tau_{max} \\
& - 4985489561a_j^2x_j\tau_{max}^2 + 985840581a_j^3\tau_{max}^3) - 112640a_j^4h^2x_j^2\tau_{max}^4(-1499475957525x_j^3 \\
& + 308806889157a_jx_j^2\tau_{max} + 1322273970212a_j^2x_j\tau_{max}^2 + 35471366388a_j^3\tau_{max}^3) \\
& + 72h^6(32207175x_j^5 + 59331397125a_jx_j^4\tau_{max} - 12239785417860a_j^2x_j^3\tau_{max}^2 \\
& + 6310487300172a_j^3x_j^2\tau_{max}^3 + 7559235594328a_j^4x_j\tau_{max}^4 - 625703362520a_j^5\tau_{max}^5) \\
& + 256a_j^2h^4\tau_{max}^2(21762669291363x_j^5 - 15814160266905a_jx_j^4\tau_{max} - 92924845443144a_j^2x_j^3\tau_{max}^2 \\
& + 16196462375544a_j^3x_j^2\tau_{max}^3 + 11967447837400a_j^4x_j\tau_{max}^4 + 89226796360a_j^5\tau_{max}^5))) \\
& /((429(675675h^6 + 1623322651104a_j^2h^4\tau_{max}^2 \\
& + 49213569888000a_j^4h^2\tau_{max}^4 + 327591628066816a_j^6\tau_{max}^6))) \\
\alpha_2 = & -((7(-2413125h^{12} + 28114944a_j^6x_j^5\tau_{max}^6(79978424821x_j + 4708995478a_j\tau_{max}) \\
& + 38610h^{10}(4515x_j^2 + 3550410a_jx_j\tau_{max} - 25259534a_j^2\tau_{max}^2) \\
& - 135168a_j^4h^2x_j^3\tau_{max}^4(-2499126595875x_j^3 + 617613778314a_jx_j^2\tau_{max} \\
& + 3305684925530a_j^2x_j\tau_{max}^2 + 118237887960a_j^3\tau_{max}^3) - 4752h^8(375375x_j^4 \\
& + 574746900a_jx_j^3\tau_{max} - 29912937366a_j^2x_j^2\tau_{max}^2 + 11830086972a_j^3x_j\tau_{max}^3 \\
& + 3794229608a_j^4\tau_{max}^4) + 1536a_j^2h^4x_j\tau_{max}^2(7254223097121x_j^5 - 6325664106762a_jx_j^4\tau_{max} \\
& - 46462422721572a_j^2x_j^3\tau_{max}^2 + 10797641583696a_j^3x_j^2\tau_{max}^3 + 11967447837400a_j^4x_j\tau_{max}^4 \\
& + 178453592720a_j^5\tau_{max}^5) + 16h^6(289864575x_j^6 + 640779088950a_jx_j^5\tau_{max} \\
& - 165237103141110a_j^2x_j^4\tau_{max}^2 + 113588771403096a_j^3x_j^3\tau_{max}^3 + 204099361046856a_j^4x_j^2\tau_{max}^4 \\
& - 33787981576080a_j^5x_j\tau_{max}^5 - 5178530366512a_j^6\tau_{max}^6))) \\
& /((1716(675675h^6 + 1623322651104a_j^2h^4\tau_{max}^2 \\
& + 49213569888000a_j^4h^2\tau_{max}^4 + 327591628066816a_j^6\tau_{max}^6))) \\
\alpha_1 = & (-675675h^{12}(375x_j + 149849a_j\tau_{max}) + 140574720a_j^6x_j^6\tau_{max}^6(239935274463x_j \\
& + 16481484173a_j\tau_{max}) + 4158h^{10}(1467375x_j^3 + 1730824875a_jx_j^2\tau_{max}
\end{aligned}$$

$$\begin{aligned}
& - 24628045650a_j^2x_j\tau_{max}^2 + 5177901778a_j^3\tau_{max}^3) - 14192640a_j^4h^2x_j^4\tau_{max}^4(-357018085125x_j^3 \\
& + 102935629719a_jx_j^2\tau_{max} + 661136985106a_j^2x_j\tau_{max}^2 + 29559471990a_j^3\tau_{max}^3) \\
& - 720h^8(52026975x_j^5 + 99574900425a_jx_j^4\tau_{max} - 6909888531546a_j^2x_j^3\tau_{max}^2 \\
& + 4099125135798a_j^3x_j^2\tau_{max}^3 + 2629401118344a_j^4x_j\tau_{max}^4 - 303683537672a_j^5\tau_{max}^5) \\
& + 10752a_j^2h^4x_j^2\tau_{max}^2(15544763779545x_j^5 - 15814160266905a_jx_j^4\tau_{max} \\
& - 139387268164716a_j^2x_j^3\tau_{max}^2 + 40491155938860a_j^3x_j^2\tau_{max}^3 + 59837239187000a_j^4x_j\tau_{max}^4 \\
& + 1338401945400a_j^5\tau_{max}^5) + 560h^6(124227675x_j^7 + 320389544475a_jx_j^6\tau_{max} \\
& - 99142261884666a_j^2x_j^5\tau_{max}^2 + 85191578552322a_j^3x_j^4\tau_{max}^3 + 204099361046856a_j^4x_j^3\tau_{max}^4 \\
& - 50681972364120a_j^5x_j^2\tau_{max}^5 - 15535591099536a_j^6x_j\tau_{max}^6 - 74240090992a_j^7\tau_{max}^7)) \\
& /((12870(675675h^6 + 1623322651104a_j^2h^4\tau_{max}^2 \\
& + 49213569888000a_j^4h^2\tau_{max}^4 + 327591628066816a_j^6\tau_{max}^6)) \\
\alpha_0 = & - ((211486275h^{14} + 2249195520a_j^6x_j^7\tau_{max}^6(239935274463x_j + 18835981912a_j\tau_{max})) \\
& - 1729728h^{12}(9375x_j^2 + 7492450a_jx_j\tau_{max} - 22675009a_j^2\tau_{max}^2) \\
& - 10813440a_j^4h^2x_j^5\tau_{max}^4(-7497379787625x_j^3 + 2470455113256a_jx_j^2\tau_{max} \\
& + 18511835582968a_j^2x_j\tau_{max}^2 + 993198258864a_j^3\tau_{max}^3) + 6336h^{10}(30814875x_j^4 \\
& + 48463096500a_jx_j^3\tau_{max} - 1034377917300a_j^2x_j^2\tau_{max}^2 + 434943749352a_j^3x_j\tau_{max}^3 \\
& + 87945430928a_j^4\tau_{max}^4) + 172032a_j^2h^4x_j^3\tau_{max}^2(15544763779545x_j^5 \\
& - 18073326019320a_jx_j^4\tau_{max} - 185849690886288a_j^2x_j^3\tau_{max}^2 + 64785849502176a_j^3x_j^2\tau_{max}^3 \\
& + 119674478374000a_j^4x_j\tau_{max}^4 + 3569071854400a_j^5\tau_{max}^5) - 5120h^8(156080925x_j^6 \\
& + 358469641530a_jx_j^5\tau_{max} - 31094498391957a_j^2x_j^4\tau_{max}^2 + 24594750814788a_j^3x_j^3\tau_{max}^3 \\
& + 23664610065096a_j^4x_j^2\tau_{max}^4 - 5466303678096a_j^5x_j\tau_{max}^5 - 321570260696a_j^6\tau_{max}^6) \\
& + 1792h^6x_j(621138375x_j^7 + 1830797397000a_jx_j^6\tau_{max} - 660948412564440a_j^2x_j^5\tau_{max}^2 \\
& + 681532628418576a_j^3x_j^4\tau_{max}^3 + 2040993610468560a_j^4x_j^3\tau_{max}^4 - 675759631521600a_j^5x_j^2\tau_{max}^5 \\
& - 310711821990720a_j^6x_j\tau_{max}^6 - 2969603639680a_j^7\tau_{max}^7))
\end{aligned}$$

$$\begin{aligned}
& /((1647360(675675h^6 + 1623322651104a_j^2h^4\tau_{max}^2 \\
& + 49213569888000a_j^4h^2\tau_{max}^4 + 327591628066816a_j^6\tau_{max}^6))) \\
\beta_7 = & (8(4096a_{j+\frac{1}{2}}^6\tau_{max}^6(2354497739a_{j+\frac{1}{2}}\tau_{max} + 239935274463x_{j+\frac{1}{2}}) \\
& - 6912a_{j+\frac{1}{2}}^4h^2\tau_{max}^4(879791707a_{j+\frac{1}{2}}\tau_{max} - 21360056375x_{j+\frac{1}{2}}) \\
& - 54432a_{j+\frac{1}{2}}^2h^4\tau_{max}^2(13002767a_{j+\frac{1}{2}}\tau_{max} - 89468841x_{j+\frac{1}{2}}) \\
& + 51975h^6(14369a_{j+\frac{1}{2}}\tau_{max} + 39x_{j+\frac{1}{2}}))) \\
& /((3(327591628066816a_{j+\frac{1}{2}}^6\tau_{max}^6 + 49213569888000a_{j+\frac{1}{2}}^4h^2\tau_{max}^4 \\
& + 1623322651104a_{j+\frac{1}{2}}^2h^4\tau_{max}^2 + 675675h^6)) \\
\beta_6 = & (28(-20480a_{j+\frac{1}{2}}^6\tau_{max}^6x_{j+\frac{1}{2}}(4708995478a_{j+\frac{1}{2}}\tau_{max} + 239935274463x_{j+\frac{1}{2}}) \\
& - 945h^6(-407585262a_{j+\frac{1}{2}}^2\tau_{max}^2 + 7902950a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}} + 10725x_{j+\frac{1}{2}}^2) \\
& + 288a_{j+\frac{1}{2}}^2h^4\tau_{max}^2(36101338556a_{j+\frac{1}{2}}^2\tau_{max}^2 + 24575229630a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}} - 84548054745x_{j+\frac{1}{2}}^2) \\
& + 1280a_{j+\frac{1}{2}}^4h^2\tau_{max}^4(50856691162a_{j+\frac{1}{2}}^2\tau_{max}^2 + 47508752178a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}} \\
& - 576721522125x_{j+\frac{1}{2}}^2) + 259875h^8)) \\
& /((15(327591628066816a_{j+\frac{1}{2}}^6\tau_{max}^6 + 49213569888000a_{j+\frac{1}{2}}^4h^2\tau_{max}^4 \\
& + 1623322651104a_{j+\frac{1}{2}}^2h^4\tau_{max}^2 + 675675h^6)) \\
\beta_5 = & -(56(-266240a_{j+\frac{1}{2}}^6\tau_{max}^6x_{j+\frac{1}{2}}^2(2354497739a_{j+\frac{1}{2}}\tau_{max} + 79978424821x_{j+\frac{1}{2}}) \\
& - 9h^6(95613443942a_{j+\frac{1}{2}}^3\tau_{max}^3 - 556353882630a_{j+\frac{1}{2}}^2\tau_{max}^2x_{j+\frac{1}{2}} + 5393763375a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}^2 \\
& + 4879875x_{j+\frac{1}{2}}^3) - 288a_{j+\frac{1}{2}}^2h^4\tau_{max}^2(27266771676a_{j+\frac{1}{2}}^3\tau_{max}^3 - 469317401228a_{j+\frac{1}{2}}^2\tau_{max}^2x_{j+\frac{1}{2}} \\
& - 159738992595a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}^2 + 366374903895x_{j+\frac{1}{2}}^3) + 1280a_{j+\frac{1}{2}}^4h^2\tau_{max}^4(5911894398a_{j+\frac{1}{2}}^3\tau_{max}^3 \\
& + 661136985106a_{j+\frac{1}{2}}^2\tau_{max}^2x_{j+\frac{1}{2}} + 308806889157a_{j+\frac{1}{2}}\tau_{max}x_{j+\frac{1}{2}}^2 - 2499126595875x_{j+\frac{1}{2}}^3) \\
& + 61425h^8(21053a_{j+\frac{1}{2}}\tau_{max} + 55x_{j+\frac{1}{2}}))) \\
& /((65(327591628066816a_{j+\frac{1}{2}}^6\tau_{max}^6 + 49213569888000a_{j+\frac{1}{2}}^4h^2\tau_{max}^4 \\
& + 1623322651104a_{j+\frac{1}{2}}^2h^4\tau_{max}^2 + 675675h^6)) \\
\beta_4 = & -((7(2641275h^{10} + 2129920a_{j+\frac{1}{2}}^6x_{j+\frac{1}{2}}^3\tau_{max}^6(239935274463x_{j+\frac{1}{2}} + 9417990956a_{j+\frac{1}{2}}\tau_{max}))
\end{aligned}$$

$$\begin{aligned}
& - 5616h^8(28875x_{j+\frac{1}{2}}^2 + 22105650a_{j+\frac{1}{2}}x_{j+\frac{1}{2}}\tau_{max} - 383499197a_{j+\frac{1}{2}}^2\tau_{max}^2) \\
& - 30720a_{j+\frac{1}{2}}^4h^2x_{j+\frac{1}{2}}\tau_{max}^4(-2499126595875x_{j+\frac{1}{2}}^3 + 411742518876a_{j+\frac{1}{2}}x_{j+\frac{1}{2}}^2\tau_{max} \\
& + 1322273970212a_{j+\frac{1}{2}}^2x_{j+\frac{1}{2}}\tau_{max}^2 + 23647577592a_{j+\frac{1}{2}}^3\tau_{max}^3) \\
& + 72h^6(14639625x_{j+\frac{1}{2}}^4 + 21575053500a_{j+\frac{1}{2}}x_{j+\frac{1}{2}}^3\tau_{max} - 3338123295780a_{j+\frac{1}{2}}^2x_{j+\frac{1}{2}}^2\tau_{max}^2 \\
& + 1147361327304a_{j+\frac{1}{2}}^3x_{j+\frac{1}{2}}\tau_{max}^3 + 687203235848a_{j+\frac{1}{2}}^4\tau_{max}^4) \\
& + 256a_{j+\frac{1}{2}}^2h^4\tau_{max}^2(9892122405165x_{j+\frac{1}{2}}^4 - 5750603733420a_{j+\frac{1}{2}}x_{j+\frac{1}{2}}^3\tau_{max} \\
& - 25343139666312a_{j+\frac{1}{2}}^2x_{j+\frac{1}{2}}^2\tau_{max}^2 + 2944811341008a_{j+\frac{1}{2}}^3x_{j+\frac{1}{2}}\tau_{max}^3 \\
& + 1087949803400a_{j+\frac{1}{2}}^4\tau_{max}^4)) \\
& /((156(675675h^6 + 1623322651104a_{j+\frac{1}{2}}^2h^4\tau_{max}^2 \\
& + 49213569888000a_{j+\frac{1}{2}}^4h^2\tau_{max}^4 + 327591628066816a_{j+\frac{1}{2}}^6\tau_{max}^6))) \\
\beta_3 = & (7(96525h^{10}(301x_{j+\frac{1}{2}} + 118347a_{j+\frac{1}{2}}\tau_{max}) + 4685824a_{j+\frac{1}{2}}^6x_{j+\frac{1}{2}}^4\tau_{max}^6(239935274463x_{j+\frac{1}{2}} \\
& + 11772488695a_{j+\frac{1}{2}}\tau_{max}) - 4752h^8(125125x_{j+\frac{1}{2}}^3 + 143686725a_{j+\frac{1}{2}}x_{j+\frac{1}{2}}^2\tau_{max} \\
& - 4985489561a_{j+\frac{1}{2}}^2x_{j+\frac{1}{2}}\tau_{max}^2 + 985840581a_{j+\frac{1}{2}}^3\tau_{max}^3) \\
& - 112640a_{j+\frac{1}{2}}^4h^2x_{j+\frac{1}{2}}^2\tau_{max}^4(-1499475957525x_{j+\frac{1}{2}}^3 \\
& + 308806889157a_{j+\frac{1}{2}}x_{j+\frac{1}{2}}^2\tau_{max} + 1322273970212a_{j+\frac{1}{2}}^2x_{j+\frac{1}{2}}\tau_{max}^2 + 35471366388a_{j+\frac{1}{2}}^3\tau_{max}^3) \\
& + 72h^6(32207175x_{j+\frac{1}{2}}^5 + 59331397125a_{j+\frac{1}{2}}x_{j+\frac{1}{2}}^4\tau_{max} - 12239785417860a_{j+\frac{1}{2}}^2x_{j+\frac{1}{2}}^3\tau_{max}^2 \\
& + 6310487300172a_{j+\frac{1}{2}}^3x_{j+\frac{1}{2}}^2\tau_{max}^3 + 7559235594328a_{j+\frac{1}{2}}^4x_{j+\frac{1}{2}}\tau_{max}^4 - 625703362520a_{j+\frac{1}{2}}^5\tau_{max}^5) \\
& + 256a_{j+\frac{1}{2}}^2h^4\tau_{max}^2(21762669291363x_{j+\frac{1}{2}}^5 - 15814160266905a_{j+\frac{1}{2}}x_{j+\frac{1}{2}}^4\tau_{max} \\
& - 92924845443144a_{j+\frac{1}{2}}^2x_{j+\frac{1}{2}}^3\tau_{max}^2 \\
& + 16196462375544a_{j+\frac{1}{2}}^3x_{j+\frac{1}{2}}^2\tau_{max}^3 + 11967447837400a_{j+\frac{1}{2}}^4x_{j+\frac{1}{2}}\tau_{max}^4 + 89226796360a_{j+\frac{1}{2}}^5\tau_{max}^5))) \\
& /((429(675675h^6 + 1623322651104a_{j+\frac{1}{2}}^2h^4\tau_{max}^2 \\
& + 49213569888000a_{j+\frac{1}{2}}^4h^2\tau_{max}^4 + 327591628066816a_{j+\frac{1}{2}}^6\tau_{max}^6))) \\
\beta_2 = & - ((7(-2413125h^{12} + 28114944a_{j+\frac{1}{2}}^6x_{j+\frac{1}{2}}^5\tau_{max}^6(79978424821x_{j+\frac{1}{2}} + 4708995478a_{j+\frac{1}{2}}\tau_{max}) \\
& + 38610h^{10}(4515x_{j+\frac{1}{2}}^2 + 3550410a_{j+\frac{1}{2}}x_{j+\frac{1}{2}}\tau_{max} - 25259534a_{j+\frac{1}{2}}^2\tau_{max}^2)
\end{aligned}$$

$$\begin{aligned}
& - 135168a_{j+\frac{1}{2}}^4 h^2 x_{j+\frac{1}{2}}^3 \tau_{max}^4 (-2499126595875x_{j+\frac{1}{2}}^3 + 617613778314a_{j+\frac{1}{2}}x_{j+\frac{1}{2}}^2 \tau_{max} \\
& + 3305684925530a_{j+\frac{1}{2}}^2 x_{j+\frac{1}{2}} \tau_{max}^2 + 118237887960a_{j+\frac{1}{2}}^3 \tau_{max}^3) - 4752h^8 (375375x_{j+\frac{1}{2}}^4 \\
& + 574746900a_{j+\frac{1}{2}}x_{j+\frac{1}{2}}^3 \tau_{max} - 29912937366a_{j+\frac{1}{2}}^2 x_{j+\frac{1}{2}}^2 \tau_{max}^2 + 11830086972a_{j+\frac{1}{2}}^3 x_{j+\frac{1}{2}} \tau_{max}^3 \\
& + 3794229608a_{j+\frac{1}{2}}^4 \tau_{max}^4) + 1536a_{j+\frac{1}{2}}^2 h^4 x_{j+\frac{1}{2}} \tau_{max}^2 (7254223097121x_{j+\frac{1}{2}}^5 \\
& - 6325664106762a_{j+\frac{1}{2}}x_{j+\frac{1}{2}}^4 \tau_{max} \\
& - 46462422721572a_{j+\frac{1}{2}}^2 x_{j+\frac{1}{2}}^3 \tau_{max}^2 + 10797641583696a_{j+\frac{1}{2}}^3 x_{j+\frac{1}{2}}^2 \tau_{max}^3 \\
& + 11967447837400a_{j+\frac{1}{2}}^4 x_{j+\frac{1}{2}} \tau_{max}^4 \\
& + 178453592720a_{j+\frac{1}{2}}^5 \tau_{max}^5) + 16h^6 (289864575x_{j+\frac{1}{2}}^6 + 640779088950a_{j+\frac{1}{2}}x_{j+\frac{1}{2}}^5 \tau_{max} \\
& - 165237103141110a_{j+\frac{1}{2}}^2 x_{j+\frac{1}{2}}^4 \tau_{max}^2 + 113588771403096a_{j+\frac{1}{2}}^3 x_{j+\frac{1}{2}}^3 \tau_{max}^3 \\
& + 204099361046856a_{j+\frac{1}{2}}^4 x_{j+\frac{1}{2}}^2 \tau_{max}^4 \\
& - 33787981576080a_{j+\frac{1}{2}}^5 x_{j+\frac{1}{2}} \tau_{max}^5 - 5178530366512a_{j+\frac{1}{2}}^6 \tau_{max}^6))) \\
& /((1716(675675h^6 + 1623322651104a_{j+\frac{1}{2}}^2 h^4 \tau_{max}^2 \\
& + 49213569888000a_{j+\frac{1}{2}}^4 h^2 \tau_{max}^4 + 327591628066816a_{j+\frac{1}{2}}^6 \tau_{max}^6))) \\
\beta_1 = & (-675675h^{12}(375x_{j+\frac{1}{2}} + 149849a_{j+\frac{1}{2}}\tau_{max}) + 140574720a_{j+\frac{1}{2}}^6 x_{j+\frac{1}{2}}^6 \tau_{max}^6 (239935274463x_{j+\frac{1}{2}} \\
& + 16481484173a_{j+\frac{1}{2}}\tau_{max}) + 4158h^{10}(1467375x_{j+\frac{1}{2}}^3 + 1730824875a_{j+\frac{1}{2}}x_{j+\frac{1}{2}}^2 \tau_{max} \\
& - 24628045650a_{j+\frac{1}{2}}^2 x_{j+\frac{1}{2}} \tau_{max}^2 + 5177901778a_{j+\frac{1}{2}}^3 \tau_{max}^3) \\
& - 14192640a_{j+\frac{1}{2}}^4 h^2 x_{j+\frac{1}{2}}^4 \tau_{max}^4 (-357018085125x_{j+\frac{1}{2}}^3 \\
& + 102935629719a_{j+\frac{1}{2}}x_{j+\frac{1}{2}}^2 \tau_{max} + 661136985106a_{j+\frac{1}{2}}^2 x_{j+\frac{1}{2}} \tau_{max}^2 + 29559471990a_{j+\frac{1}{2}}^3 \tau_{max}^3) \\
& - 720h^8 (52026975x_{j+\frac{1}{2}}^5 + 99574900425a_{j+\frac{1}{2}}x_{j+\frac{1}{2}}^4 \tau_{max} - 6909888531546a_{j+\frac{1}{2}}^2 x_{j+\frac{1}{2}}^3 \tau_{max}^2 \\
& + 4099125135798a_{j+\frac{1}{2}}^3 x_{j+\frac{1}{2}}^2 \tau_{max}^3 + 2629401118344a_{j+\frac{1}{2}}^4 x_{j+\frac{1}{2}} \tau_{max}^4 - 303683537672a_{j+\frac{1}{2}}^5 \tau_{max}^5) \\
& + 10752a_{j+\frac{1}{2}}^2 h^4 x_{j+\frac{1}{2}}^2 \tau_{max}^2 (15544763779545x_{j+\frac{1}{2}}^5 - 15814160266905a_{j+\frac{1}{2}}x_{j+\frac{1}{2}}^4 \tau_{max} \\
& - 139387268164716a_{j+\frac{1}{2}}^2 x_{j+\frac{1}{2}}^3 \tau_{max}^2 + 40491155938860a_{j+\frac{1}{2}}^3 x_{j+\frac{1}{2}}^2 \tau_{max}^3 \\
& + 59837239187000a_{j+\frac{1}{2}}^4 x_{j+\frac{1}{2}} \tau_{max}^4 \\
& + 1338401945400a_{j+\frac{1}{2}}^5 \tau_{max}^5) + 560h^6 (124227675x_{j+\frac{1}{2}}^7 + 320389544475a_{j+\frac{1}{2}}x_{j+\frac{1}{2}}^6 \tau_{max}
\end{aligned}$$

$$\begin{aligned}
& - 99142261884666a_{j+\frac{1}{2}}^2x_{j+\frac{1}{2}}^5\tau_{max}^2 + 85191578552322a_{j+\frac{1}{2}}^3x_{j+\frac{1}{2}}^4\tau_{max}^3 \\
& + 204099361046856a_{j+\frac{1}{2}}^4x_{j+\frac{1}{2}}^3\tau_{max}^4 \\
& - 50681972364120a_{j+\frac{1}{2}}^5x_{j+\frac{1}{2}}^2\tau_{max}^5 - 15535591099536a_{j+\frac{1}{2}}^6x_{j+\frac{1}{2}}\tau_{max}^6 - 74240090992a_{j+\frac{1}{2}}^7\tau_{max}^7)) \\
& /((12870(675675h^6 + 1623322651104a_{j+\frac{1}{2}}^2h^4\tau_{max}^2 \\
& + 49213569888000a_{j+\frac{1}{2}}^4h^2\tau_{max}^4 + 327591628066816a_{j+\frac{1}{2}}^6\tau_{max}^6)) \\
\beta_0 = & - ((211486275h^{14} + 2249195520a_{j+\frac{1}{2}}^6x_{j+\frac{1}{2}}^7\tau_{max}^6(239935274463x_{j+\frac{1}{2}} + 18835981912a_{j+\frac{1}{2}}\tau_{max}) \\
& - 1729728h^{12}(9375x_{j+\frac{1}{2}}^2 + 7492450a_{j+\frac{1}{2}}x_{j+\frac{1}{2}}\tau_{max} - 22675009a_{j+\frac{1}{2}}^2\tau_{max}^2) \\
& - 10813440a_{j+\frac{1}{2}}^4h^2x_{j+\frac{1}{2}}^5\tau_{max}^4(-7497379787625x_{j+\frac{1}{2}}^3 + 2470455113256a_{j+\frac{1}{2}}x_{j+\frac{1}{2}}^2\tau_{max} \\
& + 18511835582968a_{j+\frac{1}{2}}^2x_{j+\frac{1}{2}}\tau_{max}^2 + 993198258864a_{j+\frac{1}{2}}^3\tau_{max}^3) + 6336h^{10}(30814875x_{j+\frac{1}{2}}^4 \\
& + 48463096500a_{j+\frac{1}{2}}x_{j+\frac{1}{2}}^3\tau_{max} - 1034377917300a_{j+\frac{1}{2}}^2x_{j+\frac{1}{2}}^2\tau_{max}^2 + 434943749352a_{j+\frac{1}{2}}^3x_{j+\frac{1}{2}}\tau_{max}^3 \\
& + 87945430928a_{j+\frac{1}{2}}^4\tau_{max}^4) + 172032a_{j+\frac{1}{2}}^2h^4x_{j+\frac{1}{2}}^3\tau_{max}^2(15544763779545x_{j+\frac{1}{2}}^5 \\
& - 18073326019320a_{j+\frac{1}{2}}x_{j+\frac{1}{2}}^4\tau_{max} - 185849690886288a_{j+\frac{1}{2}}^2x_{j+\frac{1}{2}}^3\tau_{max}^2 \\
& + 64785849502176a_{j+\frac{1}{2}}^3x_{j+\frac{1}{2}}^2\tau_{max}^3 \\
& + 119674478374000a_{j+\frac{1}{2}}^4x_{j+\frac{1}{2}}\tau_{max}^4 + 3569071854400a_{j+\frac{1}{2}}^5\tau_{max}^5) \\
& - 5120h^8(156080925x_{j+\frac{1}{2}}^6 \\
& + 358469641530a_{j+\frac{1}{2}}x_{j+\frac{1}{2}}^5\tau_{max} - 31094498391957a_{j+\frac{1}{2}}^2x_{j+\frac{1}{2}}^4\tau_{max}^2 \\
& + 24594750814788a_{j+\frac{1}{2}}^3x_{j+\frac{1}{2}}^3\tau_{max}^3 \\
& + 23664610065096a_{j+\frac{1}{2}}^4x_{j+\frac{1}{2}}^2\tau_{max}^4 - 5466303678096a_{j+\frac{1}{2}}^5x_{j+\frac{1}{2}}\tau_{max}^5 - 321570260696a_{j+\frac{1}{2}}^6\tau_{max}^6) \\
& + 1792h^6x_{j+\frac{1}{2}}(621138375x_{j+\frac{1}{2}}^7 + 1830797397000a_{j+\frac{1}{2}}x_{j+\frac{1}{2}}^6\tau_{max} \\
& - 660948412564440a_{j+\frac{1}{2}}^2x_{j+\frac{1}{2}}^5\tau_{max}^2 \\
& + 681532628418576a_{j+\frac{1}{2}}^3x_{j+\frac{1}{2}}^4\tau_{max}^3 + 2040993610468560a_{j+\frac{1}{2}}^4x_{j+\frac{1}{2}}^3\tau_{max}^4 \\
& - 675759631521600a_{j+\frac{1}{2}}^5x_{j+\frac{1}{2}}^2\tau_{max}^5 \\
& - 310711821990720a_{j+\frac{1}{2}}^6x_{j+\frac{1}{2}}\tau_{max}^6 - 2969603639680a_{j+\frac{1}{2}}^7\tau_{max}^7)) \\
& /((1647360(675675h^6 + 1623322651104a_{j+\frac{1}{2}}^2h^4\tau_{max}^2
\end{aligned}$$

$$+ 49213569888000a_{j+\frac{1}{2}}^4 h^2 \tau_{max}^4 + 327591628066816a_{j+\frac{1}{2}}^6 \tau_{max}^6))$$

Hence, we have

$$\begin{aligned} & \int_{x_j}^{x_{j+\frac{1}{2}}} (v_I - x^8 - u_I(x - \frac{h}{2}) + (x - \frac{h}{2})^8)^2 \\ = & ((a_j - a_{j+\frac{1}{2}})^2 h^{15} \tau_{max}^2 (6407611697217187127041124660156250h^{24} \\ & + 149695974574807265373449136315234375(a_j + a_{j+\frac{1}{2}})\tau h^{23} \\ & + 14193290712100500000(102198405596032324406a_j^2 \\ & + 102198405596032324406a_{j+\frac{1}{2}}^2)\tau_{max}^2 h^{22} - 5125354979369625000(342475000791495976813a_j^3 \\ & + 70854261497225701973887a_{j+\frac{1}{2}}a_j^2 + 70854261497225701973887a_{j+\frac{1}{2}}^2 a_j \\ & + 342475000791495976813a_{j+\frac{1}{2}}^3)\tau_{max}^3 h^{21} + 7646217218640000(11652918878137704874257261a_j^4 \\ & - 356825898865465222780684354a_{j+\frac{1}{2}}a_j^3 + 4836984475588643523349889892863a_{j+\frac{1}{2}}^2 a_j^2 \\ & - 356825898865465222780684354a_{j+\frac{1}{2}}^3 a_j + 11652918878137704874257261a_{j+\frac{1}{2}}^4)\tau_{max}^4 h^{20} \\ & - 84108389405040000(915648732630103373510011a_j^5 - 9407889926703309649033127a_{j+\frac{1}{2}}a_j^4 \\ & + 50390082001645985867302150063a_{j+\frac{1}{2}}^2 a_j^3 + 50390082001645985867302150063a_{j+\frac{1}{2}}^3 a_j^2 \\ & - 9407889926703309649033127a_{j+\frac{1}{2}}^4 a_j + 915648732630103373510011a_{j+\frac{1}{2}}^5)\tau_{max}^5 h^{19} \\ & + 11950131916800(164884185222005333805319210300a_j^6 \\ & - 6630342706616501593421808756675a_{j+\frac{1}{2}}a_j^5 \\ & + 188076239951110923388959402053344698a_{j+\frac{1}{2}}^2 a_j^4 \\ & - 208575150081849679648577147724521436a_{j+\frac{1}{2}}^3 a_j^3 \\ & + 188076239951110923388959402053344698a_{j+\frac{1}{2}}^4 a_j^2 \\ & - 6630342706616501593421808756675a_{j+\frac{1}{2}}^5 a_j \\ & + 164884185222005333805319210300a_{j+\frac{1}{2}}^6)\tau_{max}^6 h^{18} \\ & - 116513786188800(9030355544280902800247270875a_j^7 \\ & - 3739708129728410820201855381800a_{j+\frac{1}{2}}a_j^6 \\ & + 1639671962272060644440386615182608a_{j+\frac{1}{2}}^2 a_j^5 \end{aligned}$$

$$\begin{aligned}
& - 468108435937358839882855115553109999a_{j+\frac{1}{2}}^3a_j^4 \\
& - 468108435937358839882855115553109999a_{j+\frac{1}{2}}^4a_j^3 \\
& + 1639671962272060644440386615182608a_{j+\frac{1}{2}}^5a_j^2 \\
& - 3739708129728410820201855381800a_{j+\frac{1}{2}}^6a_j \\
& + 9030355544280902800247270875a_{j+\frac{1}{2}}^7)\tau_{max}^7h^{17} \\
& + 1042920603648(17693229844358932539826204876250a_j^8 \\
& - 851016926324609061380223519230000a_{j+\frac{1}{2}}^7a_j^7 \\
& + 47125344545016585156950522109397583050a_{j+\frac{1}{2}}^2a_j^6 \\
& - 97529619282774141346719350823793132925a_{j+\frac{1}{2}}^3a_j^5 \\
& + 5255857505572670614682746527402276260122a_{j+\frac{1}{2}}^4a_j^4 \\
& - 97529619282774141346719350823793132925a_{j+\frac{1}{2}}^5a_j^3 \\
& + 47125344545016585156950522109397583050a_{j+\frac{1}{2}}^6a_j^2 \\
& - 851016926324609061380223519230000a_{j+\frac{1}{2}}^7a_j \\
& + 17693229844358932539826204876250a_{j+\frac{1}{2}}^8)\tau_{max}^8h^{16} \\
& - 49712548773888(101040772023733232617286205625a_j^9 \\
& - 140105212582522805843160693143750a_{j+\frac{1}{2}}^8a_j^8 \\
& + 51582530719180029823261911267645250a_{j+\frac{1}{2}}^2a_j^7 \\
& - 44163649657013283522468049574357842925a_{j+\frac{1}{2}}^3a_j^6 \\
& + 244708639996747076704243816052072844257a_{j+\frac{1}{2}}^4a_j^5 \\
& + 244708639996747076704243816052072844257a_{j+\frac{1}{2}}^5a_j^4 \\
& - 44163649657013283522468049574357842925a_{j+\frac{1}{2}}^6a_j^3 \\
& + 51582530719180029823261911267645250a_{j+\frac{1}{2}}^7a_j^2 \\
& - 140105212582522805843160693143750a_{j+\frac{1}{2}}^8a_j \\
& + 101040772023733232617286205625a_{j+\frac{1}{2}}^9)\tau_{max}^9h^{15}
\end{aligned}$$

$$\begin{aligned}
& + 1854081073152(33584331291564604301081858670000a_j^{10} \\
& - 1906180194915312599272907575693750a_{j+\frac{1}{2}}^9 a_j^9 \\
& + 245764139312547176860059537900301659800a_{j+\frac{1}{2}}^2 a_j^8 \\
& - 707663942797545329144651859631895512600a_{j+\frac{1}{2}}^3 a_j^7 \\
& + 134163063816345518949074583247554347787928a_{j+\frac{1}{2}}^4 a_j^6 \\
& + 26378601392924543232613045305256772435407a_{j+\frac{1}{2}}^5 a_j^5 \\
& + 134163063816345518949074583247554347787928a_{j+\frac{1}{2}}^6 a_j^4 \\
& - 707663942797545329144651859631895512600a_{j+\frac{1}{2}}^7 a_j^3 \\
& + 245764139312547176860059537900301659800a_{j+\frac{1}{2}}^8 a_j^2 \\
& - 1906180194915312599272907575693750a_{j+\frac{1}{2}}^9 a_j \\
& + 33584331291564604301081858670000a_{j+\frac{1}{2}}^{10})\tau_{max}^{10} h^{14} \\
& - 927040536576(4806491069858999270093254178125a_j^{11} \\
& - 32905119135827149176686917250697500a_{j+\frac{1}{2}}^{10} a_j^{10} \\
& + 11947255513222701108328435117536305200a_{j+\frac{1}{2}}^2 a_j^9 \\
& - 29516372734495368634508174842749428453225a_{j+\frac{1}{2}}^3 a_j^8 \\
& + 71647191143541128835098649254728570811155a_{j+\frac{1}{2}}^4 a_j^7 \\
& - 515027236544736694707493486699984121084966a_{j+\frac{1}{2}}^5 a_j^6 \\
& - 515027236544736694707493486699984121084966a_{j+\frac{1}{2}}^6 a_j^5 \\
& + 71647191143541128835098649254728570811155a_{j+\frac{1}{2}}^7 a_j^4 \\
& - 29516372734495368634508174842749428453225a_{j+\frac{1}{2}}^8 a_j^3 \\
& + 11947255513222701108328435117536305200a_{j+\frac{1}{2}}^9 a_j^2 \\
& - 32905119135827149176686917250697500a_{j+\frac{1}{2}}^{10} a_j \\
& + 4806491069858999270093254178125a_{j+\frac{1}{2}}^{11})\tau_{max}^{11} h^{13} \\
& + 117719433216(15416045445800001639655121228125a_j^{12}
\end{aligned}$$

$$\begin{aligned}
& - 11248959872916054394746123802068750a_{j+\frac{1}{2}}a_j^{11} \\
& + 12916292928683882713171200164090274872775a_{j+\frac{1}{2}}^2a_j^{10} \\
& - 48929883458195191471671338307385182264000a_{j+\frac{1}{2}}^3a_j^9 \\
& + 29090074466973164035719501925737910175109849a_{j+\frac{1}{2}}^4a_j^8 \\
& + 8725725200094031043177646404131246876282890a_{j+\frac{1}{2}}^5a_j^7 \\
& + 87005910929471657463937678864353442773705631a_{j+\frac{1}{2}}^6a_j^6 \\
& + 8725725200094031043177646404131246876282890a_{j+\frac{1}{2}}^7a_j^5 \\
& + 29090074466973164035719501925737910175109849a_{j+\frac{1}{2}}^8a_j^4 \\
& - 48929883458195191471671338307385182264000a_{j+\frac{1}{2}}^9a_j^3 \\
& + 12916292928683882713171200164090274872775a_{j+\frac{1}{2}}^{10}a_j^2 \\
& - 11248959872916054394746123802068750a_{j+\frac{1}{2}}^{11}a_j \\
& + 15416045445800001639655121228125a_{j+\frac{1}{2}}^{12})\tau_{max}^{12}h^{12} \\
& - 706316599296a_j^2a_{j+\frac{1}{2}}^2(4265942923575175849094670782573247875a_j^9 \\
& - 153862613744907003545930019277911586979725a_{j+\frac{1}{2}}^8a_j^8 \\
& - 3474807299101645926175246054336032343331902a_{j+\frac{1}{2}}^7a_j^7 \\
& - 18372351946354404380564325292881086972629438a_{j+\frac{1}{2}}^6a_j^6 \\
& - 23225220972302688843109493083062064356608183a_{j+\frac{1}{2}}^5a_j^5 \\
& - 23225220972302688843109493083062064356608183a_{j+\frac{1}{2}}^5a_j^4 \\
& - 18372351946354404380564325292881086972629438a_{j+\frac{1}{2}}^6a_j^3 \\
& - 3474807299101645926175246054336032343331902a_{j+\frac{1}{2}}^7a_j^2 \\
& - 153862613744907003545930019277911586979725a_{j+\frac{1}{2}}^8a_j \\
& + 4265942923575175849094670782573247875a_{j+\frac{1}{2}}^9)\tau_{max}^{13}h^{11} \\
& + 2054739197952a_j^2a_{j+\frac{1}{2}}^2(4243863750274013736917056418445662625a_j^{10} \\
& - 1547703983492716454974956935236843394625a_{j+\frac{1}{2}}^9a_j^9
\end{aligned}$$

$$\begin{aligned}
& + 7727279661360167998007807567476980062903913a_{j+\frac{1}{2}}^2a_j^8 \\
& + 2721712884336354977154964974396418615333980a_{j+\frac{1}{2}}^3a_j^7 \\
& + 66282203069516127842005808949034716669693715a_{j+\frac{1}{2}}^4a_j^6 \\
& + 10699144073071358121723076542916022358448199a_{j+\frac{1}{2}}^5a_j^5 \\
& + 66282203069516127842005808949034716669693715a_{j+\frac{1}{2}}^6a_j^4 \\
& + 2721712884336354977154964974396418615333980a_{j+\frac{1}{2}}^7a_j^3 \\
& + 7727279661360167998007807567476980062903913a_{j+\frac{1}{2}}^8a_j^2 \\
& - 1547703983492716454974956935236843394625a_{j+\frac{1}{2}}^9a_j \\
& + 4243863750274013736917056418445662625a_{j+\frac{1}{2}}^{10})\tau_{max}^{14}h^{10} \\
& + 15068087451648a_j^4a_{j+\frac{1}{2}}^4(1226440005433813283100595708201779028363970a_j^7 \\
& + 4637165397258385176176955537957113705171438a_{j+\frac{1}{2}}^6 \\
& + 12794089326376582464931337782336108624614973a_{j+\frac{1}{2}}^2a_j^5 \\
& + 22775067494022410799181373159209801943311225a_{j+\frac{1}{2}}^3a_j^4 \\
& + 22775067494022410799181373159209801943311225a_{j+\frac{1}{2}}^4a_j^3 \\
& + 12794089326376582464931337782336108624614973a_{j+\frac{1}{2}}^5a_j^2 \\
& + 4637165397258385176176955537957113705171438a_{j+\frac{1}{2}}^6a_j \\
& + 1226440005433813283100595708201779028363970a_{j+\frac{1}{2}}^7)\tau_{max}^{15}h^9 \\
& + 1826434842624a_j^4a_{j+\frac{1}{2}}^4(5735376305254471497397007185633322552332115a_j^8 \\
& + 1709501352109512008062597478578959919914800a_{j+\frac{1}{2}}^7 \\
& + 355614542756479161277327105199371564782709331a_{j+\frac{1}{2}}^2a_j^6 \\
& + 70479522127743247319870522725960510761860785a_{j+\frac{1}{2}}^3a_j^5 \\
& + 1002829664307157485423082511164031759353903459a_{j+\frac{1}{2}}^4a_j^4 \\
& + 70479522127743247319870522725960510761860785a_{j+\frac{1}{2}}^5a_j^3 \\
& + 355614542756479161277327105199371564782709331a_{j+\frac{1}{2}}^6a_j^2
\end{aligned}$$

$$\begin{aligned}
& + 1709501352109512008062597478578959919914800a_{j+\frac{1}{2}}^7 a_j \\
& + 5735376305254471497397007185633322552332115a_{j+\frac{1}{2}}^8 \tau_{max}^{16} h^8 \\
& + 3652869685248a_j^6 a_{j+\frac{1}{2}}^6 (203384565657636861119423563747531084300163875a_j^5 \\
& + 472300903909347413055630862958395552513750987a_{j+\frac{1}{2}}^4 a_j^4 \\
& + 845439059698435708279323900172583571971402815a_{j+\frac{1}{2}}^2 a_j^3 \\
& + 845439059698435708279323900172583571971402815a_{j+\frac{1}{2}}^3 a_j^2 \\
& + 472300903909347413055630862958395552513750987a_{j+\frac{1}{2}}^4 a_j \\
& + 203384565657636861119423563747531084300163875a_{j+\frac{1}{2}}^5 \tau_{max}^{17} h^7 \\
& + 463856467968a_j^6 a_{j+\frac{1}{2}}^6 (1369249982086609549862770039779934986011183875a_j^6 \\
& + 278139799007723218202002921462902585975963495a_{j+\frac{1}{2}}^5 a_j^5 \\
& + 20485983393544401265668693110581367857751110947a_{j+\frac{1}{2}}^2 a_j^4 \\
& + 1906917249724283981846234551210738981775181093a_{j+\frac{1}{2}}^3 a_j^3 \\
& + 20485983393544401265668693110581367857751110947a_{j+\frac{1}{2}}^4 a_j^2 \\
& + 278139799007723218202002921462902585975963495a_{j+\frac{1}{2}}^5 a_j \\
& + 1369249982086609549862770039779934986011183875a_{j+\frac{1}{2}}^6 \tau_{max}^{18} h^6 \\
& + 5566277615616a_j^8 a_{j+\frac{1}{2}}^8 (1664951346635098223961807292973869362140155765a_j^3 \\
& + 2526101350040567759312558807452293135318345361a_{j+\frac{1}{2}}^2 a_j^2 \\
& + 2526101350040567759312558807452293135318345361a_{j+\frac{1}{2}}^2 a_j \\
& + 1664951346635098223961807292973869362140155765a_{j+\frac{1}{2}}^3 \tau_{max}^{19} h^5 \\
& + 11132555231232a_j^8 a_{j+\frac{1}{2}}^8 (1244577120142977931757886187009203449137132565a_j^4 \\
& + 145703135473265603367263926004750654965009150a_{j+\frac{1}{2}}^3 a_j^3 \\
& + 5277782995676146054115129935872458903443701703a_{j+\frac{1}{2}}^2 a_j^2 \\
& + 145703135473265603367263926004750654965009150a_{j+\frac{1}{2}}^3 a_j \\
& + 1244577120142977931757886187009203449137132565a_{j+\frac{1}{2}}^4 \tau_{max}^{20} h^4
\end{aligned}$$

$$\begin{aligned}
& + 36865115691022402100861417388552566459381060053952888832000a_j^{10}a_{j+\frac{1}{2}}^{10}(a_j + a_{j+\frac{1}{2}})\tau_{max}^{21}h^3 \\
& + 49531972439734470923977749503756206080a_j^{10}a_{j+\frac{1}{2}}^{10}(2587661596660313943875a_j^2 \\
& + 130692634564583157233a_{j+\frac{1}{2}}a_j + 2587661596660313943875a_{j+\frac{1}{2}}^2)\tau_{max}^{22}h^2 \\
& + 426590355493017768656985138297061848222462989929694793564160a_j^{12}a_{j+\frac{1}{2}}^{12}\tau_{max}^{24} \\
& / (127209139200(675675h^6 + 1623322651104a_j^2\tau_{max}^2h^4 + 49213569888000a_j^4\tau_{max}^4h^2 \\
& + 327591628066816a_j^6\tau_{max}^6)^2(675675h^6 + 1623322651104a_{j+\frac{1}{2}}^2\tau_{max}^2h^4 \\
& + 49213569888000a_{j+\frac{1}{2}}^4\tau_{max}^4h^2 + 327591628066816a_{j+\frac{1}{2}}^6\tau_{max}^6)^2) \\
& = O(h^{19})
\end{aligned}$$

9. $k = 8, u = x^9$, by the definition

$$\begin{aligned}
\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_I dx &= \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} x^9 dx \\
\tilde{P}_h(u_I; x^m; f, u)_j &= \tilde{P}_h(x^9; x^m; f, u)_j, \quad m = 1, \dots, 8 \\
\int_{x_j}^{x_{j+1}} v_I dx &= \int_{x_j}^{x_{j+1}} x^8 dx \\
\tilde{Q}_h(v_I; x^m; f, u)_{j+\frac{1}{2}} &= \tilde{Q}_h(x^8; x^m; f, u)_{j+\frac{1}{2}}, \quad m = 1, \dots, 8
\end{aligned} \tag{1.19}$$

then we have

$$\begin{aligned}
u_I &= \sum_{m=0}^8 \alpha_m x^m, \quad \forall x \in I_j. \\
v_I &= \sum_{m=0}^8 \beta_m x^m, \quad \forall x \in I_{j+\frac{1}{2}}.
\end{aligned} \tag{1.20}$$

By similar calculation we have

$$\begin{aligned}
& \int_{x_j}^{x_{j+\frac{1}{2}}} (v_I - x^9 - u_I(x - \frac{h}{2}) + (x - \frac{h}{2})^9)^2 \\
& = (3(a_j - a_{j+\frac{1}{2}})^2 h^{21} \tau_{max}^2 (1100247162677562372226747959055445824218750 h^{28} \\
& + 36271643488641303971494745633646826318359375(a_j + a_{j+\frac{1}{2}})\tau_{max} h^{27} \\
& + 47592581951289375000(603358537378470717071922052091a_j^2 \\
& - 18776909426726176784250157368370a_{j+\frac{1}{2}}a_j \\
& + 603358537378470717071922052091a_{j+\frac{1}{2}}^2)\tau_{max}^2 h^{26}
\end{aligned}$$

$$\begin{aligned}
& - 17799625649782226250000(392265732763604422564840797a_j^3 \\
& + 881527885756504264593056283964a_{j+\frac{1}{2}}a_j^2 \\
& + 881527885756504264593056283964a_{j+\frac{1}{2}}^2a_j \\
& + 392265732763604422564840797a_{j+\frac{1}{2}}^3)\tau_{max}^3h^{25} \\
& + 1562012946093600000(2476844634069685910790681532329585a_j^4 \\
& - 101288246902584516669721282319307240a_{j+\frac{1}{2}}a_j^3 \\
& + 131580620158454441035102761180088544783174a_{j+\frac{1}{2}}^2a_j^2 \\
& - 101288246902584516669721282319307240a_{j+\frac{1}{2}}^3a_j \\
& + 2476844634069685910790681532329585a_{j+\frac{1}{2}}^4)\tau_{max}^4h^{24} \\
& - 86301215271671400000(7750842255561192531347154033305a_j^5 \\
& - 24002984726861235502615709120783935a_{j+\frac{1}{2}}a_j^4 \\
& + 34968196700601573740765634573416756678a_{j+\frac{1}{2}}^2a_j^3 \\
& + 34968196700601573740765634573416756678a_{j+\frac{1}{2}}^3a_j^2 \\
& - 24002984726861235502615709120783935a_{j+\frac{1}{2}}^4a_j \\
& + 7750842255561192531347154033305a_{j+\frac{1}{2}}^5)\tau_{max}^5h^{23} \\
& + 119010510178560000(1381088270529089528757979201207871490a_j^6 \\
& - 76457860683796307522246724105451833700a_{j+\frac{1}{2}}a_j^5 \\
& + 232527795648485692055919616044493067786121087a_{j+\frac{1}{2}}^2a_j^4 \\
& - 137642650758131781459680385519949473952301362a_{j+\frac{1}{2}}^3a_j^3 \\
& + 232527795648485692055919616044493067786121087a_{j+\frac{1}{2}}^4a_j^2 \\
& - 76457860683796307522246724105451833700a_{j+\frac{1}{2}}^5a_j \\
& + 1381088270529089528757979201207871490a_{j+\frac{1}{2}}^6)\tau_{max}^6h^{22} \\
& - 2023178673035520000(9072423166234218203840328672221325a_j^7 \\
& - 109016676753568143536347011053427742455a_{j+\frac{1}{2}}a_j^6
\end{aligned}$$

$$\begin{aligned}
& + 141471341084999001425393433217531978820294a_{j+\frac{1}{2}}^2a_j^5 \\
& - 893618706668429901997509362349101515272669984a_{j+\frac{1}{2}}^3a_j^4 \\
& - 893618706668429901997509362349101515272669984a_{j+\frac{1}{2}}^4a_j^3 \\
& + 141471341084999001425393433217531978820294a_{j+\frac{1}{2}}^5a_j^2 \\
& - 109016676753568143536347011053427742455a_{j+\frac{1}{2}}^6a_j \\
& + 9072423166234218203840328672221325a_{j+\frac{1}{2}}^7)\tau_{max}^7h^{21} \\
& + 177544817049600(12998137365405447658144174189512481017500a_j^8 \\
& - 1135156802616869989367341847124668260937500a_{j+\frac{1}{2}}^7 \\
& + 6614772780157165536692795537288316997608604578870a_{j+\frac{1}{2}}^2a_j^6 \\
& - 972288558714083444869815406558779721533966291360a_{j+\frac{1}{2}}^3a_j^5 \\
& + 2311879244110624884078502406341183990639503622335887a_{j+\frac{1}{2}}^4a_j^4 \\
& - 972288558714083444869815406558779721533966291360a_{j+\frac{1}{2}}^5a_j^3 \\
& + 6614772780157165536692795537288316997608604578870a_{j+\frac{1}{2}}^6a_j^2 \\
& - 1135156802616869989367341847124668260937500a_{j+\frac{1}{2}}^7a_j \\
& + 12998137365405447658144174189512481017500a_{j+\frac{1}{2}}^8)\tau_{max}^8h^{20} \\
& - 215805725123788800(888457065097635122437710169588178125a_j^9 \\
& - 13958448974879714404915130494435784889375a_{j+\frac{1}{2}}^8 \\
& + 34904708339364654174924652814938752680359150a_{j+\frac{1}{2}}^2a_j^7 \\
& - 621180411470546269732340215917410738499047359350a_{j+\frac{1}{2}}^3a_j^6 \\
& - 6137736630581073285363696286829426419551582444623a_{j+\frac{1}{2}}^4a_j^5 \\
& - 6137736630581073285363696286829426419551582444623a_{j+\frac{1}{2}}^5a_j^4 \\
& - 621180411470546269732340215917410738499047359350a_{j+\frac{1}{2}}^6a_j^3 \\
& + 34904708339364654174924652814938752680359150a_{j+\frac{1}{2}}^7a_j^2 \\
& - 13958448974879714404915130494435784889375a_{j+\frac{1}{2}}^8a_j
\end{aligned}$$

$$\begin{aligned}
& + 888457065097635122437710169588178125a_{j+\frac{1}{2}}^9) \tau_{max}^9 h^{19} \\
& + 6763612078080(1634528636805173906473549391917644025040625a_j^{10} \\
& - 282552484263505693659935098779956210575968750a_{j+\frac{1}{2}}a_j^9 \\
& + 2432160368250335153225589561159951054248108071082375a_{j+\frac{1}{2}}^2a_j^8 \\
& - 8310677190402830036475253936167190697765323270636500a_{j+\frac{1}{2}}^3a_j^7 \\
& + 7356127678092772811353630086301910924963391405438894565a_{j+\frac{1}{2}}^4a_j^6 \\
& + 768997102910832792382112464333824085699163231116331694a_{j+\frac{1}{2}}^5a_j^5 \\
& + 7356127678092772811353630086301910924963391405438894565a_{j+\frac{1}{2}}^6a_j^4 \\
& - 8310677190402830036475253936167190697765323270636500a_{j+\frac{1}{2}}^7a_j^3 \\
& + 2432160368250335153225589561159951054248108071082375a_{j+\frac{1}{2}}^8a_j^2 \\
& - 282552484263505693659935098779956210575968750a_{j+\frac{1}{2}}^9a_j \\
& + 1634528636805173906473549391917644025040625a_{j+\frac{1}{2}}^{10}) \tau_{max}^{10} h^{18} \\
& - 229962810654720(2878123627484681181624264388736389250000a_j^{11} \\
& - 14812024869257169580296362818104402721140625a_{j+\frac{1}{2}}a_j^{10} \\
& + 319879411192695470451622148491886177201553056875a_{j+\frac{1}{2}}^2a_j^9 \\
& - 8019635559103183879993033159378623968758372354886250a_{j+\frac{1}{2}}^3a_j^8 \\
& - 757642843520594158062329994358407558498610442132921650a_{j+\frac{1}{2}}^4a_j^7 \\
& - 1052041039947118762333138560053367878357723741168180491a_{j+\frac{1}{2}}^5a_j^6 \\
& - 1052041039947118762333138560053367878357723741168180491a_{j+\frac{1}{2}}^6a_j^5 \\
& - 757642843520594158062329994358407558498610442132921650a_{j+\frac{1}{2}}^7a_j^4 \\
& - 8019635559103183879993033159378623968758372354886250a_{j+\frac{1}{2}}^8a_j^3 \\
& + 319879411192695470451622148491886177201553056875a_{j+\frac{1}{2}}^9a_j^2 \\
& - 14812024869257169580296362818104402721140625a_{j+\frac{1}{2}}^{10}a_j \\
& + 2878123627484681181624264388736389250000a_{j+\frac{1}{2}}^{11}) \tau_{max}^{11} h^{17}
\end{aligned}$$

$$\begin{aligned}
& + 28858078199808(294268492226031378667969966077916478281250a_j^{12} \\
& - 236161506481849546070233976106274177112500000a_{j+\frac{1}{2}}a_j^{11} \\
& + 2713350071991957938204503838551143251438027831688750a_{j+\frac{1}{2}}^2a_j^{10} \\
& - 23530211804947575547179775426357352139634642830010000a_{j+\frac{1}{2}}^3a_j^9 \\
& + 63725509242974404237389260236210954684251780492952630125a_{j+\frac{1}{2}}^4a_j^8 \\
& + 8744721099627909795672887753416351889485809029059518800a_{j+\frac{1}{2}}^5a_j^7 \\
& + 209273760833645673576341575313464830902246817989905451396a_{j+\frac{1}{2}}^6a_j^6 \\
& + 8744721099627909795672887753416351889485809029059518800a_{j+\frac{1}{2}}^7a_j^5 \\
& + 63725509242974404237389260236210954684251780492952630125a_{j+\frac{1}{2}}^8a_j^4 \\
& - 23530211804947575547179775426357352139634642830010000a_{j+\frac{1}{2}}^9a_j^3 \\
& + 2713350071991957938204503838551143251438027831688750a_{j+\frac{1}{2}}^{10}a_j^2 \\
& - 236161506481849546070233976106274177112500000a_{j+\frac{1}{2}}^{11}a_j \\
& + 294268492226031378667969966077916478281250a_{j+\frac{1}{2}}^{12})\tau_{max}h^{16} \\
& + 6132341617459200(31138870323868746449369591656183703125a_j^{13} \\
& - 11734483363541331378666623294564819685771875a_{j+\frac{1}{2}}a_j^{12} \\
& - 33967905504115394998356045976621383135254595250a_{j+\frac{1}{2}}^2a_j^{11} \\
& + 660614725980244601792215805789889135943431963986550a_{j+\frac{1}{2}}^3a_j^{10} \\
& + 656921004685650654596836595699758385868303459796998655a_{j+\frac{1}{2}}^4a_j^9 \\
& + 828238349517872069712092208307559854269603908735034523a_{j+\frac{1}{2}}^5a_j^8 \\
& + 2907846615912738424611579188259045667564783472133520852a_{j+\frac{1}{2}}^6a_j^7 \\
& + 2907846615912738424611579188259045667564783472133520852a_{j+\frac{1}{2}}^7a_j^6 \\
& + 828238349517872069712092208307559854269603908735034523a_{j+\frac{1}{2}}^8a_j^5 \\
& + 656921004685650654596836595699758385868303459796998655a_{j+\frac{1}{2}}^9a_j^4 \\
& + 660614725980244601792215805789889135943431963986550a_{j+\frac{1}{2}}^{10}a_j^3
\end{aligned}$$

$$\begin{aligned}
& - 33967905504115394998356045976621383135254595250a_{j+\frac{1}{2}}^{11}a_j^2 \\
& - 11734483363541331378666623294564819685771875a_{j+\frac{1}{2}}^{12}a_j \\
& + 31138870323868746449369591656183703125a_{j+\frac{1}{2}}^{13})\tau_{max}^{13}h^{15} \\
& + 2690729902080(657772539900895770133421475843368737375000a_j^{14} \\
& - 935792378625641846254166367499934604773750000a_{j+\frac{1}{2}}^{13} \\
& + 21300219977070328338528245279899276449119296331851250a_{j+\frac{1}{2}}^2a_j^{12} \\
& - 1055002499130830585822194606146619189707195916981872500a_{j+\frac{1}{2}}^3a_j^{11} \\
& + 8813375535711277110376291351085383021901842286499079376825a_{j+\frac{1}{2}}^4a_j^{10} \\
& + 1289864642882741809338863672653955489273400847636955815570a_{j+\frac{1}{2}}^5a_j^9 \\
& + 87444575568666764600564927595166198846135850421382242058732a_{j+\frac{1}{2}}^6a_j^8 \\
& + 6101710921402747694131746838483255439633765384124783957150a_{j+\frac{1}{2}}^7a_j^7 \\
& + 87444575568666764600564927595166198846135850421382242058732a_{j+\frac{1}{2}}^8a_j^6 \\
& + 1289864642882741809338863672653955489273400847636955815570a_{j+\frac{1}{2}}^9a_j^5 \\
& + 8813375535711277110376291351085383021901842286499079376825a_{j+\frac{1}{2}}^{10}a_j^4 \\
& - 1055002499130830585822194606146619189707195916981872500a_{j+\frac{1}{2}}^{11}a_j^3 \\
& + 21300219977070328338528245279899276449119296331851250a_{j+\frac{1}{2}}^{12}a_j^2 \\
& - 935792378625641846254166367499934604773750000a_{j+\frac{1}{2}}^{13}a_j \\
& + 657772539900895770133421475843368737375000a_{j+\frac{1}{2}}^{14})\tau_{max}^{14}h^{14} \\
& - 13082328783912960a_ja_{j+\frac{1}{2}}(2316518916288108165584169195913809747875000a_j^{13} \\
& - 23009335833927844596786769820307182449456956250a_{j+\frac{1}{2}}^{12} \\
& + 2234076898252230587581171210332767321591221886615000a_{j+\frac{1}{2}}^2a_j^{11} \\
& - 2569262692700314105474432502299419529611490298684243500a_{j+\frac{1}{2}}^3a_j^{10} \\
& - 2610100289178123801574956327658860133849487175164481335a_{j+\frac{1}{2}}^4a_j^9 \\
& - 31210213643687675237037019874204852537483967904732637840a_{j+\frac{1}{2}}^5a_j^8
\end{aligned}$$

$$\begin{aligned}
& - 30882306708740486599466467604869866303812925806329378094a_{j+\frac{1}{2}}^6 a_j^7 \\
& - 30882306708740486599466467604869866303812925806329378094a_{j+\frac{1}{2}}^7 a_j^6 \\
& - 31210213643687675237037019874204852537483967904732637840a_{j+\frac{1}{2}}^8 a_j^5 \\
& - 2610100289178123801574956327658860133849487175164481335a_{j+\frac{1}{2}}^9 a_j^4 \\
& - 2569262692700314105474432502299419529611490298684243500a_{j+\frac{1}{2}}^{10} a_j^3 \\
& + 2234076898252230587581171210332767321591221886615000a_{j+\frac{1}{2}}^{11} a_j^2 \\
& - 23009335833927844596786769820307182449456956250a_{j+\frac{1}{2}}^{12} a_j \\
& + 2316518916288108165584169195913809747875000a_{j+\frac{1}{2}}^{13} \tau_{max}^{15} h^{13} \\
& + 28701118955520a_j^2 a_{j+\frac{1}{2}}^2 \\
& (398300550488038340685997391635330992385175359210000a_j^{12} \\
& - 37180359759375534401449830283255044573383668595730000a_{j+\frac{1}{2}}^{11} \\
& + 3573832402210526483256799138604423351086859453677682305375a_{j+\frac{1}{2}}^2 a_j^{10} \\
& + 517281761558761920542340699747782447916007900186874589300a_{j+\frac{1}{2}}^3 a_j^9 \\
& + 108752310025750705460582991819978419235883663099732700371042a_{j+\frac{1}{2}}^4 a_j^8 \\
& + 8667841143565661368886481081798822559742292006303275152980a_{j+\frac{1}{2}}^5 a_j^7 \\
& + 334416288697967943954453597407311821589951489108850085896415a_{j+\frac{1}{2}}^6 a_j^6 \\
& + 8667841143565661368886481081798822559742292006303275152980a_{j+\frac{1}{2}}^7 a_j^5 \\
& + 108752310025750705460582991819978419235883663099732700371042a_{j+\frac{1}{2}}^8 a_j^4 \\
& + 517281761558761920542340699747782447916007900186874589300a_{j+\frac{1}{2}}^9 a_j^3 \\
& + 3573832402210526483256799138604423351086859453677682305375a_{j+\frac{1}{2}}^{10} a_j^2 \\
& - 37180359759375534401449830283255044573383668595730000a_{j+\frac{1}{2}}^{11} a_j \\
& + 398300550488038340685997391635330992385175359210000a_{j+\frac{1}{2}}^{12} \tau_{max}^{16} h^{12} \\
& - 2439595111219200a_j^3 a_{j+\frac{1}{2}}^3 \\
& (5216829605376663971736869510023198676502432824073000a_j^{11}
\end{aligned}$$

$$\begin{aligned}
& - 38707754973779034490714085515864523123278422485573649750a_{j+\frac{1}{2}}a_j^{10} \\
& - 23487662557951040805765305902058031966287280339025143620a_{j+\frac{1}{2}}^2a_j^9 \\
& - 1546401127625692389171782922169708978725595100239117516145a_{j+\frac{1}{2}}^3a_j^8 \\
& - 1382701115905465059538867484794329337189500181444526941897a_{j+\frac{1}{2}}^4a_j^7 \\
& - 4778708299270025542183899586959224165904566721735817959751a_{j+\frac{1}{2}}^5a_j^6 \\
& - 4778708299270025542183899586959224165904566721735817959751a_{j+\frac{1}{2}}^6a_j^5 \\
& - 1382701115905465059538867484794329337189500181444526941897a_{j+\frac{1}{2}}^7a_j^4 \\
& - 1546401127625692389171782922169708978725595100239117516145a_{j+\frac{1}{2}}^8a_j^3 \\
& - 23487662557951040805765305902058031966287280339025143620a_{j+\frac{1}{2}}^9a_j^2 \\
& - 38707754973779034490714085515864523123278422485573649750a_{j+\frac{1}{2}}^{10}a_j \\
& + 5216829605376663971736869510023198676502432824073000a_{j+\frac{1}{2}}^{11})\tau_{max}^{17}h^{11} \\
& + 191340793036800a_j^4a_{j+\frac{1}{2}}^4 \\
& (191632468709253965344733237878070249880033496630104999900a_j^{10} \\
& + 8116371865709560306059587699709842737421761537367178840a_{j+\frac{1}{2}}a_j^9 \\
& + 71697229107767367741227498609551487321421952618993554381338a_{j+\frac{1}{2}}^2a_j^8 \\
& + 5675568589635422387689518297559965163646556548627006376620a_{j+\frac{1}{2}}^3a_j^7 \\
& + 678874571969404486593651361170436732182185553225373227273135a_{j+\frac{1}{2}}^4a_j^6 \\
& + 19415404434414986619967581740528472380543827907484817568330a_{j+\frac{1}{2}}^5a_j^5 \\
& + 678874571969404486593651361170436732182185553225373227273135a_{j+\frac{1}{2}}^6a_j^4 \\
& + 5675568589635422387689518297559965163646556548627006376620a_{j+\frac{1}{2}}^7a_j^3 \\
& + 71697229107767367741227498609551487321421952618993554381338a_{j+\frac{1}{2}}^8a_j^2 \\
& + 8116371865709560306059587699709842737421761537367178840a_{j+\frac{1}{2}}^9a_j \\
& + 191632468709253965344733237878070249880033496630104999900a_{j+\frac{1}{2}}^{10})\tau_{max}^{18}h^{10} \\
& - 6505586963251200a_j^5a_{j+\frac{1}{2}}^5
\end{aligned}$$

$$\begin{aligned}
& (7563227775089851309390916262351036341355712749517571940a_j^9 \\
& - 1894070184847691274286147261026921980758932210292314197170a_{j+\frac{1}{2}}a_j^8 \\
& - 1357157414949644435765512984388378233144690375592421558228a_{j+\frac{1}{2}}^2a_j^7 \\
& - 20213820140081346715907495724816240952055856286119940483478a_{j+\frac{1}{2}}^3a_j^6 \\
& - 19060018445278558110564613913870923066811341796082681421417a_{j+\frac{1}{2}}^4a_j^5 \\
& - 19060018445278558110564613913870923066811341796082681421417a_{j+\frac{1}{2}}^5a_j^4 \\
& - 20213820140081346715907495724816240952055856286119940483478a_{j+\frac{1}{2}}^6a_j^3 \\
& - 1357157414949644435765512984388378233144690375592421558228a_{j+\frac{1}{2}}^7a_j^2 \\
& - 1894070184847691274286147261026921980758932210292314197170a_{j+\frac{1}{2}}^8a_j \\
& + 7563227775089851309390916262351036341355712749517571940a_{j+\frac{1}{2}}^9)\tau_{max}^{19}h^9 \\
& + 4081936918118400a_j^6a_{j+\frac{1}{2}}^6 \\
& (1203527824719757084737294693245305249155503269095733533992a_j^8 \\
& - 32007392047825031083599698065137591100139159789058029440a_{j+\frac{1}{2}}a_j^7 \\
& + 141305873585123356093489161870497239419385588116437348159905a_{j+\frac{1}{2}}^2a_j^6 \\
& + 3432170557744974325621663946640372968979056882320258136600a_{j+\frac{1}{2}}^3a_j^5 \\
& + 434177154524320603888523162597427546054438914788557665283103a_{j+\frac{1}{2}}^4a_j^4 \\
& + 3432170557744974325621663946640372968979056882320258136600a_{j+\frac{1}{2}}^5a_j^3 \\
& + 141305873585123356093489161870497239419385588116437348159905a_{j+\frac{1}{2}}^6a_j^2 \\
& - 32007392047825031083599698065137591100139159789058029440a_{j+\frac{1}{2}}^7a_j \\
& + 1203527824719757084737294693245305249155503269095733533992a_{j+\frac{1}{2}}^8)\tau_{max}^{20}h^8 \\
& - 346964638040064000a_j^7a_{j+\frac{1}{2}}^7 \\
& (17041150655545150324249372730994204430805308898697369160a_j^7 \\
& - 1448756723322317642969137006701205107298486867431840063050a_{j+\frac{1}{2}}a_j^6 \\
& - 1209932516900506352119038564861078883913741360316360114652a_{j+\frac{1}{2}}^2a_j^5
\end{aligned}$$

$$\begin{aligned}
& - 4591142009662562218661157109342537794887196412421142278041a_{j+\frac{1}{2}}^3a_j^4 \\
& - 4591142009662562218661157109342537794887196412421142278041a_{j+\frac{1}{2}}^4a_j^3 \\
& - 1209932516900506352119038564861078883913741360316360114652a_{j+\frac{1}{2}}^5a_j^2 \\
& - 1448756723322317642969137006701205107298486867431840063050a_{j+\frac{1}{2}}^6a_j \\
& + 17041150655545150324249372730994204430805308898697369160a_{j+\frac{1}{2}}^7)\tau_{max}^{21}h^7 \\
& + 4947802324992000a_j^8a_{j+\frac{1}{2}}^8 \\
& (41815281802728355449000442728158149563514789136281327994372a_j^6 \\
& - 3521262674996309902623997935056318204886660844829942964360a_{j+\frac{1}{2}}^5 \\
& + 1597146344868207686202475698182219152691284677984635969958305a_{j+\frac{1}{2}}^2a_j^4 \\
& + 3319251232414997554730059395556932700081199710353978221190a_{j+\frac{1}{2}}^3a_j^3 \\
& + 1597146344868207686202475698182219152691284677984635969958305a_{j+\frac{1}{2}}^4a_j^2 \\
& - 3521262674996309902623997935056318204886660844829942964360a_{j+\frac{1}{2}}^5a_j \\
& + 41815281802728355449000442728158149563514789136281327994372a_{j+\frac{1}{2}}^6)\tau_{max}^{22}h^6 \\
& - 9820315020625503584256000a_j^9a_{j+\frac{1}{2}}^9 \\
& (22814556272812811845517404036661702594338050585004a_j^5 \\
& - 668297996560155590676317152735549526774837532959801a_{j+\frac{1}{2}}^4 \\
& - 603669597926841584614775500456871785196468044576094a_{j+\frac{1}{2}}^2a_j^3 \\
& - 603669597926841584614775500456871785196468044576094a_{j+\frac{1}{2}}^3a_j^2 \\
& - 668297996560155590676317152735549526774837532959801a_{j+\frac{1}{2}}^4a_j \\
& + 22814556272812811845517404036661702594338050585004a_{j+\frac{1}{2}}^5)\tau_{max}^{23}h^5 \\
& + 158329674399744000a_j^{10}a_{j+\frac{1}{2}}^{10} \\
& (17948094745355790130444601146873057041226358101845959097106a_j^4 \\
& - 1892909327941571061240358583626021836510074192394703594020a_{j+\frac{1}{2}}^3 \\
& + 223070092924324361474885042695147848967177268115932704945375a_{j+\frac{1}{2}}^2a_j^2
\end{aligned}$$

$$\begin{aligned}
& - 1892909327941571061240358583626021836510074192394703594020a_{j+\frac{1}{2}}^3a_j \\
& + 17948094745355790130444601146873057041226358101845959097106a_{j+\frac{1}{2}}^4)\tau_{max}^{24}h^4 \\
& - 7935608097475154411520000a_j^{11}a_{j+\frac{1}{2}}^{11} \\
& (363331815402149950652486299926872555176673859991678a_j^3 \\
& - 3220712031035671122093763244347187829310069793792281a_{j+\frac{1}{2}}^2a_j^2 \\
& - 3220712031035671122093763244347187829310069793792281a_{j+\frac{1}{2}}^2a_j^2 \\
& + 363331815402149950652486299926872555176673859991678a_{j+\frac{1}{2}}^3)\tau_{max}^{25}h^3 \\
& + 3303022287195888570398146560000a_j^{12}a_{j+\frac{1}{2}}^{12} \\
& (3870943533213155837623585590140549971658824411a_j^2 \\
& - 424244558726974861595284900718368831038583530a_{j+\frac{1}{2}}a_j \\
& + 3870943533213155837623585590140549971658824411a_{j+\frac{1}{2}}^2)\tau_{max}^{26}h^2 \\
& - 11851203731627063757893102598331140802206341446727910665292307164100034560000 \\
& a_j^{13}a_{j+\frac{1}{2}}^{13}(a_j + a_{j+\frac{1}{2}})\tau_{max}^{27}h \\
& + 5106607953901786020558499586889949715841821574914547542078723480595988480000 \\
& a_j^{14}a_{j+\frac{1}{2}}^{14}\tau_{max}^{28})) \\
& /((10287711948800(10135125h^8 + 4380486227076960a_j^2\tau_{max}^2h^6 \\
& + 294877585914354432a_j^4\tau_{max}^4h^4 + 2587519954603806720a_j^6\tau_{max}^6h^2 \\
& + 1025393073507860480a_j^8\tau_{max}^8)^2(10135125h^8 + 4380486227076960a_{j+\frac{1}{2}}^2\tau_{max}^2h^6 \\
& + 294877585914354432a_{j+\frac{1}{2}}^4\tau_{max}^4h^4 + 2587519954603806720a_{j+\frac{1}{2}}^6\tau_{max}^6h^2 \\
& + 1025393073507860480a_{j+\frac{1}{2}}^8\tau_{max}^8)^2) \\
& =O(h^{21})
\end{aligned}$$

2 Proof of Lemma 3.2

Here we give the detail proof of Lemma 3.2, we will only give the formulas with $u = x^{k+1}$, $k = 0, 1, \dots, 8$ since for $u = y^{k+1}$ in two-dimensional case the formulas are

symmetric to those of $u = x^{k+1}$ by switching x and y (i and j).

1. $k = 0, u = x$, by the definition of the projection,

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} u_I dx dy = \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} x dx dy \quad (2.21)$$

$$\int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} v_I dx dy = \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} x dx dy \quad (2.22)$$

then we have

$$u_I = x_i, \quad \forall (x, y) \in K_{i,j} \quad (2.23)$$

$$v_I = x_{i+\frac{1}{2}}, \quad \forall (x, y) \in K_{i+\frac{1}{2}, j+\frac{1}{2}} \quad (2.24)$$

Hence, for $(x, y) \in (x_i, x_{i+\frac{1}{2}}) \times (y_j, y_{j+\frac{1}{2}})$,

$$v_I - x - u_I(x - \frac{h}{2}) + (x - \frac{h}{2}) = x_{i+\frac{1}{2}} - x_i - \frac{h}{2} = 0 \quad (2.25)$$

Similarly, we have

$$v_I - x - u_I(x - \frac{h}{2}) + (x - \frac{h}{2}) = 0, \quad (x, y) \in (x_i, x_{i+\frac{1}{2}}) \times (y_{j-\frac{1}{2}}, y_j)$$

$$v_I - x - u_I(x + \frac{h}{2}) + (x + \frac{h}{2}) = 0, \quad (x, y) \in (x_{i-\frac{1}{2}}, x_i) \times (y_j, y_{j+\frac{1}{2}})$$

$$v_I - x - u_I(x + \frac{h}{2}) + (x + \frac{h}{2}) = 0, \quad (x, y) \in (x_{i-\frac{1}{2}}, x_i) \times (y_{j-\frac{1}{2}}, y_j)$$

2. $k = 1, u = x^2$, by the definition of the projection,

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} u_I dx dy = \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} x^2 dx dy$$

$$\tilde{P}_h(u_I; x; f, g, u)_{i,j} = \tilde{P}_h(x^2; x; f, g, u)_{i,j}$$

$$\tilde{P}_h(u_I; y; f, g, u)_{i,j} = \tilde{P}_h(x^2; y; f, g, u)_{i,j}$$

$$\tilde{P}_h(u_I; xy; f, g, u)_{i,j} = \tilde{P}_h(x^2; xy; f, g, u)_{i,j}$$

$$\int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} v_I dx dy = \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} x^2 dx dy$$

$$\tilde{Q}_h(v_I; x; f, g, u)_{i+\frac{1}{2}, j+\frac{1}{2}} = \tilde{Q}_h(x^2; x; f, g, u)_{i+\frac{1}{2}, j+\frac{1}{2}}$$

$$\tilde{Q}_h(v_I; y; f, g, u)_{i+\frac{1}{2}, j+\frac{1}{2}} = \tilde{Q}_h(x^2; y; f, g, u)_{i+\frac{1}{2}, j+\frac{1}{2}}$$

$$\tilde{Q}_h(v_I; xy; f, g, u)_{i+\frac{1}{2}, j+\frac{1}{2}} = \tilde{Q}_h(x^2; xy; f, g, u)_{i+\frac{1}{2}, j+\frac{1}{2}}$$

then by solving the above linear system we have

$$\begin{aligned} u_I(x, y) &= \alpha_0 + \alpha_1 x + \alpha_2 y + \alpha_3 xy \\ v_I(x, y) &= \beta_0 + \beta_1 x + \beta_2 y + \alpha_3 xy \end{aligned} \tag{2.26}$$

where

$$\begin{aligned} \alpha_3 &= 0 \\ \alpha_2 &= 0 \\ \alpha_1 &= 2x_i - \frac{2\tau_{max}a_{i,j}}{3} \\ \alpha_0 &= \frac{1}{12}(8\tau_{max}x_i a_{i,j} + h^2 - 12x_i^2) \\ \beta_3 &= 0 \\ \beta_2 &= 0 \\ \beta_1 &= 2x_{i+\frac{1}{2}} - \frac{2\tau_{max}a_{i+\frac{1}{2}, j+\frac{1}{2}}}{3} \\ \beta_0 &= \frac{1}{12}(8\tau_{max}x_{i+\frac{1}{2}} a_{i+\frac{1}{2}, j+\frac{1}{2}} + h^2 - 12x_{i+\frac{1}{2}}^2) \end{aligned} \tag{2.27}$$

Hence,

$$\begin{aligned} &\int_{x_i}^{x_{i+\frac{1}{2}}} \int_{y_j}^{y_{j+\frac{1}{2}}} (v_I(x, y) - x^2 - u_I(x - \frac{h}{2}, y - \frac{h}{2}) + (x - \frac{h}{2})^2)^2 dx dy \\ &= \frac{1}{108} h^4 \tau_{max}^2 (a_{i+\frac{1}{2}, j+\frac{1}{2}} - a_{i,j})^2 \\ &= O(h^8) \end{aligned} \tag{2.28}$$

Similarly, we have

$$\begin{aligned} &\int_{x_i}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_j} (v_I(x, y) - x^2 - u_I(x - \frac{h}{2}, y + \frac{h}{2}) + (x - \frac{h}{2})^2)^2 dx dy = O(h^8), \\ &\int_{x_{i-\frac{1}{2}}}^{x_i} \int_{y_j}^{y_{j+\frac{1}{2}}} (v_I(x, y) - x^2 - u_I(x + \frac{h}{2}, y - \frac{h}{2}) + (x + \frac{h}{2})^2)^2 dx dy = O(h^8), \\ &\int_{x_{i-\frac{1}{2}}}^{x_i} \int_{y_j}^{y_{j+\frac{1}{2}}} (v_I(x, y) - x^2 - u_I(x + \frac{h}{2}, y + \frac{h}{2}) + (x + \frac{h}{2})^2)^2 dx dy = O(h^8). \end{aligned}$$

3. $k = 2, u = x^3$, by the definition of the projection,

$$\begin{aligned} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} u_I dx dy &= \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} x^3 dx dy \\ \tilde{P}_h(u_I; x^k y^l; f, g, u)_{i,j} &= \tilde{P}_h(x^3; x^k y^l; f, g, u)_{i,j}, \quad k = 0, 1, 2, \quad l = 0, 1, 2 \\ \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} v_I dx dy &= \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} x^3 dx dy \\ \tilde{Q}_h(v_I; x^k y^l; f, g, u)_{i+\frac{1}{2}, j+\frac{1}{2}} &= \tilde{Q}_h(x^3; x^k y^l; f, g, u)_{i+\frac{1}{2}, j+\frac{1}{2}}, \quad k = 0, 1, 2, \quad l = 0, 1, 2 \end{aligned}$$

then by solving the above linear system we have

$$\begin{aligned} u_I(x, y) &= \sum_{k=0}^2 \sum_{l=0}^2 \alpha_{k,l} x^k y^l \\ v_I(x, y) &= \sum_{k=0}^2 \sum_{l=0}^2 \beta_{k,l} x^k y^l \end{aligned} \tag{2.29}$$

where

$$\alpha_{k,l} = 0, \quad k = 0, 1, 2, \quad l = 1, 2$$

$$\alpha_{2,0} = (-6h^2 \tau_{max} a_{i,j} + 240\tau_{max}^2 x_i a_{i,j}^2 + 45h^2 x_i) / (80\tau_{max}^2 a_{i,j}^2 + 15h^2)$$

$$\begin{aligned} \alpha_{1,0} &= (48\tau_{max}^2 a_{i,j}^2 (h^2 - 40x_i^2) + 96h^2 \tau_{max} x_i a_{i,j} + 15(h^4 - 24h^2 x_i^2)) \\ &\quad / (40(16\tau_{max}^2 a_{i,j}^2 + 3h^2)) \end{aligned}$$

$$\begin{aligned} \alpha_{0,0} &= (16\tau_{max}^2 x_i a_{i,j}^2 (40x_i^2 - 3h^2) + 4h^2 \tau_{max} a_{i,j} (h^2 - 12x_i^2) \\ &\quad - 15h^2 x_i (h^2 - 8x_i^2)) / (40(16\tau_{max}^2 a_{i,j}^2 + 3h^2)) \end{aligned}$$

$$\beta_{k,l} = 0, \quad k = 0, 1, 2, \quad l = 1, 2$$

$$\begin{aligned} \beta_{2,0} &= (-6h^2 \tau_{max} a_{i+\frac{1}{2}, j+\frac{1}{2}} + 240\tau_{max}^2 x_{i+\frac{1}{2}} a_{i+\frac{1}{2}, j+\frac{1}{2}}^2 + 45h^2 x_{i+\frac{1}{2}}) \\ &\quad / (80\tau_{max}^2 a_{i+\frac{1}{2}, j+\frac{1}{2}}^2 + 15h^2) \end{aligned}$$

$$\begin{aligned} \beta_{1,0} &= (48\tau_{max}^2 a_{i+\frac{1}{2}, j+\frac{1}{2}}^2 (h^2 - 40x_{i+\frac{1}{2}}^2) + 96h^2 \tau_{max} x_{i+\frac{1}{2}} a_{i+\frac{1}{2}, j+\frac{1}{2}} \\ &\quad + 15(h^4 - 24h^2 x_{i+\frac{1}{2}}^2)) / (40(16\tau_{max}^2 a_{i+\frac{1}{2}, j+\frac{1}{2}}^2 + 3h^2)) \end{aligned}$$

$$\begin{aligned} \beta_{0,0} &= (16\tau_{max}^2 x_{i+\frac{1}{2}} a_{i+\frac{1}{2}, j+\frac{1}{2}}^2 (40x_{i+\frac{1}{2}}^2 - 3h^2) + 4h^2 \tau_{max} a_{i+\frac{1}{2}, j+\frac{1}{2}} (h^2 - 12x_{i+\frac{1}{2}}^2) \\ &\quad - 15h^2 x_{i+\frac{1}{2}} (h^2 - 8x_{i+\frac{1}{2}}^2)) / (40(16\tau_{max}^2 a_{i+\frac{1}{2}, j+\frac{1}{2}}^2 + 3h^2)) \end{aligned}$$

Hence,

$$\begin{aligned}
& \int_{x_i}^{x_{i+\frac{1}{2}}} \int_{y_j}^{y_{j+\frac{1}{2}}} (v_I(x, y) - x^3 - u_I(x - \frac{h}{2}, y - \frac{h}{2}) + (x - \frac{h}{2})^3)^2 dx dy \\
&= (h^{10} \tau_{max}^2 (a_{i+\frac{1}{2}, j+\frac{1}{2}} - a_{i,j})^2 (\tau_{max} (3ha_{i+\frac{1}{2}, j+\frac{1}{2}} (16\tau_{max} a_{i,j} (5\tau_{max} a_{i,j} + h) \\
&\quad - 15h^2) + 8\tau_{max} a_{i+\frac{1}{2}, j+\frac{1}{2}}^2 (2\tau_{max} a_{i,j} (32\tau_{max} a_{i,j} + 15h) + 15h^2) \\
&\quad + 15h^2 a_{i,j} (8\tau_{max} a_{i,j} - 3h)) + 18h^4)) \\
&\quad / (1000(16\tau_{max}^2 a_{i+\frac{1}{2}, j+\frac{1}{2}}^2 + 3h^2)^2 (16\tau_{max}^2 a_{i,j}^2 + 3h^2)^2) \\
&= O(h^{10})
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
& \int_{x_i}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_j} (v_I(x, y) - x^3 - u_I(x - \frac{h}{2}, y + \frac{h}{2}) + (x - \frac{h}{2})^3)^2 dx dy = O(h^{10}), \\
& \int_{x_{i-\frac{1}{2}}}^{x_i} \int_{y_j}^{y_{j+\frac{1}{2}}} (v_I(x, y) - x^3 - u_I(x + \frac{h}{2}, y - \frac{h}{2}) + (x + \frac{h}{2})^3)^2 dx dy = O(h^{10}), \\
& \int_{x_{i-\frac{1}{2}}}^{x_i} \int_{y_j}^{y_{j+\frac{1}{2}}} (v_I(x, y) - x^3 - u_I(x + \frac{h}{2}, y + \frac{h}{2}) + (x + \frac{h}{2})^3)^2 dx dy = O(h^{10}).
\end{aligned}$$

4. $k = 3, u = x^4$, by the definition of the projection,

$$\begin{aligned}
& \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} u_I dx dy = \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} x^4 dx dy \\
& \tilde{P}_h(u_I; x^k y^l; f, g, u)_{i,j} = \tilde{P}_h(x^4; x^k y^l; f, g, u)_{i,j}, \quad k = 0, 1, 2, 3, \quad l = 0, 1, 2, 3 \\
& \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} v_I dx dy = \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} x^4 dx dy \\
& \tilde{Q}_h(v_I; x^k y^l; f, g, u)_{i+\frac{1}{2}, j+\frac{1}{2}} = \tilde{Q}_h(x^4; x^k y^l; f, g, u)_{i+\frac{1}{2}, j+\frac{1}{2}}, \quad k = 0, 1, 2, 3, \quad l = 0, 1, 2, 3
\end{aligned}$$

then by solving the above linear system we have

$$\begin{aligned}
u_I(x, y) &= \sum_{k=0}^3 \sum_{l=0}^3 \alpha_{k,l} x^k y^l \\
v_I(x, y) &= \sum_{k=0}^3 \sum_{l=0}^3 \beta_{k,l} x^k y^l
\end{aligned} \tag{2.30}$$

where

$$\alpha_{k,l} = 0, \quad k = 0, 1, 2, 3, \quad l = 1, 2, 3$$

$$\begin{aligned}
\alpha_{3,0} &= (76h^2\tau_{max}a_{i,j} + 704\tau_{max}^3a_{i,j}^3 + 4032\tau_{max}^2x_ia_{i,j}^2 + 420h^2x_i) \\
&\quad / (1008\tau_{max}^2a_{i,j}^2 + 105h^2) \\
\alpha_{2,0} &= (4\tau_{max}^2a_{i,j}^2(17h^2 - 504x_i^2) - 76h^2\tau_{max}x_ia_{i,j} - 704\tau_{max}^3x_ia_{i,j}^3 \\
&\quad + 7(h^4 - 30h^2x_i^2)) / (7(48\tau_{max}^2a_{i,j}^2 + 5h^2)) \\
\alpha_{1,0} &= - (88\tau_{max}^3a_{i,j}^3(h^2 - 40x_i^2) + 40\tau_{max}^2x_ia_{i,j}^2(17h^2 - 168x_i^2) \\
&\quad + 10h^2\tau_{max}a_{i,j}(h^2 - 38x_i^2) + 70h^2x_i(h^2 - 10x_i^2)) / (35(48\tau_{max}^2a_{i,j}^2 + 5h^2)) \\
\alpha_{0,0} &= (160h^2\tau_{max}a_{i,j}(3h^2x_i - 38x_i^3) + 1408\tau_{max}^3x_ia_{i,j}^3(3h^2 - 40x_i^2) \\
&\quad - 32\tau_{max}^2a_{i,j}^2(11h^4 - 510h^2x_i^2 + 2520x_i^4) - 35(h^6 - 48h^4x_i^2 + 240h^2x_i^4)) \\
&\quad / (1680(48\tau_{max}^2a_{i,j}^2 + 5h^2)) \\
\beta_{k,l} &= 0, \quad k = 0, 1, 2, 3, \quad l = 1, 2, 3 \\
\beta_{3,0} &= (76h^2\tau_{max}a_{i+\frac{1}{2},j+\frac{1}{2}} + 704\tau_{max}^3a_{i+\frac{1}{2},j+\frac{1}{2}}^3 + 4032\tau_{max}^2x_{i+\frac{1}{2}}a_{i+\frac{1}{2},j+\frac{1}{2}}^2 + 420h^2x_{i+\frac{1}{2}}) \\
&\quad / (1008\tau_{max}^2a_{i+\frac{1}{2},j+\frac{1}{2}}^2 + 105h^2) \\
\beta_{2,0} &= (4\tau_{max}^2a_{i+\frac{1}{2},j+\frac{1}{2}}^2(17h^2 - 504x_{i+\frac{1}{2}}^2) - 76h^2\tau_{max}x_{i+\frac{1}{2}}a_{i+\frac{1}{2},j+\frac{1}{2}} \\
&\quad - 704\tau_{max}^3x_{i+\frac{1}{2}}a_{i+\frac{1}{2},j+\frac{1}{2}}^3 + 7(h^4 - 30h^2x_{i+\frac{1}{2}}^2)) \\
&\quad / (7(48\tau_{max}^2a_{i+\frac{1}{2},j+\frac{1}{2}}^2 + 5h^2)) \\
\beta_{1,0} &= - (88\tau_{max}^3a_{i+\frac{1}{2},j+\frac{1}{2}}^3(h^2 - 40x_{i+\frac{1}{2}}^2) + 40\tau_{max}^2x_{i+\frac{1}{2}}a_{i+\frac{1}{2},j+\frac{1}{2}}^2(17h^2 - 168x_{i+\frac{1}{2}}^2) \\
&\quad + 10h^2\tau_{max}a_{i+\frac{1}{2},j+\frac{1}{2}}(h^2 - 38x_{i+\frac{1}{2}}^2) + 70h^2x_{i+\frac{1}{2}}(h^2 - 10x_{i+\frac{1}{2}}^2)) \\
&\quad / (35(48\tau_{max}^2a_{i+\frac{1}{2},j+\frac{1}{2}}^2 + 5h^2)) \\
\beta_{0,0} &= (160h^2\tau_{max}a_{i+\frac{1}{2},j+\frac{1}{2}}(3h^2x_{i+\frac{1}{2}} - 38x_{i+\frac{1}{2}}^3) + 1408\tau_{max}^3x_{i+\frac{1}{2}}a_{i+\frac{1}{2},j+\frac{1}{2}}^3(3h^2 - 40x_{i+\frac{1}{2}}^2) \\
&\quad - 32\tau_{max}^2a_{i+\frac{1}{2},j+\frac{1}{2}}^2(11h^4 - 510h^2x_{i+\frac{1}{2}}^2 + 2520x_{i+\frac{1}{2}}^4) \\
&\quad - 35(h^6 - 48h^4x_{i+\frac{1}{2}}^2 + 240h^2x_{i+\frac{1}{2}}^4)) \\
&\quad / (1680(48\tau_{max}^2a_{i+\frac{1}{2},j+\frac{1}{2}}^2 + 5h^2))
\end{aligned}$$

Hence,

$$\int_{x_i}^{x_{i+\frac{1}{2}}} \int_{y_j}^{y_{j+\frac{1}{2}}} (v_I(x, y) - x^4 - u_I(x - \frac{h}{2}, y - \frac{h}{2}) + (x - \frac{h}{2})^4)^2 dx dy$$

$$\begin{aligned}
&= (h^8 \tau_{max}^2 (a_{i+\frac{1}{2},j+\frac{1}{2}} - a_{i,j})^2 (286528 \tau_{max}^4 a_{i+\frac{1}{2},j+\frac{1}{2}}^4 (48 \tau_{max}^2 a_{i,j}^2 + 5h^2)^2 \\
&\quad - 88h^2 \tau_{max}^3 a_{i+\frac{1}{2},j+\frac{1}{2}}^3 (880h^2 \tau_{max} a_{i,j} + 11760h \tau_{max}^2 a_{i,j}^2 + 8448 \tau_{max}^3 a_{i,j}^3 + 1225h^3) \\
&\quad - 5h^4 \tau_{max} a_{i+\frac{1}{2},j+\frac{1}{2}} (1616h^2 \tau_{max} a_{i,j} + 21280h \tau_{max}^2 a_{i,j}^2 \\
&\quad + 15488 \tau_{max}^3 a_{i,j}^3 + 2275h^3) + 160h^2 \tau_{max}^2 a_{i+\frac{1}{2},j+\frac{1}{2}}^2 (-665h^3 \tau_{max} a_{i,j} \\
&\quad + 179598h^2 \tau_{max}^2 a_{i,j}^2 - 6468h \tau_{max}^3 a_{i,j}^3 + 859584 \tau_{max}^4 a_{i,j}^4 + 9381h^4) \\
&\quad + 5h^4 (-2275h^3 \tau_{max} a_{i,j} + 300192h^2 \tau_{max}^2 a_{i,j}^2 - 21560h \tau_{max}^3 a_{i,j}^3 \\
&\quad + 1432640 \tau_{max}^4 a_{i,j}^4 + 15725h^4))) \\
&\quad / (1234800 (48 \tau_{max}^2 a_{i+\frac{1}{2},j+\frac{1}{2}}^2 + 5h^2)^2 (48 \tau_{max}^2 a_{i,j}^2 + 5h^2)^2) \\
&= O(h^{12})
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\int_{x_i}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_j} (v_I(x, y) - x^4 - u_I(x - \frac{h}{2}, y + \frac{h}{2}) + (x - \frac{h}{2})^4)^2 dx dy = O(h^{12}), \\
&\int_{x_{i-\frac{1}{2}}}^{x_i} \int_{y_j}^{y_{j+\frac{1}{2}}} (v_I(x, y) - x^4 - u_I(x + \frac{h}{2}, y - \frac{h}{2}) + (x + \frac{h}{2})^4)^2 dx dy = O(h^{12}), \\
&\int_{x_{i-\frac{1}{2}}}^{x_i} \int_{y_j}^{y_{j+\frac{1}{2}}} (v_I(x, y) - x^4 - u_I(x + \frac{h}{2}, y + \frac{h}{2}) + (x + \frac{h}{2})^4)^2 dx dy = O(h^{12}).
\end{aligned}$$

5. $k = 4, u = x^5$, by the definition of the projection,

$$\begin{aligned}
&\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} u_I dx dy = \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} x^5 dx dy \\
&\tilde{P}_h(u_I; x^k y^l; f, g, u)_{i,j} = \tilde{P}_h(x^5; x^k y^l; f, g, u)_{i,j}, \quad k = 0, 1, 2, 3, 4, \quad l = 0, 1, 2, 3, 4 \\
&\int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} v_I dx dy = \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} x^5 dx dy \\
&\tilde{Q}_h(v_I; x^k y^l; f, g, u)_{i+\frac{1}{2},j+\frac{1}{2}} = \tilde{Q}_h(x^5; x^k y^l; f, g, u)_{i+\frac{1}{2},j+\frac{1}{2}}, \quad k = 0, 1, 2, 3, 4, \quad l = 0, 1, 2, 3, 4
\end{aligned}$$

then by solving the above linear system we have

$$\begin{aligned}
u_I(x, y) &= \sum_{k=0}^4 \sum_{l=0}^4 \alpha_{k,l} x^k y^l \\
v_I(x, y) &= \sum_{k=0}^4 \sum_{l=0}^4 \beta_{k,l} x^k y^l
\end{aligned} \tag{2.31}$$

where

$$\alpha_{k,l} = 0, \quad k = 0, 1, 2, 3, 4, \quad l = 1, 2, 3, 4$$

$$\begin{aligned} \alpha_{4,0} = & (464640a_{i,j}^4\tau_{max}^4x_i + 80a_{i,j}^2h^2\tau_{max}^2(178a_{i,j}\tau_{max} + 795x_i) \\ & + 75h^4(20a_{i,j}\tau_{max} + 21x_i)) \\ & / (92928a_{i,j}^4\tau_{max}^4 + 12720a_{i,j}^2h^2\tau_{max}^2 + 315h^4) \end{aligned}$$

$$\begin{aligned} \alpha_{3,0} = & (5(-2230272a_{i,j}^4\tau_{max}^4x_i^2 - 24h^4(-262a_{i,j}^2\tau_{max}^2 + 600a_{i,j}\tau_{max}x_i + 315x_i^2) \\ & + 128a_{i,j}^2h^2\tau_{max}^2(286a_{i,j}^2\tau_{max}^2 - 1068a_{i,j}\tau_{max}x_i - 2385x_i^2) + 225h^6)) \\ & / (36(30976a_{i,j}^4\tau_{max}^4 + 4240a_{i,j}^2h^2\tau_{max}^2 + 105h^4)) \end{aligned}$$

$$\begin{aligned} \alpha_{2,0} = & - (5(-5203968a_{i,j}^4\tau_{max}^4x_i^3 + 896a_{i,j}^2h^2\tau_{max}^2x_i(286a_{i,j}^2\tau_{max}^2 - 534a_{i,j}\tau_{max}x_i - 795x_i^2) \\ & - 8h^4(-1856a_{i,j}^3\tau_{max}^3 - 5502a_{i,j}^2\tau_{max}^2x_i + 6300a_{i,j}\tau_{max}x_i^2 + 2205x_i^3) \\ & + 63h^6(26a_{i,j}\tau_{max} + 25x_i))) \\ & / (84(30976a_{i,j}^4\tau_{max}^4 + 4240a_{i,j}^2h^2\tau_{max}^2 + 105h^4)) \end{aligned}$$

$$\begin{aligned} \alpha_{1,0} = & - (5(20815872a_{i,j}^4\tau_{max}^4x_i^4 - 56h^6(-86a_{i,j}^2\tau_{max}^2 + 468a_{i,j}\tau_{max}x_i + 225x_i^2) \\ & - 3584a_{i,j}^2h^2\tau_{max}^2x_i^2(572a_{i,j}^2\tau_{max}^2 - 712a_{i,j}\tau_{max}x_i - 795x_i^2) \\ & + 32h^4(440a_{i,j}^4\tau_{max}^4 - 7424a_{i,j}^3\tau_{max}^3x_i - 11004a_{i,j}^2\tau_{max}^2x_i^2 \\ & + 8400a_{i,j}\tau_{max}x_i^3 + 2205x_i^4) + 231h^8)) \\ & / (672(30976a_{i,j}^4\tau_{max}^4 + 4240a_{i,j}^2h^2\tau_{max}^2 + 105h^4)) \end{aligned}$$

$$\begin{aligned} \alpha_{0,0} = & (62447616a_{i,j}^4\tau_{max}^4x_i^5 - 17920a_{i,j}^2h^2\tau_{max}^2x_i^3(572a_{i,j}^2\tau_{max}^2 - 534a_{i,j}\tau_{max}x_i - 477x_i^2) \\ & - 8h^6(-3608a_{i,j}^3\tau_{max}^3 - 9030a_{i,j}^2\tau_{max}^2x_i + 24570a_{i,j}\tau_{max}x_i^2 + 7875x_i^3) \\ & + 480h^4x_i(440a_{i,j}^4\tau_{max}^4 - 3712a_{i,j}^3\tau_{max}^3x_i - 3668a_{i,j}^2\tau_{max}^2x_i^2 \\ & + 2100a_{i,j}\tau_{max}x_i^3 + 441x_i^4) + 315h^8(12a_{i,j}\tau_{max} + 11x_i)) \\ & / (2016(30976a_{i,j}^4\tau_{max}^4 + 4240a_{i,j}^2h^2\tau_{max}^2 + 105h^4)) \end{aligned}$$

$$\beta_{k,l} = 0, \quad k = 0, 1, 2, 3, 4, \quad l = 1, 2, 3, 4$$

$$\beta_{4,0} = (464640a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4x_{i+\frac{1}{2}} + 80a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^2\tau_{max}^2(178a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max} + 795x_{i+\frac{1}{2}}))$$

$$\begin{aligned}
& + 75h^4(20a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max} + 21x_{i+\frac{1}{2}})) \\
& / (92928a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4 + 12720a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^2\tau_{max}^2 + 315h^4) \\
\beta_{3,0} = & (5(-2230272a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4x_{i+\frac{1}{2}}^2 - 24h^4(-262a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2 \\
& + 600a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}} + 315x_{i+\frac{1}{2}}^2) \\
& + 128a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^2\tau_{max}^2(286a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2 \\
& - 1068a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}} - 2385x_{i+\frac{1}{2}}^2) + 225h^6)) \\
& / (36(30976a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4 + 4240a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^2\tau_{max}^2 + 105h^4)) \\
\beta_{2,0} = & - (5(-5203968a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4x_{i+\frac{1}{2}}^3 + 896a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^2\tau_{max}^2x_{i+\frac{1}{2}}(286a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2 \\
& - 534a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}} - 795x_{i+\frac{1}{2}}^2) \\
& - 8h^4(-1856a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3 - 5502a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2x_{i+\frac{1}{2}} \\
& + 6300a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}^2 + 2205x_{i+\frac{1}{2}}^3) + 63h^6(26a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max} + 25x_{i+\frac{1}{2}}))) \\
& / (84(30976a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4 + 4240a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^2\tau_{max}^2 + 105h^4)) \\
\beta_{1,0} = & - (5(20815872a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4x_{i+\frac{1}{2}}^4 - 56h^6(-86a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2 \\
& + 468a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}} + 225x_{i+\frac{1}{2}}^2) \\
& - 3584a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^2\tau_{max}^2x_{i+\frac{1}{2}}^2(572a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2 - 712a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}} - 795x_{i+\frac{1}{2}}^2) \\
& + 32h^4(440a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4 - 7424a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3x_{i+\frac{1}{2}} - 11004a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2x_{i+\frac{1}{2}}^2 \\
& + 8400a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}^3 + 2205x_{i+\frac{1}{2}}^4) + 231h^8)) \\
& / (672(30976a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4 + 4240a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^2\tau_{max}^2 + 105h^4)) \\
\beta_{0,0} = & (62447616a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4x_{i+\frac{1}{2}}^5 - 17920a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^2\tau_{max}^2x_{i+\frac{1}{2}}^3(572a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2 \\
& - 534a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}} - 477x_{i+\frac{1}{2}}^2) \\
& - 8h^6(-3608a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3 - 9030a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2x_{i+\frac{1}{2}} \\
& + 24570a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}^2 + 7875x_{i+\frac{1}{2}}^3) \\
& + 480h^4x_{i+\frac{1}{2}}(440a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4 - 3712a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3x_{i+\frac{1}{2}} - 3668a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2x_{i+\frac{1}{2}}^2 \\
& + 2100a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}^3 + 441x_{i+\frac{1}{2}}^4) + 315h^8(12a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max} + 11x_{i+\frac{1}{2}}))
\end{aligned}$$

$$/(2016(30976a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4 + 4240a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^2\tau_{max}^2 + 105h^4))$$

Hence, we have

$$\begin{aligned} & \int_{x_i}^{x_{i+\frac{1}{2}}} \int_{y_j}^{y_{j+\frac{1}{2}}} (v_I(x, y) - x^5 - u_I(x - \frac{h}{2}, y - \frac{h}{2}) + (x - \frac{h}{2})^5)^2 dx dy \\ = & (h^{14}\tau_{max}^2(a_{i,j} - a_{i+\frac{1}{2},j+\frac{1}{2}})^2(74297776099295232a_{i,j}^6a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^{12} \\ & - 17160990584995840a_{i,j}^5a_{i+\frac{1}{2},j+\frac{1}{2}}^5h\tau_{max}^{11}(a_{i,j} + a_{i+\frac{1}{2},j+\frac{1}{2}}) \\ & + 189000h^{10}\tau_{max}^2(1679899a_{i,j}^2 - 1244578a_{i,j}a_{i+\frac{1}{2},j+\frac{1}{2}} + 1679899a_{i+\frac{1}{2},j+\frac{1}{2}}^2) \\ & + 7938000h^9\tau_{max}^3(a_{i,j} + a_{i+\frac{1}{2},j+\frac{1}{2}})(110956a_{i,j}^2 - 146113a_{i,j}a_{i+\frac{1}{2},j+\frac{1}{2}} + 110956a_{i+\frac{1}{2},j+\frac{1}{2}}^2) \\ & + 317194240a_{i,j}^4a_{i+\frac{1}{2},j+\frac{1}{2}}^4h^2\tau_{max}^{10}(51995327a_{i,j}^2 - 6974930a_{i,j}a_{i+\frac{1}{2},j+\frac{1}{2}} + 51995327a_{i+\frac{1}{2},j+\frac{1}{2}}^2) \\ & - 7208960a_{i,j}^3a_{i+\frac{1}{2},j+\frac{1}{2}}^3h^3\tau_{max}^9(a_{i,j} + a_{i+\frac{1}{2},j+\frac{1}{2}})(411149046a_{i,j}^2 - 110513705a_{i,j}a_{i+\frac{1}{2},j+\frac{1}{2}} \\ & + 411149046a_{i+\frac{1}{2},j+\frac{1}{2}}^2) + 806400h^8\tau_{max}^4(4609881a_{i,j}^4 - 5666392a_{i,j}^3a_{i+\frac{1}{2},j+\frac{1}{2}} \\ & + 21455275a_{i,j}^2a_{i+\frac{1}{2},j+\frac{1}{2}}^2 - 5666392a_{i,j}a_{i+\frac{1}{2},j+\frac{1}{2}}^3 + 4609881a_{i+\frac{1}{2},j+\frac{1}{2}}^4) \\ & + 403200h^7\tau_{max}^5(a_{i,j} + a_{i+\frac{1}{2},j+\frac{1}{2}})(11867779a_{i,j}^4 - 59803436a_{i,j}^3a_{i+\frac{1}{2},j+\frac{1}{2}} \\ & + 53324708a_{i,j}^2a_{i+\frac{1}{2},j+\frac{1}{2}}^2 - 59803436a_{i,j}a_{i+\frac{1}{2},j+\frac{1}{2}}^3 + 11867779a_{i+\frac{1}{2},j+\frac{1}{2}}^4) \\ & - 47308800a_{i,j}a_{i+\frac{1}{2},j+\frac{1}{2}}h^5\tau_{max}^7(a_{i,j} + a_{i+\frac{1}{2},j+\frac{1}{2}})(2637558a_{i,j}^4 - 2676409a_{i,j}^3a_{i+\frac{1}{2},j+\frac{1}{2}} \\ & + 11391631a_{i,j}^2a_{i+\frac{1}{2},j+\frac{1}{2}}^2 - 2676409a_{i,j}a_{i+\frac{1}{2},j+\frac{1}{2}}^3 + 2637558a_{i+\frac{1}{2},j+\frac{1}{2}}^4) \\ & + 1966080a_{i,j}^2a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^4\tau_{max}^8(532478408a_{i,j}^4 - 225653868a_{i,j}^3a_{i+\frac{1}{2},j+\frac{1}{2}} \\ & + 1900203439a_{i,j}^2a_{i+\frac{1}{2},j+\frac{1}{2}}^2 - 225653868a_{i,j}a_{i+\frac{1}{2},j+\frac{1}{2}}^3 + 532478408a_{i+\frac{1}{2},j+\frac{1}{2}}^4) \\ & + 307200h^6\tau_{max}^6(45865050a_{i,j}^6 - 72215220a_{i,j}^5a_{i+\frac{1}{2},j+\frac{1}{2}} + 800405235a_{i,j}^4a_{i+\frac{1}{2},j+\frac{1}{2}}^2 \\ & - 293286178a_{i,j}^3a_{i+\frac{1}{2},j+\frac{1}{2}}^3 + 800405235a_{i,j}^2a_{i+\frac{1}{2},j+\frac{1}{2}}^4 \\ & - 72215220a_{i,j}a_{i+\frac{1}{2},j+\frac{1}{2}}^5 + 45865050a_{i+\frac{1}{2},j+\frac{1}{2}}^6) \\ & + 40059613125h^{11}\tau_{max}(a_{i,j} + a_{i+\frac{1}{2},j+\frac{1}{2}}) + 8661350250h^{12})) \\ & /((508032(30976a_{i,j}^4\tau_{max}^4 + 4240a_{i,j}^2h^2\tau_{max}^2 + 105h^4)^2(30976a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4 \\ & + 4240a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^2\tau_{max}^2 + 105h^4)^2) \\ = & O(h^{14}) \end{aligned}$$

Similarly, we have

$$\begin{aligned} \int_{x_i}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_j} (v_I(x, y) - x^5 - u_I(x - \frac{h}{2}, y + \frac{h}{2}) + (x - \frac{h}{2})^5)^2 dx dy &= O(h^{14}), \\ \int_{x_{i-\frac{1}{2}}}^{x_i} \int_{y_j}^{y_{j+\frac{1}{2}}} (v_I(x, y) - x^5 - u_I(x + \frac{h}{2}, y - \frac{h}{2}) + (x + \frac{h}{2})^5)^2 dx dy &= O(h^{14}), \\ \int_{x_{i-\frac{1}{2}}}^{x_i} \int_{y_j}^{y_{j+\frac{1}{2}}} (v_I(x, y) - x^5 - u_I(x + \frac{h}{2}, y + \frac{h}{2}) + (x + \frac{h}{2})^5)^2 dx dy &= O(h^{14}). \end{aligned}$$

6. $k = 5, u = x^6$, by the definition of the projection,

$$\begin{aligned} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} u_I dx dy &= \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} x^6 dx dy \\ \tilde{P}_h(u_I; x^k y^l; f, g, u)_{i,j} &= \tilde{P}_h(x^6; x^k y^l; f, g, u)_{i,j}, \quad k = 0, 1, \dots, 5, \quad l = 0, 1, \dots, 5 \\ \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} v_I dx dy &= \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} x^6 dx dy \\ \tilde{Q}_h(v_I; x^k y^l; f, g, u)_{i+\frac{1}{2}, j+\frac{1}{2}} &= \tilde{Q}_h(x^6; x^k y^l; f, g, u)_{i+\frac{1}{2}, j+\frac{1}{2}}, \quad k = 0, 1, \dots, 5, \quad l = 0, 1, \dots, 5 \end{aligned}$$

then by solving the above linear system we have

$$\begin{aligned} u_I(x, y) &= \sum_{k=0}^5 \sum_{l=0}^5 \alpha_{k,l} x^k y^l \\ v_I(x, y) &= \sum_{k=0}^5 \sum_{l=0}^5 \beta_{k,l} x^k y^l \end{aligned} \tag{2.32}$$

where

$$\alpha_{k,l} = 0, \quad k = 0, 1, \dots, 5, \quad l = 1, \dots, 5$$

$$\begin{aligned} \alpha_{5,0} &= (630(33x_i + 145a_{i,j}\tau_{max})h^4 + 288a_{i,j}^2\tau_{max}^2(70455x_i + 481a_{i,j}\tau_{max})h^2 \\ &\quad + 5632a_{i,j}^4\tau_{max}^4(35193x_i - 1331a_{i,j}\tau_{max})) \\ &\quad / (11(315h^4 + 307440a_{i,j}^2\tau_{max}^2h^2 + 3003136a_{i,j}^4\tau_{max}^4)) \end{aligned}$$

$$\begin{aligned} \alpha_{4,0} &= (5(3465h^6 - 12(10395x_i^2 + 91350a_{i,j}\tau_{max}x_i - 191168a_{i,j}^2\tau_{max}^2)h^4 \\ &\quad + 64a_{i,j}^2\tau_{max}^2(-1902285x_i^2 - 25974a_{i,j}\tau_{max}x_i \\ &\quad + 348082a_{i,j}^2\tau_{max}^2)h^2 + 33792a_{i,j}^4x_i\tau_{max}^4(2662a_{i,j}\tau_{max} - 35193x_i)) \\ &\quad / (132(315h^4 + 307440a_{i,j}^2\tau_{max}^2h^2 + 3003136a_{i,j}^4\tau_{max}^4)) \end{aligned}$$

$$\begin{aligned}
\alpha_{3,0} = & (5(-9(1155x_i + 5531a_{i,j}\tau_{max})h^6 + 12(10395x_i^3 + 137025a_{i,j}\tau_{max}x_i^2 \\
& - 573504a_{i,j}^2\tau_{max}^2x_i - 21856a_{i,j}^3\tau_{max}^3)h^4 + 64a_{i,j}^2\tau_{max}^2(1902285x_i^3 \\
& + 38961a_{i,j}\tau_{max}x_i^2 - 1044246a_{i,j}^2\tau_{max}^2x_i + 34606a_{i,j}^3\tau_{max}^3)h^2 \\
& - 101376a_{i,j}^4x_i^2\tau_{max}^4(1331a_{i,j}\tau_{max} - 11731x_i)) \\
& / (99(315h^4 + 307440a_{i,j}^2\tau_{max}^2h^2 + 3003136a_{i,j}^4\tau_{max}^4))
\end{aligned}$$

$$\begin{aligned}
\alpha_{2,0} = & - (5(2871h^8 - 48(3465x_i^2 + 33186a_{i,j}\tau_{max}x_i - 23006a_{i,j}^2\tau_{max}^2)h^6 \\
& + 32(31185x_i^4 + 548100a_{i,j}\tau_{max}x_i^3 - 3441024a_{i,j}^2\tau_{max}^2x_i^2 - 262272a_{i,j}^3\tau_{max}^3x_i \\
& + 329432a_{i,j}^4\tau_{max}^4)h^4 + 512a_{i,j}^2x_i\tau_{max}^2(1902285x_i^3 + 51948a_{i,j}\tau_{max}x_i^2 \\
& - 2088492a_{i,j}^2\tau_{max}^2x_i + 138424a_{i,j}^3\tau_{max}^3)h^2 \\
& - 270336a_{i,j}^4x_i^3\tau_{max}^4(5324a_{i,j}\tau_{max} - 35193x_i)) \\
& / (1056(315h^4 + 307440a_{i,j}^2\tau_{max}^2h^2 + 3003136a_{i,j}^4\tau_{max}^4))
\end{aligned}$$

$$\begin{aligned}
\alpha_{1,0} = & (3465(29x_i + 149a_{i,j}\tau_{max})h^8 - 336(5775x_i^3 + 82965a_{i,j}\tau_{max}x_i^2 \\
& - 115030a_{i,j}^2\tau_{max}^2x_i - 12178a_{i,j}^3\tau_{max}^3)h^6 + 160(43659x_i^5 + 959175a_{i,j}\tau_{max}x_i^4 \\
& - 8029056a_{i,j}^2\tau_{max}^2x_i^3 - 917952a_{i,j}^3\tau_{max}^3x_i^2 + 2306024a_{i,j}^4\tau_{max}^4x_i - 53240a_{i,j}^5\tau_{max}^5)h^4 \\
& + 17920a_{i,j}^2x_i^2\tau_{max}^2(380457x_i^3 + 12987a_{i,j}\tau_{max}x_i^2 - 696164a_{i,j}^2\tau_{max}^2x_i \\
& + 69212a_{i,j}^3\tau_{max}^3)h^2 - 1892352a_{i,j}^4x_i^4\tau_{max}^4(6655a_{i,j}\tau_{max} - 35193x_i)) \\
& / (3696(315h^4 + 307440a_{i,j}^2\tau_{max}^2h^2 + 3003136a_{i,j}^4\tau_{max}^4))
\end{aligned}$$

$$\begin{aligned}
\alpha_{0,0} = & (17325h^{10} - 84(14355x_i^2 + 147510a_{i,j}\tau_{max}x_i - 37792a_{i,j}^2\tau_{max}^2)h^8 + 32(363825x_i^4 \\
& + 6969060a_{i,j}\tau_{max}x_i^3 - 14493780a_{i,j}^2\tau_{max}^2x_i^2 - 3068856a_{i,j}^3\tau_{max}^3x_i + 873136a_{i,j}^4\tau_{max}^4)h^6 \\
& - 1920x_i(14553x_i^5 + 383670a_{i,j}\tau_{max}x_i^4 - 4014528a_{i,j}^2\tau_{max}^2x_i^3 - 611968a_{i,j}^3\tau_{max}^3x_i^2 \\
& + 2306024a_{i,j}^4\tau_{max}^4x_i - 106480a_{i,j}^5\tau_{max}^5)h^4 - 14336a_{i,j}^2x_i^3\tau_{max}^2(1902285x_i^3 \\
& + 77922a_{i,j}\tau_{max}x_i^2 - 5221230a_{i,j}^2\tau_{max}^2x_i + 692120a_{i,j}^3\tau_{max}^3)h^2 \\
& + 22708224a_{i,j}^4x_i^5\tau_{max}^4(2662a_{i,j}\tau_{max} - 11731x_i)) \\
& / (88704(315h^4 + 307440a_{i,j}^2\tau_{max}^2h^2 + 3003136a_{i,j}^4\tau_{max}^4))
\end{aligned}$$

$$\beta_{k,l} = 0, \quad k = 0, 1, \dots, 5, \quad l = 1, \dots, 5$$

$$\begin{aligned} \beta_{5,0} = & (630(33x_{i+\frac{1}{2}} + 145a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max})h^4 \\ & + 288a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2(70455x_{i+\frac{1}{2}} + 481a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max})h^2 \\ & + 5632a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4(35193x_{i+\frac{1}{2}} - 1331a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max})) \\ & /((11(315h^4 + 307440a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2h^2 + 3003136a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4)) \end{aligned}$$

$$\begin{aligned} \beta_{4,0} = & (5(3465h^6 - 12(10395x_{i+\frac{1}{2}}^2 + 91350a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}} - 191168a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2)h^4 \\ & + 64a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2(-1902285x_{i+\frac{1}{2}}^2 - 25974a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}} \\ & + 348082a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2)h^2 + 33792a_{i+\frac{1}{2},j+\frac{1}{2}}^4x_{i+\frac{1}{2}}\tau_{max}^4 \\ & (2662a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max} - 35193x_{i+\frac{1}{2}}))) \\ & /((132(315h^4 + 307440a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2h^2 + 3003136a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4)) \end{aligned}$$

$$\begin{aligned} \beta_{3,0} = & (5(-9(1155x_{i+\frac{1}{2}} + 5531a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max})h^6 + 12(10395x_{i+\frac{1}{2}}^3 + 137025a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}^2 \\ & - 573504a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2x_{i+\frac{1}{2}} - 21856a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3)h^4 + 64a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2(1902285x_{i+\frac{1}{2}}^3 \\ & + 38961a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}^2 - 1044246a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2x_{i+\frac{1}{2}} + 34606a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3)h^2 \\ & - 101376a_{i+\frac{1}{2},j+\frac{1}{2}}^4x_{i+\frac{1}{2}}^2\tau_{max}^4(1331a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max} - 11731x_{i+\frac{1}{2}}))) \\ & /((99(315h^4 + 307440a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2h^2 + 3003136a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4)) \end{aligned}$$

$$\begin{aligned} \beta_{2,0} = & -(5(2871h^8 - 48(3465x_{i+\frac{1}{2}}^2 + 33186a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}} - 23006a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2)h^6 \\ & + 32(31185x_{i+\frac{1}{2}}^4 + 548100a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}^3 - 3441024a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2x_{i+\frac{1}{2}}^2 \\ & - 262272a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3x_{i+\frac{1}{2}} + 329432a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4)h^4 \\ & + 512a_{i+\frac{1}{2},j+\frac{1}{2}}^2x_{i+\frac{1}{2}}\tau_{max}^2(1902285x_{i+\frac{1}{2}}^3 \\ & + 51948a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}^2 - 2088492a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2x_{i+\frac{1}{2}} + 138424a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3)h^2 \\ & - 270336a_{i+\frac{1}{2},j+\frac{1}{2}}^4x_{i+\frac{1}{2}}^3\tau_{max}^4(5324a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max} - 35193x_{i+\frac{1}{2}}))) \\ & /((1056(315h^4 + 307440a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2h^2 + 3003136a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4)) \end{aligned}$$

$$\begin{aligned} \beta_{1,0} = & (3465(29x_{i+\frac{1}{2}} + 149a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max})h^8 - 336(5775x_{i+\frac{1}{2}}^3 + 82965a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}^2 \\ & - 115030a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2x_{i+\frac{1}{2}} - 12178a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3)h^6 + 160(43659x_{i+\frac{1}{2}}^5 \end{aligned}$$

$$\begin{aligned}
& + 959175a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}^4 - 8029056a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2x_{i+\frac{1}{2}}^3 - 917952a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3x_{i+\frac{1}{2}}^2 \\
& + 2306024a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4x_{i+\frac{1}{2}} - 53240a_{i+\frac{1}{2},j+\frac{1}{2}}^5\tau_{max}^5)h^4 \\
& + 17920a_{i+\frac{1}{2},j+\frac{1}{2}}^2x_{i+\frac{1}{2}}^2\tau_{max}^2(380457x_{i+\frac{1}{2}}^3 \\
& + 12987a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}^2 - 696164a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2x_{i+\frac{1}{2}} \\
& + 69212a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3)h^2 - 1892352a_{i+\frac{1}{2},j+\frac{1}{2}}^4x_{i+\frac{1}{2}}^4\tau_{max}^4 \\
& (6655a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max} - 35193x_{i+\frac{1}{2}})) \\
& / (3696(315h^4 + 307440a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2h^2 + 3003136a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4)) \\
\beta_{0,0} = & (17325h^{10} - 84(14355x_{i+\frac{1}{2}}^2 + 147510a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}} - 37792a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2)h^8 \\
& + 32(363825x_{i+\frac{1}{2}}^4 + 6969060a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}^3 - 14493780a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2x_{i+\frac{1}{2}}^2 \\
& - 3068856a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3x_{i+\frac{1}{2}} + 873136a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4)h^6 \\
& - 1920x_{i+\frac{1}{2}}(14553x_{i+\frac{1}{2}}^5 + 383670a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}^4 - 4014528a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2x_{i+\frac{1}{2}}^3 \\
& - 611968a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3x_{i+\frac{1}{2}}^2 + 2306024a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4x_{i+\frac{1}{2}} - 106480a_{i+\frac{1}{2},j+\frac{1}{2}}^5\tau_{max}^5)h^4 \\
& - 14336a_{i+\frac{1}{2},j+\frac{1}{2}}^2x_{i+\frac{1}{2}}^3\tau_{max}^2(1902285x_{i+\frac{1}{2}}^3 + 77922a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}^2 \\
& - 5221230a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2x_{i+\frac{1}{2}} + 692120a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3)h^2 \\
& + 22708224a_{i+\frac{1}{2},j+\frac{1}{2}}^4x_{i+\frac{1}{2}}^5\tau_{max}^4(2662a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max} - 11731x_{i+\frac{1}{2}})) \\
& / ((88704(315h^4 + 307440a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2h^2 + 3003136a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4))
\end{aligned}$$

Hence, we have

$$\begin{aligned}
& \int_{x_i}^{x_{i+\frac{1}{2}}} \int_{y_j}^{y_{j+\frac{1}{2}}} (v_I(x, y) - x^6 - u_I(x - \frac{h}{2}, y - \frac{h}{2}) + (x - \frac{h}{2})^6)^2 dx dy \\
& = (h^{12}\tau_{max}^2(a_{i,j} - a_{i+\frac{1}{2},j+\frac{1}{2}})^2(195012000131713231589585453056a_{i,j}^8a_{i+\frac{1}{2},j+\frac{1}{2}}^8\tau_{max}^{16} \\
& - 201891950852331333916557312a_{i,j}^6a_{i+\frac{1}{2},j+\frac{1}{2}}^6h^3\tau_{max}^{13}(a_{i,j} + a_{i+\frac{1}{2},j+\frac{1}{2}}) \\
& + 1800338400h^{14}\tau_{max}^2(13438284110a_{i,j}^2 - 51320885931a_{i,j}a_{i+\frac{1}{2},j+\frac{1}{2}} \\
& + 13438284110a_{i+\frac{1}{2},j+\frac{1}{2}}^2) + 100416141472907132928a_{i,j}^6a_{i+\frac{1}{2},j+\frac{1}{2}}^6h^2\tau_{max}^{14} \\
& (397624535a_{i,j}^2 - 18450573a_{i,j}a_{i+\frac{1}{2},j+\frac{1}{2}} + 397624535a_{i+\frac{1}{2},j+\frac{1}{2}}^2) \\
& - 2200413600h^{13}\tau_{max}^3(10662687321a_{i,j}^3 + 292187561882a_{i,j}^2a_{i+\frac{1}{2},j+\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
& + 292187561882a_{i,j}a_{i+\frac{1}{2},j+\frac{1}{2}}^2 + 10662687321a_{i+\frac{1}{2},j+\frac{1}{2}}^3) \\
& + 228614400h^{12}\tau_{max}^4(2094049467654a_{i,j}^4 - 8864817048804a_{i,j}^3a_{i+\frac{1}{2},j+\frac{1}{2}} \\
& + 295334595970901a_{i,j}^2a_{i+\frac{1}{2},j+\frac{1}{2}}^2 \\
& - 8864817048804a_{i,j}a_{i+\frac{1}{2},j+\frac{1}{2}}^3 + 2094049467654a_{i+\frac{1}{2},j+\frac{1}{2}}^4) \\
& - 23253221376a_{i,j}^4a_{i+\frac{1}{2},j+\frac{1}{2}}^4h^5\tau_{max}^{11}(2316625977228695a_{i,j}^3 \\
& + 3103819172036596a_{i,j}^2a_{i+\frac{1}{2},j+\frac{1}{2}}^2 + 3103819172036596a_{i,j}a_{i+\frac{1}{2},j+\frac{1}{2}}^2 \\
& + 2316625977228695a_{i+\frac{1}{2},j+\frac{1}{2}}^3) \\
& + 1056964608a_{i,j}^4a_{i+\frac{1}{2},j+\frac{1}{2}}^4h^4\tau_{max}^{12}(1972327189988974885a_{i,j}^4 \\
& - 357721171163321350a_{i,j}^3a_{i+\frac{1}{2},j+\frac{1}{2}}^3 + 8331528893607301417a_{i,j}^2a_{i+\frac{1}{2},j+\frac{1}{2}}^2 \\
& - 357721171163321350a_{i,j}a_{i+\frac{1}{2},j+\frac{1}{2}}^3 + 1972327189988974885a_{i+\frac{1}{2},j+\frac{1}{2}}^4) \\
& - 11735539200h^{11}\tau_{max}^5(54815097606a_{i,j}^5 + 1107739544932a_{i,j}^4a_{i+\frac{1}{2},j+\frac{1}{2}} \\
& + 4065675623387a_{i,j}^3a_{i+\frac{1}{2},j+\frac{1}{2}}^2 \\
& + 4065675623387a_{i,j}^2a_{i+\frac{1}{2},j+\frac{1}{2}}^3 + 1107739544932a_{i,j}a_{i+\frac{1}{2},j+\frac{1}{2}}^4 \\
& + 54815097606a_{i+\frac{1}{2},j+\frac{1}{2}}^5) - 19074908160a_{i,j}^2a_{i+\frac{1}{2},j+\frac{1}{2}}^2 \\
& h^7\tau_{max}^9(179294959298488a_{i,j}^5 + 308410865284400a_{i,j}^4a_{i+\frac{1}{2},j+\frac{1}{2}} \\
& + 770619223416953a_{i,j}^3a_{i+\frac{1}{2},j+\frac{1}{2}}^2 + 770619223416953a_{i,j}^2a_{i+\frac{1}{2},j+\frac{1}{2}}^3 \\
& + 308410865284400a_{i,j}a_{i+\frac{1}{2},j+\frac{1}{2}}^4 \\
& + 179294959298488a_{i+\frac{1}{2},j+\frac{1}{2}}^5) + 487710720h^{10}\tau_{max}^6(5441989209404a_{i,j}^6 \\
& - 26664212415965a_{i,j}^5a_{i+\frac{1}{2},j+\frac{1}{2}}^5 + 2805981567667107a_{i,j}^4a_{i+\frac{1}{2},j+\frac{1}{2}}^2 \\
& - 492603505952389a_{i,j}^3a_{i+\frac{1}{2},j+\frac{1}{2}}^3 + 2805981567667107a_{i,j}^2a_{i+\frac{1}{2},j+\frac{1}{2}}^4 \\
& - 26664212415965a_{i,j}a_{i+\frac{1}{2},j+\frac{1}{2}}^5 + 5441989209404a_{i+\frac{1}{2},j+\frac{1}{2}}^6) \\
& + 1189085184a_{i,j}^2a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^6\tau_{max}^{10}(3522080594245675a_{i,j}^6 \\
& - 16385982633633415a_{i,j}^5a_{i+\frac{1}{2},j+\frac{1}{2}}^5 + 466662656914525505a_{i,j}^4a_{i+\frac{1}{2},j+\frac{1}{2}}^2 \\
& - 65580553513354431a_{i,j}^3a_{i+\frac{1}{2},j+\frac{1}{2}}^3 + 466662656914525505a_{i,j}^2a_{i+\frac{1}{2},j+\frac{1}{2}}^4
\end{aligned}$$

$$\begin{aligned}
& - 16385982633633415a_{i,j}a_{i+\frac{1}{2},j+\frac{1}{2}}^5 + 3522080594245675a_{i+\frac{1}{2},j+\frac{1}{2}}^6) \\
& - 1788272640h^9\tau_{max}^7(1947374673365a_{i,j}^7 + 36401168398920a_{i,j}^6a_{i+\frac{1}{2},j+\frac{1}{2}} \\
& + 422084292605924a_{i,j}^5a_{i+\frac{1}{2},j+\frac{1}{2}}^2 + 595382564504327a_{i,j}^4a_{i+\frac{1}{2},j+\frac{1}{2}}^3 \\
& + 595382564504327a_{i,j}^3a_{i+\frac{1}{2},j+\frac{1}{2}}^4 \\
& + 422084292605924a_{i,j}^2a_{i+\frac{1}{2},j+\frac{1}{2}}^5 + 36401168398920a_{i,j}a_{i+\frac{1}{2},j+\frac{1}{2}}^6 \\
& + 1947374673365a_{i+\frac{1}{2},j+\frac{1}{2}}^7) + 185794560h^8\tau_{max}^8(11547805227035a_{i,j}^8 \\
& - 106377383972860a_{i,j}^7a_{i+\frac{1}{2},j+\frac{1}{2}} + 39707200376096864a_{i,j}^6a_{i+\frac{1}{2},j+\frac{1}{2}}^2 \\
& - 22325282290601846a_{i,j}^5a_{i+\frac{1}{2},j+\frac{1}{2}}^3 + 259867644987323578a_{i,j}^4a_{i+\frac{1}{2},j+\frac{1}{2}}^4 \\
& - 22325282290601846a_{i,j}^3a_{i+\frac{1}{2},j+\frac{1}{2}}^5 + 39707200376096864a_{i,j}^2a_{i+\frac{1}{2},j+\frac{1}{2}}^6 \\
& - 106377383972860a_{i,j}a_{i+\frac{1}{2},j+\frac{1}{2}}^7 + 11547805227035a_{i+\frac{1}{2},j+\frac{1}{2}}^8) \\
& + 604417549765837500h^{15}\tau_{max}(a_{i,j} + a_{i+\frac{1}{2},j+\frac{1}{2}}) + 71270165866433625h^{16})) \\
& / (21638098944(3003136a_{i,j}^4\tau_{max}^4 + 307440a_{i,j}^2h^2\tau_{max}^2 + 315h^4)^2 \\
& (3003136a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4 + 307440a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^2\tau_{max}^2 + 315h^4)^2) = O(h^{16})
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
& \int_{x_i}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_j} (v_I(x, y) - x^6 - u_I(x - \frac{h}{2}, y + \frac{h}{2}) + (x - \frac{h}{2})^6)^2 dx dy = O(h^{16}), \\
& \int_{x_{i-\frac{1}{2}}}^{x_i} \int_{y_j}^{y_{j+\frac{1}{2}}} (v_I(x, y) - x^6 - u_I(x + \frac{h}{2}, y - \frac{h}{2}) + (x + \frac{h}{2})^6)^2 dx dy = O(h^{16}), \\
& \int_{x_{i-\frac{1}{2}}}^{x_i} \int_{y_j}^{y_{j+\frac{1}{2}}} (v_I(x, y) - x^6 - u_I(x + \frac{h}{2}, y + \frac{h}{2}) + (x + \frac{h}{2})^6)^2 dx dy = O(h^{16}).
\end{aligned}$$

7. $k = 6, u = x^7$, by the definition of the projection,

$$\begin{aligned}
& \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} u_I dx dy = \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} x^7 dx dy \\
& \tilde{P}_h(u_I; x^k y^l; f, g, u)_{i,j} = \tilde{P}_h(x^7; x^k y^l; f, g, u)_{i,j}, \quad k = 0, 1, \dots, 6, \quad l = 0, 1, \dots, 6 \\
& \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} v_I dx dy = \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} x^7 dx dy \\
& \tilde{Q}_h(v_I; x^k y^l; f, g, u)_{i+\frac{1}{2},j+\frac{1}{2}} = \tilde{Q}_h(x^7; x^k y^l; f, g, u)_{i+\frac{1}{2},j+\frac{1}{2}}, \quad k = 0, 1, \dots, 6, \quad l = 0, 1, \dots, 6
\end{aligned}$$

then by solving the above linear system we have

$$\begin{aligned} u_I(x, y) &= \sum_{k=0}^6 \sum_{l=0}^6 \alpha_{k,l} x^k y^l \\ v_I(x, y) &= \sum_{k=0}^6 \sum_{l=0}^6 \beta_{k,l} x^k y^l \end{aligned} \tag{2.33}$$

where

$$\alpha_{k,l} = 0, \quad k = 0, 1, \dots, 6, \quad l = 1, \dots, 6$$

$$\begin{aligned} \alpha_{6,0} &= (7(94332080128a_{i,j}^6 \tau_{max}^6 x_i - 1280a_{i,j}^4 h^2 \tau_{max}^4 (4415546a_{i,j} \tau_{max} \\ &\quad - 42028623x_i) - 672a_{i,j}^2 h^4 \tau_{max}^2 (789340a_{i,j} \tau_{max} - 6671691x_i) \\ &\quad + 63h^6 (67718a_{i,j} \tau_{max} + 2145x_i))) \\ &\quad / (13(7256313856a_{i,j}^6 \tau_{max}^6 + 4138202880a_{i,j}^4 h^2 \tau_{max}^4 \\ &\quad + 344875104a_{i,j}^2 h^4 \tau_{max}^2 + 10395h^6)) \end{aligned}$$

$$\begin{aligned} \alpha_{5,0} &= (21(-754656641024a_{i,j}^6 \tau_{max}^6 x_i^2 - 504h^6 (-1271288a_{i,j}^2 \tau_{max}^2 \\ &\quad + 135436a_{i,j} \tau_{max} x_i + 2145x_i^2) + 256a_{i,j}^2 h^4 \tau_{max}^2 (28301533a_{i,j}^2 \tau_{max}^2 \\ &\quad + 33152280a_{i,j} \tau_{max} x_i - 140105511x_i^2) \\ &\quad + 2048a_{i,j}^4 h^2 \tau_{max}^4 (4448202a_{i,j}^2 \tau_{max}^2 \\ &\quad + 44155460a_{i,j} \tau_{max} x_i - 210143115x_i^2) + 28665h^8)) \\ &\quad / (104(7256313856a_{i,j}^6 \tau_{max}^6 + 4138202880a_{i,j}^4 h^2 \tau_{max}^4 \\ &\quad + 344875104a_{i,j}^2 h^4 \tau_{max}^2 + 10395h^6)) \end{aligned}$$

$$\begin{aligned} \alpha_{4,0} &= - (35(-8301223051264a_{i,j}^6 \tau_{max}^6 x_i^3 + 67584a_{i,j}^4 h^2 \tau_{max}^4 x_i (4448202a_{i,j}^2 \tau_{max}^2 \\ &\quad + 22077730a_{i,j} \tau_{max} x_i - 70047705x_i^2) - 24h^6 (103797556a_{i,j}^3 \tau_{max}^3 \\ &\quad - 881002584a_{i,j}^2 \tau_{max}^2 x_i + 46928574a_{i,j} \tau_{max} x_i^2 + 495495x_i^3) \\ &\quad - 256a_{i,j}^2 h^4 \tau_{max}^2 (108151636a_{i,j}^3 \tau_{max}^3 - 933950589a_{i,j}^2 \tau_{max}^2 x_i \\ &\quad - 547012620a_{i,j} \tau_{max} x_i^2 + 1541160621x_i^3) \\ &\quad + 2079h^8 (15216a_{i,j} \tau_{max} + 455x_i))) \end{aligned}$$

$$\begin{aligned}
& /((1144(7256313856a_{i,j}^6\tau_{max}^6 + 4138202880a_{i,j}^4h^2\tau_{max}^4 \\
& + 344875104a_{i,j}^2h^4\tau_{max}^2 + 10395h^6)) \\
\alpha_{3,0} = & - (35(16602446102528a_{i,j}^6\tau_{max}^6x_i^4 - 132h^8(-6457912a_{i,j}^2\tau_{max}^2 \\
& + 1917216a_{i,j}\tau_{max}x_i + 28665x_i^2) - 45056a_{i,j}^4h^2\tau_{max}^4x_i^2(26689212a_{i,j}^2\tau_{max}^2 \\
& + 88310920a_{i,j}\tau_{max}x_i - 210143115x_i^2) + 16h^6(569473600a_{i,j}^4\tau_{max}^4 \\
& + 1245570672a_{i,j}^3\tau_{max}^3x_i - 5286015504a_{i,j}^2\tau_{max}^2x_i^2 \\
& + 187714296a_{i,j}\tau_{max}x_i^3 + 1486485x_i^4) + 512a_{i,j}^2h^4\tau_{max}^2(13427128a_{i,j}^4\tau_{max}^4 \\
& + 432606544a_{i,j}^3\tau_{max}^3x_i - 1867901178a_{i,j}^2\tau_{max}^2x_i^2 \\
& - 729350160a_{i,j}\tau_{max}x_i^3 + 1541160621x_i^4) + 63063h^{10})) \\
& /((2288(7256313856a_{i,j}^6\tau_{max}^6 + 4138202880a_{i,j}^4h^2\tau_{max}^4 \\
& + 344875104a_{i,j}^2h^4\tau_{max}^2 + 10395h^6)) \\
\alpha_{2,0} = & (7(49807338307584a_{i,j}^6\tau_{max}^6x_i^5 - 675840a_{i,j}^4h^2\tau_{max}^4x_i^3(8896404a_{i,j}^2\tau_{max}^2 \\
& + 22077730a_{i,j}\tau_{max}x_i - 42028623x_i^2) - 180h^8(7154768a_{i,j}^3\tau_{max}^3 \\
& - 71037032a_{i,j}^2\tau_{max}^2x_i + 10544688a_{i,j}\tau_{max}x_i^2 + 105105x_i^3) \\
& + 1536a_{i,j}^2h^4\tau_{max}^2x_i(67135640a_{i,j}^4\tau_{max}^4 + 1081516360a_{i,j}^3\tau_{max}^3x_i \\
& - 3113168630a_{i,j}^2\tau_{max}^2x_i^2 - 911687700a_{i,j}\tau_{max}x_i^3 \\
& + 1541160621x_i^4) + 16h^6(-999477488a_{i,j}^5\tau_{max}^5 + 8542104000a_{i,j}^4\tau_{max}^4x_i \\
& + 9341780040a_{i,j}^3\tau_{max}^3x_i^2 - 26430077520a_{i,j}^2\tau_{max}^2x_i^3 \\
& + 703928610a_{i,j}\tau_{max}x_i^4 + 4459455x_i^5) \\
& + 99h^{10}(334172a_{i,j}\tau_{max} + 9555x_i)) \\
& /((2288(7256313856a_{i,j}^6\tau_{max}^6 + 4138202880a_{i,j}^4h^2\tau_{max}^4 \\
& + 344875104a_{i,j}^2h^4\tau_{max}^2 + 10395h^6)) \\
\alpha_{1,0} = & (7(-796917412921344a_{i,j}^6\tau_{max}^6x_i^6 - 1584h^{10}(-2907158a_{i,j}^2\tau_{max}^2 \\
& + 2005032a_{i,j}\tau_{max}x_i + 28665x_i^2) + 32440320a_{i,j}^4h^2\tau_{max}^4x_i^4
\end{aligned}$$

$$\begin{aligned}
& (4448202a_{i,j}^2\tau_{max}^2 + 8831092a_{i,j}\tau_{max}x_i - 14009541x_i^2) \\
& + 864h^8(54537656a_{i,j}^4\tau_{max}^4 + 143095360a_{i,j}^3\tau_{max}^3x_i \\
& - 710370320a_{i,j}^2\tau_{max}^2x_i^2 + 70297920a_{i,j}\tau_{max}x_i^3 + 525525x_i^4) \\
& - 24576a_{i,j}^2h^4\tau_{max}^2x_i^2(201406920a_{i,j}^4\tau_{max}^4 + 2163032720a_{i,j}^3\tau_{max}^3x_i \\
& - 4669752945a_{i,j}^2\tau_{max}^2x_i^2 - 1094025240a_{i,j}\tau_{max}x_i^3 + 1541160621x_i^4) \\
& - 256h^6(-55859408a_{i,j}^6\tau_{max}^6 - 5996864928a_{i,j}^5\tau_{max}^5x_i \\
& + 25626312000a_{i,j}^4\tau_{max}^4x_i^2 + 18683560080a_{i,j}^3\tau_{max}^3x_i^3 - 39645116280a_{i,j}^2\tau_{max}^2x_i^4 \\
& + 844714332a_{i,j}\tau_{max}x_i^5 + 4459455x_i^6) + 637065h^{12})) \\
& /((109824(7256313856a_{i,j}^6\tau_{max}^6 + 4138202880a_{i,j}^4h^2\tau_{max}^4 \\
& + 344875104a_{i,j}^2h^4\tau_{max}^2 + 10395h^6)) \\
\alpha_{0,0} = & (796917412921344a_{i,j}^6\tau_{max}^6x_i^7 - 15138816a_{i,j}^4h^2\tau_{max}^4x_i^5(13344606a_{i,j}^2\tau_{max}^2 \\
& + 22077730a_{i,j}\tau_{max}x_i - 30020445x_i^2) + 1008h^{10}(1438204a_{i,j}^3\tau_{max}^3 \\
& - 31978738a_{i,j}^2\tau_{max}^2x_i + 11027676a_{i,j}\tau_{max}x_i^2 + 105105x_i^3) \\
& + 172032a_{i,j}^2h^4\tau_{max}^2x_i^3(67135640a_{i,j}^4\tau_{max}^4 + 540758180a_{i,j}^3\tau_{max}^3x_i \\
& - 933950589a_{i,j}^2\tau_{max}^2x_i^2 - 182337540a_{i,j}\tau_{max}x_i^3 + 220165803x_i^4) \\
& - 32h^8(-967818016a_{i,j}^5\tau_{max}^5 + 10307616984a_{i,j}^4\tau_{max}^4x_i \\
& + 13522511520a_{i,j}^3\tau_{max}^3x_i^2 - 44753330160a_{i,j}^2\tau_{max}^2x_i^3 + 3321576720a_{i,j}\tau_{max}x_i^4 \\
& + 19864845x_i^5) + 1792h^6x_i(-55859408a_{i,j}^6\tau_{max}^6 - 2998432464a_{i,j}^5\tau_{max}^5x_i \\
& + 8542104000a_{i,j}^4\tau_{max}^4x_i^2 + 4670890020a_{i,j}^3\tau_{max}^3x_i^3 - 7929023256a_{i,j}^2\tau_{max}^2x_i^4 \\
& + 140785722a_{i,j}\tau_{max}x_i^5 + 637065x_i^6) - 693h^{12}(232088a_{i,j}\tau_{max} + 6435x_i)) \\
& /((109824(7256313856a_{i,j}^6\tau_{max}^6 + 4138202880a_{i,j}^4h^2\tau_{max}^4 \\
& + 344875104a_{i,j}^2h^4\tau_{max}^2 + 10395h^6))
\end{aligned}$$

$$\beta_{k,l} = 0, \quad k = 0, 1, \dots, 6, \quad l = 1, \dots, 6$$

$$\beta_{6,0} = (7(94332080128a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^6x_{i+\frac{1}{2}} - 1280a_{i+\frac{1}{2},j+\frac{1}{2}}^4h^2\tau_{max}^4(4415546a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}$$

$$\begin{aligned}
& - 42028623x_{i+\frac{1}{2}}) - 672a_{i+\frac{1}{2},j+\frac{1}{2}}^2 h^4 \tau_{max}^2 (789340a_{i+\frac{1}{2},j+\frac{1}{2}} \tau_{max} \\
& - 6671691x_{i+\frac{1}{2}}) + 63h^6 (67718a_{i+\frac{1}{2},j+\frac{1}{2}} \tau_{max} + 2145x_{i+\frac{1}{2}})) \\
& / (13(7256313856a_{i+\frac{1}{2},j+\frac{1}{2}}^6 \tau_{max}^6 + 4138202880a_{i+\frac{1}{2},j+\frac{1}{2}}^4 h^2 \tau_{max}^4 \\
& + 344875104a_{i+\frac{1}{2},j+\frac{1}{2}}^2 h^4 \tau_{max}^2 + 10395h^6)) \\
\beta_{5,0} = & (21(-754656641024a_{i+\frac{1}{2},j+\frac{1}{2}}^6 \tau_{max}^6 x_{i+\frac{1}{2}}^2 - 504h^6(-1271288a_{i+\frac{1}{2},j+\frac{1}{2}}^2 \tau_{max}^2 \\
& + 135436a_{i+\frac{1}{2},j+\frac{1}{2}} \tau_{max} x_{i+\frac{1}{2}} + 2145x_{i+\frac{1}{2}}^2) \\
& + 256a_{i+\frac{1}{2},j+\frac{1}{2}}^2 h^4 \tau_{max}^2 (28301533a_{i+\frac{1}{2},j+\frac{1}{2}}^2 \tau_{max}^2 + 33152280a_{i+\frac{1}{2},j+\frac{1}{2}} \tau_{max} x_{i+\frac{1}{2}} \\
& - 140105511x_{i+\frac{1}{2}}^2) + 2048a_{i+\frac{1}{2},j+\frac{1}{2}}^4 h^2 \tau_{max}^4 (4448202a_{i+\frac{1}{2},j+\frac{1}{2}}^2 \tau_{max}^2 \\
& + 44155460a_{i+\frac{1}{2},j+\frac{1}{2}} \tau_{max} x_{i+\frac{1}{2}} - 210143115x_{i+\frac{1}{2}}^2) + 28665h^8)) \\
& / (104(7256313856a_{i+\frac{1}{2},j+\frac{1}{2}}^6 \tau_{max}^6 + 4138202880a_{i+\frac{1}{2},j+\frac{1}{2}}^4 h^2 \tau_{max}^4 \\
& + 344875104a_{i+\frac{1}{2},j+\frac{1}{2}}^2 h^4 \tau_{max}^2 + 10395h^6)) \\
\beta_{4,0} = & - (35(-8301223051264a_{i+\frac{1}{2},j+\frac{1}{2}}^6 \tau_{max}^6 x_{i+\frac{1}{2}}^3 + 67584a_{i+\frac{1}{2},j+\frac{1}{2}}^4 h^2 \tau_{max}^4 x_{i+\frac{1}{2}} \\
& (4448202a_{i+\frac{1}{2},j+\frac{1}{2}}^2 \tau_{max}^2 + 22077730a_{i+\frac{1}{2},j+\frac{1}{2}} \tau_{max} x_{i+\frac{1}{2}} \\
& - 70047705x_{i+\frac{1}{2}}^2) - 24h^6(103797556a_{i+\frac{1}{2},j+\frac{1}{2}}^3 \tau_{max}^3 - 881002584a_{i+\frac{1}{2},j+\frac{1}{2}}^2 \tau_{max}^2 x_{i+\frac{1}{2}} \\
& + 46928574a_{i+\frac{1}{2},j+\frac{1}{2}} \tau_{max} x_{i+\frac{1}{2}}^2 + 495495x_{i+\frac{1}{2}}^3) - 256a_{i+\frac{1}{2},j+\frac{1}{2}}^2 h^4 \tau_{max}^2 \\
& (108151636a_{i+\frac{1}{2},j+\frac{1}{2}}^3 \tau_{max}^3 - 933950589a_{i+\frac{1}{2},j+\frac{1}{2}}^2 \tau_{max}^2 x_{i+\frac{1}{2}} \\
& - 547012620a_{i+\frac{1}{2},j+\frac{1}{2}} \tau_{max} x_{i+\frac{1}{2}}^2 + 1541160621x_{i+\frac{1}{2}}^3) \\
& + 2079h^8(15216a_{i+\frac{1}{2},j+\frac{1}{2}} \tau_{max} + 455x_{i+\frac{1}{2}})) \\
& / (1144(7256313856a_{i+\frac{1}{2},j+\frac{1}{2}}^6 \tau_{max}^6 + 4138202880a_{i+\frac{1}{2},j+\frac{1}{2}}^4 h^2 \tau_{max}^4 \\
& + 344875104a_{i+\frac{1}{2},j+\frac{1}{2}}^2 h^4 \tau_{max}^2 + 10395h^6)) \\
\beta_{3,0} = & - (35(16602446102528a_{i+\frac{1}{2},j+\frac{1}{2}}^6 \tau_{max}^6 x_{i+\frac{1}{2}}^4 - 132h^8(-6457912a_{i+\frac{1}{2},j+\frac{1}{2}}^2 \tau_{max}^2 \\
& + 1917216a_{i+\frac{1}{2},j+\frac{1}{2}} \tau_{max} x_{i+\frac{1}{2}} + 28665x_{i+\frac{1}{2}}^2) - 45056a_{i+\frac{1}{2},j+\frac{1}{2}}^4 h^2 \tau_{max}^4 x_{i+\frac{1}{2}}^2 \\
& (26689212a_{i+\frac{1}{2},j+\frac{1}{2}}^2 \tau_{max}^2 + 88310920a_{i+\frac{1}{2},j+\frac{1}{2}} \tau_{max} x_{i+\frac{1}{2}} - 210143115x_{i+\frac{1}{2}}^2) \\
& + 16h^6(569473600a_{i+\frac{1}{2},j+\frac{1}{2}}^4 \tau_{max}^4 + 1245570672a_{i+\frac{1}{2},j+\frac{1}{2}}^3 \tau_{max}^3 x_{i+\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
& - 5286015504a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2x_{i+\frac{1}{2}}^2 + 187714296a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}^3 \\
& + 1486485x_{i+\frac{1}{2}}^4) + 512a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^4\tau_{max}^2(13427128a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4 \\
& + 432606544a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3x_{i+\frac{1}{2}} - 1867901178a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2x_{i+\frac{1}{2}}^2 \\
& - 729350160a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}^3 + 1541160621x_{i+\frac{1}{2}}^4) + 63063h^{10})) \\
& /((2288(7256313856a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^6 + 4138202880a_{i+\frac{1}{2},j+\frac{1}{2}}^4h^2\tau_{max}^4 \\
& + 344875104a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^4\tau_{max}^2 + 10395h^6)) \\
\beta_{2,0} = & (7(49807338307584a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^6x_{i+\frac{1}{2}}^5 - 675840a_{i+\frac{1}{2},j+\frac{1}{2}}^4h^2\tau_{max}^4x_{i+\frac{1}{2}}^3 \\
& (8896404a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2 + 22077730a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}} \\
& - 42028623x_{i+\frac{1}{2}}^2) - 180h^8(7154768a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3 - 71037032a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2x_{i+\frac{1}{2}} \\
& + 10544688a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}^2 + 105105x_{i+\frac{1}{2}}^3) + 1536a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^4\tau_{max}^2x_{i+\frac{1}{2}} \\
& (67135640a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4 + 1081516360a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3x_{i+\frac{1}{2}} \\
& - 3113168630a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2x_{i+\frac{1}{2}}^2 - 911687700a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}^3 \\
& + 1541160621x_{i+\frac{1}{2}}^4) + 16h^6(-999477488a_{i+\frac{1}{2},j+\frac{1}{2}}^5\tau_{max}^5 \\
& + 8542104000a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4x_{i+\frac{1}{2}} + 9341780040a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3x_{i+\frac{1}{2}}^2 \\
& - 26430077520a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2x_{i+\frac{1}{2}}^3 + 703928610a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}^4 \\
& + 4459455x_{i+\frac{1}{2}}^5) + 99h^{10}(334172a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max} + 9555x_{i+\frac{1}{2}}))) \\
& /((2288(7256313856a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^6 + 4138202880a_{i+\frac{1}{2},j+\frac{1}{2}}^4h^2\tau_{max}^4 \\
& + 344875104a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^4\tau_{max}^2 + 10395h^6)) \\
\beta_{1,0} = & (7(-796917412921344a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^6x_{i+\frac{1}{2}}^6 - 1584h^{10}(-2907158a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2 \\
& + 2005032a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}} + 28665x_{i+\frac{1}{2}}^2) + 32440320a_{i+\frac{1}{2},j+\frac{1}{2}}^4h^2\tau_{max}^4x_{i+\frac{1}{2}}^4 \\
& (4448202a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2 + 8831092a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}} \\
& - 14009541x_{i+\frac{1}{2}}^2) + 864h^8(54537656a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4 \\
& + 143095360a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3x_{i+\frac{1}{2}} - 710370320a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2x_{i+\frac{1}{2}}^2 \\
& + 70297920a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}^3 + 525525x_{i+\frac{1}{2}}^4) - 24576a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^4\tau_{max}^2x_{i+\frac{1}{2}}^2
\end{aligned}$$

$$\begin{aligned}
& (201406920a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4 + 2163032720a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3x_{i+\frac{1}{2}} \\
& - 4669752945a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2x_{i+\frac{1}{2}}^2 - 1094025240a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}^3 \\
& + 1541160621x_{i+\frac{1}{2}}^4) - 256h^6(-55859408a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^6 \\
& - 5996864928a_{i+\frac{1}{2},j+\frac{1}{2}}^5\tau_{max}^5x_{i+\frac{1}{2}} + 25626312000a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4x_{i+\frac{1}{2}}^2 \\
& + 18683560080a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3x_{i+\frac{1}{2}}^3 - 39645116280a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2x_{i+\frac{1}{2}}^4 \\
& + 844714332a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}^5 + 4459455x_{i+\frac{1}{2}}^6) + 637065h^{12})) \\
& /((109824(7256313856a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^6 + 4138202880a_{i+\frac{1}{2},j+\frac{1}{2}}^4h^2\tau_{max}^4 \\
& + 344875104a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^4\tau_{max}^2 + 10395h^6)) \\
\beta_{0,0} = & (796917412921344a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^6x_{i+\frac{1}{2}}^7 - 15138816a_{i+\frac{1}{2},j+\frac{1}{2}}^4h^2\tau_{max}^4x_{i+\frac{1}{2}}^5 \\
& (13344606a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2 + 22077730a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}} - 30020445x_{i+\frac{1}{2}}^2) \\
& + 1008h^{10}(1438204a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3 - 31978738a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2x_{i+\frac{1}{2}} \\
& + 11027676a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}^2 + 105105x_{i+\frac{1}{2}}^3) + 172032a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^4\tau_{max}^2x_{i+\frac{1}{2}}^3 \\
& (67135640a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4 + 540758180a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3x_{i+\frac{1}{2}} - 933950589a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2x_{i+\frac{1}{2}}^2 \\
& - 182337540a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}^3 + 220165803x_{i+\frac{1}{2}}^4) \\
& - 32h^8(-967818016a_{i+\frac{1}{2},j+\frac{1}{2}}^5\tau_{max}^5 + 10307616984a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4x_{i+\frac{1}{2}} \\
& + 13522511520a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3x_{i+\frac{1}{2}}^2 - 44753330160a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2x_{i+\frac{1}{2}}^3 \\
& + 3321576720a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}^4 + 19864845x_{i+\frac{1}{2}}^5) \\
& + 1792h^6x_{i+\frac{1}{2}}(-55859408a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^6 - 2998432464a_{i+\frac{1}{2},j+\frac{1}{2}}^5\tau_{max}^5x_{i+\frac{1}{2}} \\
& + 8542104000a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4x_{i+\frac{1}{2}}^2 + 4670890020a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3x_{i+\frac{1}{2}}^3 \\
& - 7929023256a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2x_{i+\frac{1}{2}}^4 + 140785722a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}^5 \\
& + 637065x_{i+\frac{1}{2}}^6) - 693h^{12}(232088a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max} + 6435x_{i+\frac{1}{2}})) \\
& /((109824(7256313856a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^6 + 4138202880a_{i+\frac{1}{2},j+\frac{1}{2}}^4h^2\tau_{max}^4 \\
& + 344875104a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^4\tau_{max}^2 + 10395h^6))
\end{aligned}$$

Hence, we have

$$\begin{aligned}
& \int_{x_i}^{x_{i+\frac{1}{2}}} \int_{y_j}^{y_{j+\frac{1}{2}}} (v_I(x, y) - x^7 - u_I(x - \frac{h}{2}, y - \frac{h}{2}) + (x - \frac{h}{2})^7)^2 dx dy \\
&= ((a_{i,j} - a_{i+\frac{1}{2},j+\frac{1}{2}})^2 h^{18} \tau_{max}^2 (437403046994236795781925 h^{20} \\
&+ 8859873555495770497489800(a_{i,j} + a_{i+\frac{1}{2},j+\frac{1}{2}}) \tau_{max} h^{19} \\
&+ 1485279180(1699073037709623145 a_{i,j}^2 \\
&- 16158953949797393574 a_{i+\frac{1}{2},j+\frac{1}{2}} a_{i,j} \\
&+ 1699073037709623145 a_{i+\frac{1}{2},j+\frac{1}{2}}^2) \tau_{max}^2 h^{18} \\
&- 2973528918360(1414536046558650 a_{i,j}^3 + 101689335736344761 a_{i+\frac{1}{2},j+\frac{1}{2}} a_{i,j}^2 \\
&+ 101689335736344761 a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j} + 1414536046558650 a_{i+\frac{1}{2},j+\frac{1}{2}}^3) \tau_{max}^3 h^{17} \\
&+ 3168595584(20961537958383611375 a_{i,j}^4 \\
&- 322425487440101544600 a_{i+\frac{1}{2},j+\frac{1}{2}} a_{i,j}^3 \\
&+ 151880015288861843693378 a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j}^2 \\
&- 322425487440101544600 a_{i+\frac{1}{2},j+\frac{1}{2}}^3 a_{i,j} \\
&+ 20961537958383611375 a_{i+\frac{1}{2},j+\frac{1}{2}}^4) \tau_{max}^4 h^{16} \\
&- 257448391200(420292950978969849 a_{i,j}^5 \\
&+ 17072570927322274705 a_{i+\frac{1}{2},j+\frac{1}{2}} a_{i,j}^4 + 586768804488348852256 a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j}^3 \\
&+ 586768804488348852256 a_{i+\frac{1}{2},j+\frac{1}{2}}^3 a_{i,j}^2 + 17072570927322274705 a_{i+\frac{1}{2},j+\frac{1}{2}}^4 a_{i,j} \\
&+ 420292950978969849 a_{i+\frac{1}{2},j+\frac{1}{2}}^5) \tau_{max}^5 h^{15} \\
&+ 3840721920(156595436000740513590 a_{i,j}^6 \\
&- 3699346840438621615628 a_{i+\frac{1}{2},j+\frac{1}{2}} a_{i,j}^5 \\
&+ 3109924558495450458531461 a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j}^4 \\
&- 3855737499325029678375014 a_{i+\frac{1}{2},j+\frac{1}{2}}^3 a_{i,j}^3 \\
&+ 3109924558495450458531461 a_{i+\frac{1}{2},j+\frac{1}{2}}^4 a_{i,j}^2 \\
&- 3699346840438621615628 a_{i+\frac{1}{2},j+\frac{1}{2}}^5 a_{i,j}
\end{aligned}$$

$$\begin{aligned}
& + 156595436000740513590a_{i+\frac{1}{2},j+\frac{1}{2}}^6)\tau_{max}^6h^{14} \\
& - 549223234560(1593068096417356625a_{i,j}^7 \\
& - 11325611182537259267a_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}^6 \\
& + 7442033876711042429922a_{i+\frac{1}{2},j+\frac{1}{2}}^2a_{i,j}^5 \\
& - 165818214087284083403612a_{i+\frac{1}{2},j+\frac{1}{2}}^3a_{i,j}^4 \\
& - 165818214087284083403612a_{i+\frac{1}{2},j+\frac{1}{2}}^4a_{i,j}^3 \\
& + 7442033876711042429922a_{i+\frac{1}{2},j+\frac{1}{2}}^5a_{i,j}^2 \\
& - 11325611182537259267a_{i+\frac{1}{2},j+\frac{1}{2}}^6a_{i,j} \\
& + 1593068096417356625a_{i+\frac{1}{2},j+\frac{1}{2}}^7)\tau_{max}^7h^{13} \\
& + 585252864(3667960856961456465300a_{i,j}^8 \\
& - 126786200646401196602900a_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}^7 \\
& + 167960141291628617113597954a_{i+\frac{1}{2},j+\frac{1}{2}}^2a_{i,j}^6 \\
& - 552419651770920469669003840a_{i+\frac{1}{2},j+\frac{1}{2}}^3a_{i,j}^5 \\
& + 11457736972910639970259979445a_{i+\frac{1}{2},j+\frac{1}{2}}^4a_{i,j}^4 \\
& - 552419651770920469669003840a_{i+\frac{1}{2},j+\frac{1}{2}}^5a_{i,j}^3 \\
& + 167960141291628617113597954a_{i+\frac{1}{2},j+\frac{1}{2}}^6a_{i,j}^2 \\
& - 126786200646401196602900a_{i+\frac{1}{2},j+\frac{1}{2}}^7a_{i,j} \\
& + 3667960856961456465300a_{i+\frac{1}{2},j+\frac{1}{2}}^8)\tau_{max}^8h^{12} \\
& - 19020718080(116883680195507193925a_{i,j}^9 \\
& - 12913068248935915017815a_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}^8 \\
& + 1986166975590076367754380a_{i+\frac{1}{2},j+\frac{1}{2}}^2a_{i,j}^7 \\
& - 107237984339516532769057736a_{i+\frac{1}{2},j+\frac{1}{2}}^3a_{i,j}^6 \\
& + 676505105051936160968017607a_{i+\frac{1}{2},j+\frac{1}{2}}^4a_{i,j}^5 \\
& + 676505105051936160968017607a_{i+\frac{1}{2},j+\frac{1}{2}}^5a_{i,j}^4
\end{aligned}$$

$$\begin{aligned}
& - 107237984339516532769057736a_{i+\frac{1}{2},j+\frac{1}{2}}^6 a_{i,j}^3 \\
& + 1986166975590076367754380a_{i+\frac{1}{2},j+\frac{1}{2}}^7 a_{i,j}^2 \\
& - 12913068248935915017815a_{i+\frac{1}{2},j+\frac{1}{2}}^8 a_{i,j} \\
& + 116883680195507193925a_{i+\frac{1}{2},j+\frac{1}{2}}^9 \tau_{max}^9 h^{11} \\
& + 3901685760(653854776598314061485a_{i,j}^{10} \\
& - 26347537693490147722742a_{i+\frac{1}{2},j+\frac{1}{2}}^9 a_{i,j}^9 \\
& + 77009004765064962422325163a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j}^8 \\
& - 527477199135104510286360772a_{i+\frac{1}{2},j+\frac{1}{2}}^3 a_{i,j}^7 \\
& + 37103073127910502944858616329a_{i+\frac{1}{2},j+\frac{1}{2}}^4 a_{i,j}^6 \\
& - 713564523544966917683070426a_{i+\frac{1}{2},j+\frac{1}{2}}^5 a_{i,j}^5 \\
& + 37103073127910502944858616329a_{i+\frac{1}{2},j+\frac{1}{2}}^6 a_{i,j}^4 \\
& - 527477199135104510286360772a_{i+\frac{1}{2},j+\frac{1}{2}}^7 a_{i,j}^3 \\
& + 77009004765064962422325163a_{i+\frac{1}{2},j+\frac{1}{2}}^8 a_{i,j}^2 \\
& - 26347537693490147722742a_{i+\frac{1}{2},j+\frac{1}{2}}^9 a_{i,j} \\
& + 653854776598314061485a_{i+\frac{1}{2},j+\frac{1}{2}}^{10} \tau_{max}^{10} h^{10} + 101443829760a_{i,j} a_{i+\frac{1}{2},j+\frac{1}{2}} \\
& (4559721353965663034369a_{i,j}^9 - 1154079108387189138422257a_{i+\frac{1}{2},j+\frac{1}{2}} a_{i,j}^8 \\
& + 129370543069772118288464708a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j}^7 \\
& - 2767084241034372123203306878a_{i+\frac{1}{2},j+\frac{1}{2}}^3 a_{i,j}^6 \\
& - 2142066506103509583429763149a_{i+\frac{1}{2},j+\frac{1}{2}}^4 a_{i,j}^5 \\
& - 2142066506103509583429763149a_{i+\frac{1}{2},j+\frac{1}{2}}^5 a_{i,j}^4 \\
& - 2767084241034372123203306878a_{i+\frac{1}{2},j+\frac{1}{2}}^6 a_{i,j}^3 \\
& + 129370543069772118288464708a_{i+\frac{1}{2},j+\frac{1}{2}}^7 a_{i,j}^2 \\
& - 1154079108387189138422257a_{i+\frac{1}{2},j+\frac{1}{2}}^8 a_{i,j} \\
& + 4559721353965663034369a_{i+\frac{1}{2},j+\frac{1}{2}}^9 \tau_{max}^{11} h^9 + 16647192576a_{i,j}^2 a_{i+\frac{1}{2},j+\frac{1}{2}}^2
\end{aligned}$$

$$\begin{aligned}
& (18461283971311741166192542a_{i,j}^8 \\
& - 183011002492177671027896240a_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}^7 \\
& + 56697160731182582899942819925a_{i+\frac{1}{2},j+\frac{1}{2}}^2a_{i,j}^6 \\
& + 2372628401241137086503337520a_{i+\frac{1}{2},j+\frac{1}{2}}^3a_{i,j}^5 \\
& + 190469561341043092690908870388a_{i+\frac{1}{2},j+\frac{1}{2}}^4a_{i,j}^4 \\
& + 2372628401241137086503337520a_{i+\frac{1}{2},j+\frac{1}{2}}^5a_{i,j}^3 \\
& + 56697160731182582899942819925a_{i+\frac{1}{2},j+\frac{1}{2}}^6a_{i,j}^2 \\
& - 183011002492177671027896240a_{i+\frac{1}{2},j+\frac{1}{2}}^7a_{i,j} \\
& + 18461283971311741166192542a_{i+\frac{1}{2},j+\frac{1}{2}}^8)\tau_{max}^{12}h^8 \\
& + 1082067517440a_{i,j}^3a_{i+\frac{1}{2},j+\frac{1}{2}}^3(18321852318992025713799428a_{i,j}^7 \\
& - 1321003108505324002163493396a_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}^6 \\
& - 498594225191753361165098406a_{i+\frac{1}{2},j+\frac{1}{2}}^2a_{i,j}^5 \\
& - 4512798124101130032009521037a_{i+\frac{1}{2},j+\frac{1}{2}}^3a_{i,j}^4 \\
& - 4512798124101130032009521037a_{i+\frac{1}{2},j+\frac{1}{2}}^4a_{i,j}^3 \\
& - 498594225191753361165098406a_{i+\frac{1}{2},j+\frac{1}{2}}^5a_{i,j}^2 \\
& - 1321003108505324002163493396a_{i+\frac{1}{2},j+\frac{1}{2}}^6a_{i,j} \\
& + 18321852318992025713799428a_{i+\frac{1}{2},j+\frac{1}{2}}^7)\tau_{max}^{13}h^7 + 31708938240a_{i,j}^4a_{i+\frac{1}{2},j+\frac{1}{2}}^4 \\
& (48802141800690392120765929181a_{i,j}^6 \\
& + 11048478168216937331149943798a_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}^5 \\
& + 651595638176670808303246681584a_{i+\frac{1}{2},j+\frac{1}{2}}^2a_{i,j}^4 \\
& + 43557999647137173043199301698a_{i+\frac{1}{2},j+\frac{1}{2}}^3a_{i,j}^3 \\
& + 651595638176670808303246681584a_{i+\frac{1}{2},j+\frac{1}{2}}^4a_{i,j}^2 \\
& + 11048478168216937331149943798a_{i+\frac{1}{2},j+\frac{1}{2}}^5a_{i,j} \\
& + 48802141800690392120765929181a_{i+\frac{1}{2},j+\frac{1}{2}}^6)\tau_{max}^{14}h^6
\end{aligned}$$

$$\begin{aligned}
& + 1097319516733440a_{i,j}^5 a_{i+\frac{1}{2},j+\frac{1}{2}}^5 (3198843901481228518023451a_{i,j}^5 \\
& - 22843767273592440900178964a_{i+\frac{1}{2},j+\frac{1}{2}} a_{i,j}^4 \\
& - 13165800443519527815052054a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j}^3 \\
& - 13165800443519527815052054a_{i+\frac{1}{2},j+\frac{1}{2}}^3 a_{i,j}^2 \\
& - 22843767273592440900178964a_{i+\frac{1}{2},j+\frac{1}{2}}^4 a_{i,j} \\
& + 3198843901481228518023451a_{i+\frac{1}{2},j+\frac{1}{2}}^5) \tau_{max}^{15} h^5 + 67645734912a_{i,j}^6 a_{i+\frac{1}{2},j+\frac{1}{2}}^6 \\
& (492258158772799267571707352258a_{i,j}^4 \\
& + 115400788701858795739154476200a_{i+\frac{1}{2},j+\frac{1}{2}} a_{i,j}^3 \\
& + 1968592966632694779138376243595a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j}^2 \\
& + 115400788701858795739154476200a_{i+\frac{1}{2},j+\frac{1}{2}}^3 a_{i,j} \\
& + 492258158772799267571707352258a_{i+\frac{1}{2},j+\frac{1}{2}}^4) \tau_{max}^{16} h^4 \\
& + 5852370755911680a_{i,j}^7 a_{i+\frac{1}{2},j+\frac{1}{2}}^7 (10862152607395926487179716a_{i,j}^3 \\
& - 13228624002089287227191317a_{i+\frac{1}{2},j+\frac{1}{2}} a_{i,j}^2 \\
& - 13228624002089287227191317a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j} \\
& + 10862152607395926487179716a_{i+\frac{1}{2},j+\frac{1}{2}}^3) \tau_{max}^{17} h^3 \\
& + 798923638576250880a_{i,j}^8 a_{i+\frac{1}{2},j+\frac{1}{2}}^8 (258161172004769893077187a_{i,j}^2 \\
& + 50057250128499149598270a_{i+\frac{1}{2},j+\frac{1}{2}} a_{i,j} \\
& + 258161172004769893077187a_{i+\frac{1}{2},j+\frac{1}{2}}^2) \tau_{max}^{18} h^2 \\
& + 304539049496485812590397344184454726287360a_{i,j}^9 a_{i+\frac{1}{2},j+\frac{1}{2}}^9 (a_{i,j} + a_{i+\frac{1}{2},j+\frac{1}{2}}) \\
& \tau_{max}^{19} h + 276774410375026167841490291916617329672192a_{i,j}^{10} a_{i+\frac{1}{2},j+\frac{1}{2}}^{10} \tau_{max}^{20}))) \\
& / (12249768960(10395h^6 + 344875104a_{i,j}^2 \tau_{max}^2 h^4 + 4138202880a_{i,j}^4 \tau_{max}^4 h^2 \\
& + 7256313856a_{i,j}^6 \tau_{max}^6) (10395h^6 + 344875104a_{i+\frac{1}{2},j+\frac{1}{2}}^2 \tau_{max}^2 h^4 \\
& + 4138202880a_{i+\frac{1}{2},j+\frac{1}{2}}^4 \tau_{max}^4 h^2 + 7256313856a_{i+\frac{1}{2},j+\frac{1}{2}}^6 \tau_{max}^6) = O(h^{18})
\end{aligned}$$

Similarly, we have

$$\begin{aligned} \int_{x_i}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_j} (v_I(x, y) - x^7 - u_I(x - \frac{h}{2}, y + \frac{h}{2}) + (x - \frac{h}{2})^7)^2 dx dy &= O(h^{18}), \\ \int_{x_{i-\frac{1}{2}}}^{x_i} \int_{y_j}^{y_{j+\frac{1}{2}}} (v_I(x, y) - x^7 - u_I(x + \frac{h}{2}, y - \frac{h}{2}) + (x + \frac{h}{2})^7)^2 dx dy &= O(h^{18}), \\ \int_{x_{i-\frac{1}{2}}}^{x_i} \int_{y_j}^{y_{j+\frac{1}{2}}} (v_I(x, y) - x^7 - u_I(x + \frac{h}{2}, y + \frac{h}{2}) + (x + \frac{h}{2})^7)^2 dx dy &= O(h^{18}). \end{aligned}$$

8. $k = 7, u = x^8$, by the definition of the projection,

$$\begin{aligned} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} u_I dx dy &= \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} x^8 dx dy \\ \tilde{P}_h(u_I; x^k y^l; f, g, u)_{i,j} &= \tilde{P}_h(x^8; x^k y^l; f, g, u)_{i,j}, \quad k = 0, 1, \dots, 7, \quad l = 0, 1, \dots, 7 \\ \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} v_I dx dy &= \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} x^8 dx dy \\ \tilde{Q}_h(v_I; x^k y^l; f, g, u)_{i+\frac{1}{2}, j+\frac{1}{2}} &= \tilde{Q}_h(x^8; x^k y^l; f, g, u)_{i+\frac{1}{2}, j+\frac{1}{2}}, \quad k = 0, 1, \dots, 7, \quad l = 0, 1, \dots, 7 \end{aligned}$$

then by solving the above linear system we have

$$\begin{aligned} u_I(x, y) &= \sum_{k=0}^7 \sum_{l=0}^7 \alpha_{k,l} x^k y^l \\ v_I(x, y) &= \sum_{k=0}^7 \sum_{l=0}^7 \beta_{k,l} x^k y^l \end{aligned} \tag{2.34}$$

where

$$\alpha_{k,l} = 0, \quad k = 0, 1, \dots, 7, \quad l = 1, \dots, 7$$

$$\begin{aligned} \alpha_{7,0} &= (8(4096a_{i,j}^6 \tau_{max}^6 (2354497739a_{i,j} \tau_{max} + 239935274463x_i) \\ &\quad - 6912a_{i,j}^4 h^2 \tau_{max}^4 (879791707a_{i,j} \tau_{max} - 21360056375x_i) \\ &\quad - 54432a_{i,j}^2 h^4 \tau_{max}^2 (13002767a_{i,j} \tau_{max} - 89468841x_i) \\ &\quad + 51975h^6 (14369a_{i,j} \tau_{max} + 39x_i))) \\ &\quad / (3(327591628066816a_{i,j}^6 \tau_{max}^6 + 49213569888000a_{i,j}^4 h^2 \tau_{max}^4 \\ &\quad + 1623322651104a_{i,j}^2 h^4 \tau_{max}^2 + 675675h^6)) \end{aligned}$$

$$\alpha_{6,0} = (28(-20480a_{i,j}^6 \tau_{max}^6 x_i (4708995478a_{i,j} \tau_{max} + 239935274463x_i)$$

$$\begin{aligned}
& - 945h^6(-407585262a_{i,j}^2\tau_{max}^2 + 7902950a_{i,j}\tau_{max}x_i + 10725x_i^2) \\
& + 288a_{i,j}^2h^4\tau_{max}^2(36101338556a_{i,j}^2\tau_{max}^2 + 24575229630a_{i,j}\tau_{max}x_i \\
& - 84548054745x_i^2) + 1280a_{i,j}^4h^2\tau_{max}^4(50856691162a_{i,j}^2\tau_{max}^2 \\
& + 47508752178a_{i,j}\tau_{max}x_i - 576721522125x_i^2) + 259875h^8)) \\
& /((15(327591628066816a_{i,j}^6\tau_{max}^6 + 49213569888000a_{i,j}^4h^2\tau_{max}^4 \\
& + 1623322651104a_{i,j}^2h^4\tau_{max}^2 + 675675h^6)) \\
\alpha_{5,0} = & - (56(-266240a_{i,j}^6\tau_{max}^6x_i^2(2354497739a_{i,j}\tau_{max} + 79978424821x_i) \\
& - 9h^6(95613443942a_{i,j}^3\tau_{max}^3 - 556353882630a_{i,j}^2\tau_{max}^2x_i + 5393763375a_{i,j}\tau_{max}x_i^2 \\
& + 4879875x_i^3) - 288a_{i,j}^2h^4\tau_{max}^2(27266771676a_{i,j}^3\tau_{max}^3 - 469317401228a_{i,j}^2\tau_{max}^2x_i \\
& - 159738992595a_{i,j}\tau_{max}x_i^2 + 366374903895x_i^3) + 1280a_{i,j}^4h^2\tau_{max}^4(5911894398a_{i,j}^3\tau_{max}^3 \\
& + 661136985106a_{i,j}^2\tau_{max}^2x_i + 308806889157a_{i,j}\tau_{max}x_i^2 - 2499126595875x_i^3) \\
& + 61425h^8(21053a_{i,j}\tau_{max} + 55x_i))) \\
& /((65(327591628066816a_{i,j}^6\tau_{max}^6 + 49213569888000a_{i,j}^4h^2\tau_{max}^4 \\
& + 1623322651104a_{i,j}^2h^4\tau_{max}^2 + 675675h^6)) \\
\alpha_{4,0} = & - ((7(2641275h^{10} + 2129920a_{i,j}^6x_i^3\tau_{max}^6(239935274463x_i + 9417990956a_{i,j}\tau_{max}) \\
& - 5616h^8(28875x_i^2 + 22105650a_{i,j}x_i\tau_{max} - 383499197a_{i,j}^2\tau_{max}^2) \\
& - 30720a_{i,j}^4h^2x_i\tau_{max}^4(-2499126595875x_i^3 + 411742518876a_{i,j}x_i^2\tau_{max} \\
& + 1322273970212a_{i,j}^2x_i\tau_{max}^2 + 23647577592a_{i,j}^3\tau_{max}^3) \\
& + 72h^6(14639625x_i^4 + 21575053500a_{i,j}x_i^3\tau_{max} - 3338123295780a_{i,j}^2x_i^2\tau_{max}^2 \\
& + 1147361327304a_{i,j}^3x_i\tau_{max}^3 + 687203235848a_{i,j}^4\tau_{max}^4) \\
& + 256a_{i,j}^2h^4\tau_{max}^2(9892122405165x_i^4 - 5750603733420a_{i,j}x_i^3\tau_{max} \\
& - 25343139666312a_{i,j}^2x_i^2\tau_{max}^2 + 2944811341008a_{i,j}^3x_i\tau_{max}^3 \\
& + 1087949803400a_{i,j}^4\tau_{max}^4))) \\
& /((156(675675h^6 + 1623322651104a_{i,j}^2h^4\tau_{max}^2
\end{aligned}$$

$$\begin{aligned}
& + 49213569888000a_{i,j}^4h^2\tau_{max}^4 + 327591628066816a_{i,j}^6\tau_{max}^6)) \\
\alpha_{3,0} = & (7(96525h^{10}(301x_i + 118347a_{i,j}\tau_{max}) + 4685824a_{i,j}^6x_i^4\tau_{max}^6(239935274463x_i \\
& + 11772488695a_{i,j}\tau_{max}) - 4752h^8(125125x_i^3 + 143686725a_{i,j}x_i^2\tau_{max} \\
& - 4985489561a_{i,j}^2x_i\tau_{max}^2 + 985840581a_{i,j}^3\tau_{max}^3) \\
& - 112640a_{i,j}^4h^2x_i^2\tau_{max}^4(-1499475957525x_i^3 \\
& + 308806889157a_{i,j}x_i^2\tau_{max} + 1322273970212a_{i,j}^2x_i\tau_{max}^2 + 35471366388a_{i,j}^3\tau_{max}^3) \\
& + 72h^6(32207175x_i^5 + 59331397125a_{i,j}x_i^4\tau_{max} - 12239785417860a_{i,j}^2x_i^3\tau_{max}^2 \\
& + 6310487300172a_{i,j}^3x_i^2\tau_{max}^3 + 7559235594328a_{i,j}^4x_i\tau_{max}^4 - 625703362520a_{i,j}^5\tau_{max}^5) \\
& + 256a_{i,j}^2h^4\tau_{max}^2(21762669291363x_i^5 \\
& - 15814160266905a_{i,j}x_i^4\tau_{max} - 92924845443144a_{i,j}^2x_i^3\tau_{max}^2 \\
& + 16196462375544a_{i,j}^3x_i^2\tau_{max}^3 + 11967447837400a_{i,j}^4x_i\tau_{max}^4 + 89226796360a_{i,j}^5\tau_{max}^5))) \\
& / (429(675675h^6 + 1623322651104a_{i,j}^2h^4\tau_{max}^2 \\
& + 49213569888000a_{i,j}^4h^2\tau_{max}^4 + 327591628066816a_{i,j}^6\tau_{max}^6)) \\
\alpha_{2,0} = & - ((7(-2413125h^{12} + 28114944a_{i,j}^6x_i^5\tau_{max}^6(79978424821x_i + 4708995478a_{i,j}\tau_{max}) \\
& + 38610h^{10}(4515x_i^2 + 3550410a_{i,j}x_i\tau_{max} - 25259534a_{i,j}^2\tau_{max}^2) \\
& - 135168a_{i,j}^4h^2x_i^3\tau_{max}^4(-2499126595875x_i^3 + 617613778314a_{i,j}x_i^2\tau_{max} \\
& + 3305684925530a_{i,j}^2x_i\tau_{max}^2 + 118237887960a_{i,j}^3\tau_{max}^3) - 4752h^8(375375x_i^4 \\
& + 574746900a_{i,j}x_i^3\tau_{max} - 29912937366a_{i,j}^2x_i^2\tau_{max}^2 + 11830086972a_{i,j}^3x_i\tau_{max}^3 \\
& + 3794229608a_{i,j}^4\tau_{max}^4) + 1536a_{i,j}^2h^4x_i\tau_{max}^2 \\
& (7254223097121x_i^5 - 6325664106762a_{i,j}x_i^4\tau_{max} \\
& - 46462422721572a_{i,j}^2x_i^3\tau_{max}^2 + 10797641583696a_{i,j}^3x_i^2\tau_{max}^3 \\
& + 11967447837400a_{i,j}^4x_i\tau_{max}^4 \\
& + 178453592720a_{i,j}^5\tau_{max}^5) + 16h^6(289864575x_i^6 + 640779088950a_{i,j}x_i^5\tau_{max} \\
& - 165237103141110a_{i,j}^2x_i^4\tau_{max}^2 + 113588771403096a_{i,j}^3x_i^3\tau_{max}^3
\end{aligned}$$

$$\begin{aligned}
& + 204099361046856a_{i,j}^4x_i^2\tau_{max}^4 \\
& - 33787981576080a_{i,j}^5x_i\tau_{max}^5 - 5178530366512a_{i,j}^6\tau_{max}^6)) \\
& /((1716(675675h^6 + 1623322651104a_{i,j}^2h^4\tau_{max}^2 \\
& + 49213569888000a_{i,j}^4h^2\tau_{max}^4 + 327591628066816a_{i,j}^6\tau_{max}^6))) \\
\alpha_{1,0} = & (-675675h^{12}(375x_i + 149849a_{i,j}\tau_{max}) + 140574720a_{i,j}^6x_i^6\tau_{max}^6(239935274463x_i \\
& + 16481484173a_{i,j}\tau_{max}) + 4158h^{10}(1467375x_i^3 + 1730824875a_{i,j}x_i^2\tau_{max} \\
& - 24628045650a_{i,j}^2x_i\tau_{max}^2 + 5177901778a_{i,j}^3\tau_{max}^3) \\
& - 14192640a_{i,j}^4h^2x_i^4\tau_{max}^4(-357018085125x_i^3 \\
& + 102935629719a_{i,j}x_i^2\tau_{max} + 661136985106a_{i,j}^2x_i\tau_{max}^2 + 29559471990a_{i,j}^3\tau_{max}^3) \\
& - 720h^8(52026975x_i^5 + 99574900425a_{i,j}x_i^4\tau_{max} - 6909888531546a_{i,j}^2x_i^3\tau_{max}^2 \\
& + 4099125135798a_{i,j}^3x_i^2\tau_{max}^3 + 2629401118344a_{i,j}^4x_i\tau_{max}^4 - 303683537672a_{i,j}^5\tau_{max}^5) \\
& + 10752a_{i,j}^2h^4x_i^2\tau_{max}^2(15544763779545x_i^5 - 15814160266905a_{i,j}x_i^4\tau_{max} \\
& - 139387268164716a_{i,j}^2x_i^3\tau_{max}^2 + 40491155938860a_{i,j}^3x_i^2\tau_{max}^3 \\
& + 59837239187000a_{i,j}^4x_i\tau_{max}^4 + 1338401945400a_{i,j}^5\tau_{max}^5) \\
& + 560h^6(124227675x_i^7 + 320389544475a_{i,j}x_i^6\tau_{max} \\
& - 99142261884666a_{i,j}^2x_i^5\tau_{max}^2 + 85191578552322a_{i,j}^3x_i^4\tau_{max}^3 \\
& + 204099361046856a_{i,j}^4x_i^3\tau_{max}^4 - 50681972364120a_{i,j}^5x_i^2\tau_{max}^5 \\
& - 15535591099536a_{i,j}^6x_i\tau_{max}^6 - 74240090992a_{i,j}^7\tau_{max}^7)) \\
& /((12870(675675h^6 + 1623322651104a_{i,j}^2h^4\tau_{max}^2 \\
& + 49213569888000a_{i,j}^4h^2\tau_{max}^4 + 327591628066816a_{i,j}^6\tau_{max}^6))) \\
\alpha_{0,0} = & -((211486275h^{14} + 2249195520a_{i,j}^6x_i^7\tau_{max}^6 \\
& (239935274463x_i + 18835981912a_{i,j}\tau_{max}) \\
& - 1729728h^{12}(9375x_i^2 + 7492450a_{i,j}x_i\tau_{max} - 22675009a_{i,j}^2\tau_{max}^2) \\
& - 10813440a_{i,j}^4h^2x_i^5\tau_{max}^4(-7497379787625x_i^3 + 2470455113256a_{i,j}x_i^2\tau_{max}
\end{aligned}$$

$$\begin{aligned}
& + 18511835582968a_{i,j}^2x_i\tau_{max}^2 + 993198258864a_{i,j}^3\tau_{max}^3) + 6336h^{10}(30814875x_i^4 \\
& + 48463096500a_{i,j}x_i^3\tau_{max} - 1034377917300a_{i,j}^2x_i^2\tau_{max}^2 + 434943749352a_{i,j}^3x_i\tau_{max}^3 \\
& + 87945430928a_{i,j}^4\tau_{max}^4) + 172032a_{i,j}^2h^4x_i^3\tau_{max}^2(15544763779545x_i^5 \\
& - 18073326019320a_{i,j}x_i^4\tau_{max} - 185849690886288a_{i,j}^2x_i^3\tau_{max}^2 \\
& + 64785849502176a_{i,j}^3x_i^2\tau_{max}^3 + 119674478374000a_{i,j}^4x_i\tau_{max}^4 \\
& + 3569071854400a_{i,j}^5\tau_{max}^5) - 5120h^8(156080925x_i^6 + 358469641530a_{i,j}x_i^5\tau_{max} \\
& - 31094498391957a_{i,j}^2x_i^4\tau_{max}^2 + 24594750814788a_{i,j}^3x_i^3\tau_{max}^3 \\
& + 23664610065096a_{i,j}^4x_i^2\tau_{max}^4 - 5466303678096a_{i,j}^5x_i\tau_{max}^5 - 321570260696a_{i,j}^6\tau_{max}^6) \\
& + 1792h^6x_i(621138375x_i^7 + 1830797397000a_{i,j}x_i^6\tau_{max} \\
& - 660948412564440a_{i,j}^2x_i^5\tau_{max}^2 + 681532628418576a_{i,j}^3x_i^4\tau_{max}^3 \\
& + 2040993610468560a_{i,j}^4x_i^3\tau_{max}^4 - 675759631521600a_{i,j}^5x_i^2\tau_{max}^5 \\
& - 310711821990720a_{i,j}^6x_i\tau_{max}^6 - 2969603639680a_{i,j}^7\tau_{max}^7)) \\
& /((1647360(675675h^6 + 1623322651104a_{i,j}^2h^4\tau_{max}^2 \\
& + 49213569888000a_{i,j}^4h^2\tau_{max}^4 + 327591628066816a_{i,j}^6\tau_{max}^6)))
\end{aligned}$$

$$\beta_{k,l} = 0, \quad k = 0, 1, \dots, 7, \quad l = 1, \dots, 7$$

$$\begin{aligned}
\beta_{7,0} = & (8(4096a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^6(2354497739a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max} + 239935274463x_{i+\frac{1}{2}}) \\
& - 6912a_{i+\frac{1}{2},j+\frac{1}{2}}^4h^2\tau_{max}^4(879791707a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max} - 21360056375x_{i+\frac{1}{2}}) \\
& - 54432a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^4\tau_{max}^2(13002767a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max} - 89468841x_{i+\frac{1}{2}}) \\
& + 51975h^6(14369a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max} + 39x_{i+\frac{1}{2}}))) \\
& /((3(327591628066816a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^6 + 49213569888000a_{i+\frac{1}{2},j+\frac{1}{2}}^4h^2\tau_{max}^4 \\
& + 1623322651104a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^4\tau_{max}^2 + 675675h^6))
\end{aligned}$$

$$\begin{aligned}
\beta_{6,0} = & (28(-20480a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^6x_{i+\frac{1}{2}}(4708995478a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max} + 239935274463x_{i+\frac{1}{2}}) \\
& - 945h^6(-407585262a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2 + 7902950a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}} + 10725x_{i+\frac{1}{2}}^2) \\
& + 288a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^4\tau_{max}^2(36101338556a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2
\end{aligned}$$

$$\begin{aligned}
& + 24575229630a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}} - 84548054745x_{i+\frac{1}{2}}^2) \\
& + 1280a_{i+\frac{1}{2},j+\frac{1}{2}}^4h^2\tau_{max}^4(50856691162a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2 \\
& + 47508752178a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}} - 576721522125x_{i+\frac{1}{2}}^2) + 259875h^8)) \\
& /((15(327591628066816a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^6 + 49213569888000a_{i+\frac{1}{2},j+\frac{1}{2}}^4h^2\tau_{max}^4 \\
& + 1623322651104a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^4\tau_{max}^2 + 675675h^6)) \\
\beta_{5,0} = & - (56(-266240a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^6x_{i+\frac{1}{2}}^2(2354497739a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max} + 79978424821x_{i+\frac{1}{2}}) \\
& - 9h^6(95613443942a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3 - 556353882630a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2x_{i+\frac{1}{2}} \\
& + 5393763375a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}^2 + 4879875x_{i+\frac{1}{2}}^3) - 288a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^4\tau_{max}^2 \\
& (27266771676a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3 - 469317401228a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2x_{i+\frac{1}{2}} \\
& - 159738992595a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}^2 + 366374903895x_{i+\frac{1}{2}}^3) \\
& + 1280a_{i+\frac{1}{2},j+\frac{1}{2}}^4h^2\tau_{max}^4(5911894398a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3 \\
& + 661136985106a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2x_{i+\frac{1}{2}} \\
& + 308806889157a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}x_{i+\frac{1}{2}}^2 - 2499126595875x_{i+\frac{1}{2}}^3) \\
& + 61425h^8(21053a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max} + 55x_{i+\frac{1}{2}}))) \\
& /((65(327591628066816a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^6 + 49213569888000a_{i+\frac{1}{2},j+\frac{1}{2}}^4h^2\tau_{max}^4 \\
& + 1623322651104a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^4\tau_{max}^2 + 675675h^6)) \\
\beta_{4,0} = & - ((7(2641275h^{10} + 2129920a_{i+\frac{1}{2},j+\frac{1}{2}}^6x_{i+\frac{1}{2}}^3\tau_{max}^6 \\
& (239935274463x_{i+\frac{1}{2}} + 9417990956a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}) \\
& - 5616h^8(28875x_{i+\frac{1}{2}}^2 + 22105650a_{i+\frac{1}{2},j+\frac{1}{2}}x_{i+\frac{1}{2}}\tau_{max} - 383499197a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2) \\
& - 30720a_{i+\frac{1}{2},j+\frac{1}{2}}^4h^2x_{i+\frac{1}{2}}\tau_{max}^4(-2499126595875x_{i+\frac{1}{2}}^3 \\
& + 411742518876a_{i+\frac{1}{2},j+\frac{1}{2}}x_{i+\frac{1}{2}}^2\tau_{max} + 1322273970212a_{i+\frac{1}{2},j+\frac{1}{2}}^2x_{i+\frac{1}{2}}\tau_{max}^2 \\
& + 23647577592a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3) + 72h^6(14639625x_{i+\frac{1}{2}}^4 \\
& + 21575053500a_{i+\frac{1}{2},j+\frac{1}{2}}x_{i+\frac{1}{2}}^3\tau_{max} - 3338123295780a_{i+\frac{1}{2},j+\frac{1}{2}}^2x_{i+\frac{1}{2}}\tau_{max}^2 \\
& + 1147361327304a_{i+\frac{1}{2},j+\frac{1}{2}}^3x_{i+\frac{1}{2}}\tau_{max}^3 + 687203235848a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4)
\end{aligned}$$

$$\begin{aligned}
& + 256a_{i+\frac{1}{2},j+\frac{1}{2}}^2 h^4 \tau_{max}^2 (9892122405165x_{i+\frac{1}{2}}^4 - 5750603733420a_{i+\frac{1}{2},j+\frac{1}{2}}x_{i+\frac{1}{2}}^3 \tau_{max} \\
& - 25343139666312a_{i+\frac{1}{2},j+\frac{1}{2}}^2 x_{i+\frac{1}{2}}^2 \tau_{max}^2 + 2944811341008a_{i+\frac{1}{2},j+\frac{1}{2}}^3 x_{i+\frac{1}{2}} \tau_{max}^3 \\
& + 1087949803400a_{i+\frac{1}{2},j+\frac{1}{2}}^4 \tau_{max}^4)) \\
& /((156(675675h^6 + 1623322651104a_{i+\frac{1}{2},j+\frac{1}{2}}^2 h^4 \tau_{max}^2 \\
& + 49213569888000a_{i+\frac{1}{2},j+\frac{1}{2}}^4 h^2 \tau_{max}^4 + 327591628066816a_{i+\frac{1}{2},j+\frac{1}{2}}^6 \tau_{max}^6))) \\
\beta_{3,0} = & (7(96525h^{10}(301x_{i+\frac{1}{2}} + 118347a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}) \\
& + 4685824a_{i+\frac{1}{2},j+\frac{1}{2}}^6 x_{i+\frac{1}{2}}^4 \tau_{max}^6 (239935274463x_{i+\frac{1}{2}} \\
& + 11772488695a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}) - 4752h^8(125125x_{i+\frac{1}{2}}^3 + 143686725a_{i+\frac{1}{2},j+\frac{1}{2}}x_{i+\frac{1}{2}}^2 \tau_{max} \\
& - 4985489561a_{i+\frac{1}{2},j+\frac{1}{2}}^2 x_{i+\frac{1}{2}} \tau_{max}^2 + 985840581a_{i+\frac{1}{2},j+\frac{1}{2}}^3 \tau_{max}^3) \\
& - 112640a_{i+\frac{1}{2},j+\frac{1}{2}}^4 h^2 x_{i+\frac{1}{2}}^2 \tau_{max}^4 (-1499475957525x_{i+\frac{1}{2}}^3 \\
& + 308806889157a_{i+\frac{1}{2},j+\frac{1}{2}}x_{i+\frac{1}{2}}^2 \tau_{max} + 1322273970212a_{i+\frac{1}{2},j+\frac{1}{2}}^2 x_{i+\frac{1}{2}} \tau_{max}^2 \\
& + 35471366388a_{i+\frac{1}{2},j+\frac{1}{2}}^3 \tau_{max}^3) + 72h^6(32207175x_{i+\frac{1}{2}}^5 \\
& + 59331397125a_{i+\frac{1}{2},j+\frac{1}{2}}x_{i+\frac{1}{2}}^4 \tau_{max} - 12239785417860a_{i+\frac{1}{2},j+\frac{1}{2}}^2 x_{i+\frac{1}{2}}^3 \tau_{max}^2 \\
& + 6310487300172a_{i+\frac{1}{2},j+\frac{1}{2}}^3 x_{i+\frac{1}{2}}^2 \tau_{max}^3 + 7559235594328a_{i+\frac{1}{2},j+\frac{1}{2}}^4 x_{i+\frac{1}{2}} \tau_{max}^4 \\
& - 625703362520a_{i+\frac{1}{2},j+\frac{1}{2}}^5 \tau_{max}^5) + 256a_{i+\frac{1}{2},j+\frac{1}{2}}^2 h^4 \tau_{max}^2 (21762669291363x_{i+\frac{1}{2}}^5 \\
& - 15814160266905a_{i+\frac{1}{2},j+\frac{1}{2}}x_{i+\frac{1}{2}}^4 \tau_{max} - 92924845443144a_{i+\frac{1}{2},j+\frac{1}{2}}^2 x_{i+\frac{1}{2}}^3 \tau_{max}^2 \\
& + 16196462375544a_{i+\frac{1}{2},j+\frac{1}{2}}^3 x_{i+\frac{1}{2}}^2 \tau_{max}^3 + 11967447837400a_{i+\frac{1}{2},j+\frac{1}{2}}^4 x_{i+\frac{1}{2}} \tau_{max}^4 \\
& + 89226796360a_{i+\frac{1}{2},j+\frac{1}{2}}^5 \tau_{max}^5))) \\
& /((429(675675h^6 + 1623322651104a_{i+\frac{1}{2},j+\frac{1}{2}}^2 h^4 \tau_{max}^2 \\
& + 49213569888000a_{i+\frac{1}{2},j+\frac{1}{2}}^4 h^2 \tau_{max}^4 + 327591628066816a_{i+\frac{1}{2},j+\frac{1}{2}}^6 \tau_{max}^6))) \\
\beta_{2,0} = & - ((7(-2413125h^{12} + 28114944a_{i+\frac{1}{2},j+\frac{1}{2}}^6 x_{i+\frac{1}{2}}^5 \tau_{max}^6 \\
& (79978424821x_{i+\frac{1}{2}} + 4708995478a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}) \\
& + 38610h^{10}(4515x_{i+\frac{1}{2}}^2 + 3550410a_{i+\frac{1}{2},j+\frac{1}{2}}x_{i+\frac{1}{2}}\tau_{max} - 25259534a_{i+\frac{1}{2},j+\frac{1}{2}}^2 \tau_{max}^2) \\
& - 135168a_{i+\frac{1}{2},j+\frac{1}{2}}^4 h^2 x_{i+\frac{1}{2}}^3 \tau_{max}^4 (-2499126595875x_{i+\frac{1}{2}}^3
\end{aligned}$$

$$\begin{aligned}
& + 617613778314a_{i+\frac{1}{2},j+\frac{1}{2}}x_{i+\frac{1}{2}}^2\tau_{max} + 3305684925530a_{i+\frac{1}{2},j+\frac{1}{2}}^2x_{i+\frac{1}{2}}\tau_{max}^2 \\
& + 118237887960a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3) - 4752h^8(375375x_{i+\frac{1}{2}}^4 \\
& + 574746900a_{i+\frac{1}{2},j+\frac{1}{2}}x_{i+\frac{1}{2}}^3\tau_{max} - 29912937366a_{i+\frac{1}{2},j+\frac{1}{2}}^2x_{i+\frac{1}{2}}\tau_{max}^2 \\
& + 11830086972a_{i+\frac{1}{2},j+\frac{1}{2}}^3x_{i+\frac{1}{2}}\tau_{max}^3 \\
& + 3794229608a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4) + 1536a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^4x_{i+\frac{1}{2}}\tau_{max}^2 \\
& (7254223097121x_{i+\frac{1}{2}}^5 - 6325664106762a_{i+\frac{1}{2},j+\frac{1}{2}}x_{i+\frac{1}{2}}^4\tau_{max} \\
& - 46462422721572a_{i+\frac{1}{2},j+\frac{1}{2}}^2x_{i+\frac{1}{2}}^3\tau_{max}^2 + 10797641583696a_{i+\frac{1}{2},j+\frac{1}{2}}^3x_{i+\frac{1}{2}}^2\tau_{max}^3 \\
& + 11967447837400a_{i+\frac{1}{2},j+\frac{1}{2}}^4x_{i+\frac{1}{2}}\tau_{max}^4 + 178453592720a_{i+\frac{1}{2},j+\frac{1}{2}}^5\tau_{max}^5) \\
& + 16h^6(289864575x_{i+\frac{1}{2}}^6 + 640779088950a_{i+\frac{1}{2},j+\frac{1}{2}}x_{i+\frac{1}{2}}^5\tau_{max} \\
& - 165237103141110a_{i+\frac{1}{2},j+\frac{1}{2}}^2x_{i+\frac{1}{2}}^4\tau_{max}^2 + 113588771403096a_{i+\frac{1}{2},j+\frac{1}{2}}^3x_{i+\frac{1}{2}}^3\tau_{max}^3 \\
& + 204099361046856a_{i+\frac{1}{2},j+\frac{1}{2}}^4x_{i+\frac{1}{2}}^2\tau_{max}^4 \\
& - 33787981576080a_{i+\frac{1}{2},j+\frac{1}{2}}^5x_{i+\frac{1}{2}}\tau_{max}^5 - 5178530366512a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^6))) \\
& /((1716(675675h^6 + 1623322651104a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^4\tau_{max}^2 \\
& + 49213569888000a_{i+\frac{1}{2},j+\frac{1}{2}}^4h^2\tau_{max}^4 + 327591628066816a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^6))) \\
\beta_{1,0} = & (-675675h^{12}(375x_{i+\frac{1}{2}} + 149849a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}) \\
& + 140574720a_{i+\frac{1}{2},j+\frac{1}{2}}^6x_{i+\frac{1}{2}}^6\tau_{max}^6(239935274463x_{i+\frac{1}{2}} \\
& + 16481484173a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max}) + 4158h^{10} \\
& (1467375x_{i+\frac{1}{2}}^3 + 1730824875a_{i+\frac{1}{2},j+\frac{1}{2}}x_{i+\frac{1}{2}}^2\tau_{max} \\
& - 24628045650a_{i+\frac{1}{2},j+\frac{1}{2}}^2x_{i+\frac{1}{2}}\tau_{max}^2 + 5177901778a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3) \\
& - 14192640a_{i+\frac{1}{2},j+\frac{1}{2}}^4h^2x_{i+\frac{1}{2}}^4\tau_{max}^4(-357018085125x_{i+\frac{1}{2}}^3 \\
& + 102935629719a_{i+\frac{1}{2},j+\frac{1}{2}}x_{i+\frac{1}{2}}^2\tau_{max} + 661136985106a_{i+\frac{1}{2},j+\frac{1}{2}}^2x_{i+\frac{1}{2}}\tau_{max}^2 \\
& + 29559471990a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3) - 720h^8(52026975x_{i+\frac{1}{2}}^5 \\
& + 99574900425a_{i+\frac{1}{2},j+\frac{1}{2}}x_{i+\frac{1}{2}}^4\tau_{max} - 6909888531546a_{i+\frac{1}{2},j+\frac{1}{2}}^2x_{i+\frac{1}{2}}^3\tau_{max}^2 \\
& + 4099125135798a_{i+\frac{1}{2},j+\frac{1}{2}}^3x_{i+\frac{1}{2}}^2\tau_{max}^3 + 2629401118344a_{i+\frac{1}{2},j+\frac{1}{2}}^4x_{i+\frac{1}{2}}\tau_{max}^4
\end{aligned}$$

$$\begin{aligned}
& - 303683537672a_{i+\frac{1}{2},j+\frac{1}{2}}^5\tau_{max}^5 + 10752a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^4x_{i+\frac{1}{2}}^2\tau_{max}^2 \\
& (15544763779545x_{i+\frac{1}{2}}^5 - 15814160266905a_{i+\frac{1}{2},j+\frac{1}{2}}x_{i+\frac{1}{2}}^4\tau_{max} \\
& - 139387268164716a_{i+\frac{1}{2},j+\frac{1}{2}}^2x_{i+\frac{1}{2}}^3\tau_{max}^2 + 40491155938860a_{i+\frac{1}{2},j+\frac{1}{2}}^3x_{i+\frac{1}{2}}^2\tau_{max}^3 \\
& + 59837239187000a_{i+\frac{1}{2},j+\frac{1}{2}}^4x_{i+\frac{1}{2}}\tau_{max}^4 + 1338401945400a_{i+\frac{1}{2},j+\frac{1}{2}}^5\tau_{max}^5) \\
& + 560h^6(124227675x_{i+\frac{1}{2}}^7 + 320389544475a_{i+\frac{1}{2},j+\frac{1}{2}}x_{i+\frac{1}{2}}^6\tau_{max} \\
& - 99142261884666a_{i+\frac{1}{2},j+\frac{1}{2}}^2x_{i+\frac{1}{2}}^5\tau_{max}^2 + 85191578552322a_{i+\frac{1}{2},j+\frac{1}{2}}^3x_{i+\frac{1}{2}}^4\tau_{max}^3 \\
& + 204099361046856a_{i+\frac{1}{2},j+\frac{1}{2}}^4x_{i+\frac{1}{2}}^3\tau_{max}^4 - 50681972364120a_{i+\frac{1}{2},j+\frac{1}{2}}^5x_{i+\frac{1}{2}}^2\tau_{max}^5 \\
& - 15535591099536a_{i+\frac{1}{2},j+\frac{1}{2}}^6x_{i+\frac{1}{2}}\tau_{max}^6 - 74240090992a_{i+\frac{1}{2},j+\frac{1}{2}}^7\tau_{max}^7)) \\
& /((12870(675675h^6 + 1623322651104a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^4\tau_{max}^2 \\
& + 49213569888000a_{i+\frac{1}{2},j+\frac{1}{2}}^4h^2\tau_{max}^4 + 327591628066816a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^6)) \\
\beta_{0,0} = & - ((211486275h^14 + 2249195520a_{i+\frac{1}{2},j+\frac{1}{2}}^6x_{i+\frac{1}{2}}^7\tau_{max}^6 \\
& (239935274463x_{i+\frac{1}{2}} + 18835981912a_{i+\frac{1}{2},j+\frac{1}{2}}\tau_{max})) \\
& - 1729728h^{12}(9375x_{i+\frac{1}{2}}^2 + 7492450a_{i+\frac{1}{2},j+\frac{1}{2}}x_{i+\frac{1}{2}}\tau_{max} - 22675009a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2) \\
& - 10813440a_{i+\frac{1}{2},j+\frac{1}{2}}^4h^2x_{i+\frac{1}{2}}^5\tau_{max}^4(-7497379787625x_{i+\frac{1}{2}}^3 \\
& + 2470455113256a_{i+\frac{1}{2},j+\frac{1}{2}}x_{i+\frac{1}{2}}^2\tau_{max} + 18511835582968a_{i+\frac{1}{2},j+\frac{1}{2}}^2x_{i+\frac{1}{2}}\tau_{max}^2 \\
& + 993198258864a_{i+\frac{1}{2},j+\frac{1}{2}}^3\tau_{max}^3) + 6336h^{10}(30814875x_{i+\frac{1}{2}}^4 \\
& + 48463096500a_{i+\frac{1}{2},j+\frac{1}{2}}x_{i+\frac{1}{2}}^3\tau_{max} - 1034377917300a_{i+\frac{1}{2},j+\frac{1}{2}}^2x_{i+\frac{1}{2}}^2\tau_{max}^2 \\
& + 434943749352a_{i+\frac{1}{2},j+\frac{1}{2}}^3x_{i+\frac{1}{2}}\tau_{max}^3 + 87945430928a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4) \\
& + 172032a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^4x_{i+\frac{1}{2}}^3\tau_{max}^2(15544763779545x_{i+\frac{1}{2}}^5 \\
& - 18073326019320a_{i+\frac{1}{2},j+\frac{1}{2}}x_{i+\frac{1}{2}}^4\tau_{max} - 185849690886288a_{i+\frac{1}{2},j+\frac{1}{2}}^2x_{i+\frac{1}{2}}^3\tau_{max}^2 \\
& + 64785849502176a_{i+\frac{1}{2},j+\frac{1}{2}}^3x_{i+\frac{1}{2}}^2\tau_{max}^3 + 119674478374000a_{i+\frac{1}{2},j+\frac{1}{2}}^4x_{i+\frac{1}{2}}\tau_{max}^4 \\
& + 3569071854400a_{i+\frac{1}{2},j+\frac{1}{2}}^5\tau_{max}^5) - 5120h^8(156080925x_{i+\frac{1}{2}}^6 \\
& + 358469641530a_{i+\frac{1}{2},j+\frac{1}{2}}x_{i+\frac{1}{2}}^5\tau_{max} - 31094498391957a_{i+\frac{1}{2},j+\frac{1}{2}}^2x_{i+\frac{1}{2}}^4\tau_{max}^2 \\
& + 24594750814788a_{i+\frac{1}{2},j+\frac{1}{2}}^3x_{i+\frac{1}{2}}^3\tau_{max}^3 + 23664610065096a_{i+\frac{1}{2},j+\frac{1}{2}}^4x_{i+\frac{1}{2}}^2\tau_{max}^4
\end{aligned}$$

$$\begin{aligned}
& - 5466303678096a_{i+\frac{1}{2},j+\frac{1}{2}}^5x_{i+\frac{1}{2}}\tau_{max}^5 - 321570260696a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^6) \\
& + 1792h^6x_{i+\frac{1}{2}}(621138375x_{i+\frac{1}{2}}^7 + 1830797397000a_{i+\frac{1}{2},j+\frac{1}{2}}x_{i+\frac{1}{2}}^6\tau_{max} \\
& - 660948412564440a_{i+\frac{1}{2},j+\frac{1}{2}}^2x_{i+\frac{1}{2}}^5\tau_{max}^2 + 681532628418576a_{i+\frac{1}{2},j+\frac{1}{2}}^3x_{i+\frac{1}{2}}^4\tau_{max}^3 \\
& + 2040993610468560a_{i+\frac{1}{2},j+\frac{1}{2}}^4x_{i+\frac{1}{2}}^3\tau_{max}^4 - 675759631521600a_{i+\frac{1}{2},j+\frac{1}{2}}^5x_{i+\frac{1}{2}}^2\tau_{max}^5 \\
& - 310711821990720a_{i+\frac{1}{2},j+\frac{1}{2}}^6x_{i+\frac{1}{2}}\tau_{max}^6 - 2969603639680a_{i+\frac{1}{2},j+\frac{1}{2}}^7\tau_{max}^7)) \\
& /((1647360(675675h^6 + 1623322651104a_{i+\frac{1}{2},j+\frac{1}{2}}^2h^4\tau_{max}^2 \\
& + 49213569888000a_{i+\frac{1}{2},j+\frac{1}{2}}^4h^2\tau_{max}^4 + 327591628066816a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^6)))
\end{aligned}$$

Hence, we have

$$\begin{aligned}
& \int_{x_i}^{x_{i+\frac{1}{2}}} \int_{y_j}^{y_{j+\frac{1}{2}}} (v_I(x, y) - x^8 - u_I(x - \frac{h}{2}, y - \frac{h}{2}) + (x - \frac{h}{2})^8)^2 dx dy \\
& = ((a_{i,j} - a_{i+\frac{1}{2},j+\frac{1}{2}})^2 h^{16} \tau_{max}^2 (6407611697217187127041124660156250h^{24} \\
& + 149695974574807265373449136315234375(a_{i,j} + a_{i+\frac{1}{2},j+\frac{1}{2}})\tau h^{23} \\
& + 14193290712100500000(102198405596032324406a_{i,j}^2 \\
& - 1964913426422720209639a_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j} \\
& + 102198405596032324406a_{i+\frac{1}{2},j+\frac{1}{2}}^2)\tau_{max}^2 h^{22} \\
& - 5125354979369625000(342475000791495976813a_{i,j}^3 \\
& + 70854261497225701973887a_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}^2 \\
& + 70854261497225701973887a_{i+\frac{1}{2},j+\frac{1}{2}}^2a_{i,j} \\
& + 342475000791495976813a_{i+\frac{1}{2},j+\frac{1}{2}}^3)\tau_{max}^3 h^{21} \\
& + 7646217218640000(11652918878137704874257261a_{i,j}^4 \\
& - 356825898865465222780684354a_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j}^3 \\
& + 4836984475588643523349889892863a_{i+\frac{1}{2},j+\frac{1}{2}}^2a_{i,j}^2 \\
& - 356825898865465222780684354a_{i+\frac{1}{2},j+\frac{1}{2}}^3a_{i,j} \\
& + 11652918878137704874257261a_{i+\frac{1}{2},j+\frac{1}{2}}^4)\tau_{max}^4 h^{20} \\
& - 84108389405040000(915648732630103373510011a_{i,j}^5
\end{aligned}$$

$$\begin{aligned}
& - 9407889926703309649033127a_{i+\frac{1}{2},j+\frac{1}{2}}^4 a_{i,j}^4 \\
& + 50390082001645985867302150063a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j}^3 \\
& + 50390082001645985867302150063a_{i+\frac{1}{2},j+\frac{1}{2}}^3 a_{i,j}^2 \\
& - 9407889926703309649033127a_{i+\frac{1}{2},j+\frac{1}{2}}^4 a_{i,j}^4 \\
& + 915648732630103373510011a_{i+\frac{1}{2},j+\frac{1}{2}}^5 \tau_{max}^5 h^{19} \\
& + 11950131916800(164884185222005333805319210300a_{i,j}^6 \\
& - 6630342706616501593421808756675a_{i+\frac{1}{2},j+\frac{1}{2}}^5 a_{i,j}^5 \\
& + 188076239951110923388959402053344698a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j}^4 \\
& - 208575150081849679648577147724521436a_{i+\frac{1}{2},j+\frac{1}{2}}^3 a_{i,j}^3 \\
& + 188076239951110923388959402053344698a_{i+\frac{1}{2},j+\frac{1}{2}}^4 a_{i,j}^2 \\
& - 6630342706616501593421808756675a_{i+\frac{1}{2},j+\frac{1}{2}}^5 a_{i,j}^5 \\
& + 164884185222005333805319210300a_{i+\frac{1}{2},j+\frac{1}{2}}^6 \tau_{max}^6 h^{18} \\
& - 116513786188800(9030355544280902800247270875a_{i,j}^7 \\
& - 3739708129728410820201855381800a_{i+\frac{1}{2},j+\frac{1}{2}}^6 a_{i,j}^6 \\
& + 1639671962272060644440386615182608a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j}^5 \\
& - 468108435937358839882855115553109999a_{i+\frac{1}{2},j+\frac{1}{2}}^3 a_{i,j}^4 \\
& - 468108435937358839882855115553109999a_{i+\frac{1}{2},j+\frac{1}{2}}^4 a_{i,j}^3 \\
& + 1639671962272060644440386615182608a_{i+\frac{1}{2},j+\frac{1}{2}}^5 a_{i,j}^2 \\
& - 3739708129728410820201855381800a_{i+\frac{1}{2},j+\frac{1}{2}}^6 a_{i,j}^6 \\
& + 9030355544280902800247270875a_{i+\frac{1}{2},j+\frac{1}{2}}^7 \tau_{max}^7 h^{17} \\
& + 1042920603648(17693229844358932539826204876250a_{i,j}^8 \\
& - 851016926324609061380223519230000a_{i+\frac{1}{2},j+\frac{1}{2}}^7 a_{i,j}^7 \\
& + 47125344545016585156950522109397583050a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j}^6 \\
& - 97529619282774141346719350823793132925a_{i+\frac{1}{2},j+\frac{1}{2}}^3 a_{i,j}^5
\end{aligned}$$

$$\begin{aligned}
& + 5255857505572670614682746527402276260122a_{i+\frac{1}{2},j+\frac{1}{2}}^4a_{i,j}^4 \\
& - 97529619282774141346719350823793132925a_{i+\frac{1}{2},j+\frac{1}{2}}^5a_{i,j}^3 \\
& + 47125344545016585156950522109397583050a_{i+\frac{1}{2},j+\frac{1}{2}}^6a_{i,j}^2 \\
& - 851016926324609061380223519230000a_{i+\frac{1}{2},j+\frac{1}{2}}^7a_{i,j} \\
& + 17693229844358932539826204876250a_{i+\frac{1}{2},j+\frac{1}{2}}^8)\tau_{max}^8h^{16} \\
& - 49712548773888(101040772023733232617286205625a_{i,j}^9 \\
& - 140105212582522805843160693143750a_{i+\frac{1}{2},j+\frac{1}{2}}^8a_{i,j}^8 \\
& + 51582530719180029823261911267645250a_{i+\frac{1}{2},j+\frac{1}{2}}^2a_{i,j}^7 \\
& - 44163649657013283522468049574357842925a_{i+\frac{1}{2},j+\frac{1}{2}}^3a_{i,j}^6 \\
& + 244708639996747076704243816052072844257a_{i+\frac{1}{2},j+\frac{1}{2}}^4a_{i,j}^5 \\
& + 244708639996747076704243816052072844257a_{i+\frac{1}{2},j+\frac{1}{2}}^5a_{i,j}^4 \\
& - 44163649657013283522468049574357842925a_{i+\frac{1}{2},j+\frac{1}{2}}^6a_{i,j}^3 \\
& + 51582530719180029823261911267645250a_{i+\frac{1}{2},j+\frac{1}{2}}^7a_{i,j}^2 \\
& - 140105212582522805843160693143750a_{i+\frac{1}{2},j+\frac{1}{2}}^8a_{i,j} \\
& + 101040772023733232617286205625a_{i+\frac{1}{2},j+\frac{1}{2}}^9)\tau_{max}^9h^{15} \\
& + 1854081073152(33584331291564604301081858670000a_{i,j}^{10} \\
& - 1906180194915312599272907575693750a_{i+\frac{1}{2},j+\frac{1}{2}}^9a_{i,j}^9 \\
& + 245764139312547176860059537900301659800a_{i+\frac{1}{2},j+\frac{1}{2}}^2a_{i,j}^8 \\
& - 707663942797545329144651859631895512600a_{i+\frac{1}{2},j+\frac{1}{2}}^3a_{i,j}^7 \\
& + 134163063816345518949074583247554347787928a_{i+\frac{1}{2},j+\frac{1}{2}}^4a_{i,j}^6 \\
& + 26378601392924543232613045305256772435407a_{i+\frac{1}{2},j+\frac{1}{2}}^5a_{i,j}^5 \\
& + 134163063816345518949074583247554347787928a_{i+\frac{1}{2},j+\frac{1}{2}}^6a_{i,j}^4 \\
& - 707663942797545329144651859631895512600a_{i+\frac{1}{2},j+\frac{1}{2}}^7a_{i,j}^3 \\
& + 245764139312547176860059537900301659800a_{i+\frac{1}{2},j+\frac{1}{2}}^8a_{i,j}^2
\end{aligned}$$

$$\begin{aligned}
& - 1906180194915312599272907575693750a_{i+\frac{1}{2},j+\frac{1}{2}}^9 a_{i,j} \\
& + 33584331291564604301081858670000a_{i+\frac{1}{2},j+\frac{1}{2}}^{10} \tau_{max}^{10} h^{14} \\
& - 927040536576(4806491069858999270093254178125a_{i,j}^{11} \\
& - 32905119135827149176686917250697500a_{i+\frac{1}{2},j+\frac{1}{2}}^{10} a_{i,j}^{10} \\
& + 11947255513222701108328435117536305200a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j}^9 \\
& - 29516372734495368634508174842749428453225a_{i+\frac{1}{2},j+\frac{1}{2}}^3 a_{i,j}^8 \\
& + 71647191143541128835098649254728570811155a_{i+\frac{1}{2},j+\frac{1}{2}}^4 a_{i,j}^7 \\
& - 515027236544736694707493486699984121084966a_{i+\frac{1}{2},j+\frac{1}{2}}^5 a_{i,j}^6 \\
& - 515027236544736694707493486699984121084966a_{i+\frac{1}{2},j+\frac{1}{2}}^6 a_{i,j}^5 \\
& + 71647191143541128835098649254728570811155a_{i+\frac{1}{2},j+\frac{1}{2}}^7 a_{i,j}^4 \\
& - 29516372734495368634508174842749428453225a_{i+\frac{1}{2},j+\frac{1}{2}}^8 a_{i,j}^3 \\
& + 11947255513222701108328435117536305200a_{i+\frac{1}{2},j+\frac{1}{2}}^9 a_{i,j}^2 \\
& - 32905119135827149176686917250697500a_{i+\frac{1}{2},j+\frac{1}{2}}^{10} a_{i,j}^{10} \\
& + 4806491069858999270093254178125a_{i+\frac{1}{2},j+\frac{1}{2}}^{11} \tau_{max}^{11} h^{13} \\
& + 117719433216(15416045445800001639655121228125a_{i,j}^{12} \\
& - 11248959872916054394746123802068750a_{i+\frac{1}{2},j+\frac{1}{2}}^{11} a_{i,j}^{11} \\
& + 12916292928683882713171200164090274872775a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j}^{10} \\
& - 48929883458195191471671338307385182264000a_{i+\frac{1}{2},j+\frac{1}{2}}^3 a_{i,j}^9 \\
& + 29090074466973164035719501925737910175109849a_{i+\frac{1}{2},j+\frac{1}{2}}^4 a_{i,j}^8 \\
& + 8725725200094031043177646404131246876282890a_{i+\frac{1}{2},j+\frac{1}{2}}^5 a_{i,j}^7 \\
& + 87005910929471657463937678864353442773705631a_{i+\frac{1}{2},j+\frac{1}{2}}^6 a_{i,j}^6 \\
& + 8725725200094031043177646404131246876282890a_{i+\frac{1}{2},j+\frac{1}{2}}^7 a_{i,j}^5 \\
& + 29090074466973164035719501925737910175109849a_{i+\frac{1}{2},j+\frac{1}{2}}^8 a_{i,j}^4 \\
& - 48929883458195191471671338307385182264000a_{i+\frac{1}{2},j+\frac{1}{2}}^9 a_{i,j}^3
\end{aligned}$$

$$\begin{aligned}
& + 12916292928683882713171200164090274872775a_{i+\frac{1}{2},j+\frac{1}{2}}^{10}a_{i,j}^2 \\
& - 11248959872916054394746123802068750a_{i+\frac{1}{2},j+\frac{1}{2}}^{11}a_{i,j} \\
& + 15416045445800001639655121228125a_{i+\frac{1}{2},j+\frac{1}{2}}^{12})\tau_{max}^{12}h^{12} \\
& - 706316599296a_{i,j}^2a_{i+\frac{1}{2},j+\frac{1}{2}}^2(4265942923575175849094670782573247875a_{i,j}^9 \\
& - 153862613744907003545930019277911586979725a_{i+\frac{1}{2},j+\frac{1}{2}}^8a_{i,j}^8 \\
& - 3474807299101645926175246054336032343331902a_{i+\frac{1}{2},j+\frac{1}{2}}^2a_{i,j}^7 \\
& - 18372351946354404380564325292881086972629438a_{i+\frac{1}{2},j+\frac{1}{2}}^3a_{i,j}^6 \\
& - 23225220972302688843109493083062064356608183a_{i+\frac{1}{2},j+\frac{1}{2}}^4a_{i,j}^5 \\
& - 23225220972302688843109493083062064356608183a_{i+\frac{1}{2},j+\frac{1}{2}}^5a_{i,j}^4 \\
& - 18372351946354404380564325292881086972629438a_{i+\frac{1}{2},j+\frac{1}{2}}^6a_{i,j}^3 \\
& - 3474807299101645926175246054336032343331902a_{i+\frac{1}{2},j+\frac{1}{2}}^7a_{i,j}^2 \\
& - 153862613744907003545930019277911586979725a_{i+\frac{1}{2},j+\frac{1}{2}}^8a_{i,j} \\
& + 4265942923575175849094670782573247875a_{i+\frac{1}{2},j+\frac{1}{2}}^9)\tau_{max}^{13}h^{11} \\
& + 2054739197952a_{i,j}^2a_{i+\frac{1}{2},j+\frac{1}{2}}^2(4243863750274013736917056418445662625a_{i,j}^{10} \\
& - 1547703983492716454974956935236843394625a_{i+\frac{1}{2},j+\frac{1}{2}}^9a_{i,j}^9 \\
& + 7727279661360167998007807567476980062903913a_{i+\frac{1}{2},j+\frac{1}{2}}^2a_{i,j}^8 \\
& + 2721712884336354977154964974396418615333980a_{i+\frac{1}{2},j+\frac{1}{2}}^3a_{i,j}^7 \\
& + 66282203069516127842005808949034716669693715a_{i+\frac{1}{2},j+\frac{1}{2}}^4a_{i,j}^6 \\
& + 10699144073071358121723076542916022358448199a_{i+\frac{1}{2},j+\frac{1}{2}}^5a_{i,j}^5 \\
& + 66282203069516127842005808949034716669693715a_{i+\frac{1}{2},j+\frac{1}{2}}^6a_{i,j}^4 \\
& + 2721712884336354977154964974396418615333980a_{i+\frac{1}{2},j+\frac{1}{2}}^7a_{i,j}^3 \\
& + 7727279661360167998007807567476980062903913a_{i+\frac{1}{2},j+\frac{1}{2}}^8a_{i,j}^2 \\
& - 1547703983492716454974956935236843394625a_{i+\frac{1}{2},j+\frac{1}{2}}^9a_{i,j} \\
& + 4243863750274013736917056418445662625a_{i+\frac{1}{2},j+\frac{1}{2}}^{10})\tau_{max}^{14}h^{10}
\end{aligned}$$

$$\begin{aligned}
& + 15068087451648a_{i,j}^4a_{i+\frac{1}{2},j+\frac{1}{2}}^4 \\
& (1226440005433813283100595708201779028363970a_{i,j}^7 \\
& + 4637165397258385176176955537957113705171438a_{i+\frac{1}{2},j+\frac{1}{2}}^6a_{i,j} \\
& + 12794089326376582464931337782336108624614973a_{i+\frac{1}{2},j+\frac{1}{2}}^2a_{i,j}^5 \\
& + 22775067494022410799181373159209801943311225a_{i+\frac{1}{2},j+\frac{1}{2}}^3a_{i,j}^4 \\
& + 22775067494022410799181373159209801943311225a_{i+\frac{1}{2},j+\frac{1}{2}}^4a_{i,j}^3 \\
& + 12794089326376582464931337782336108624614973a_{i+\frac{1}{2},j+\frac{1}{2}}^5a_{i,j}^2 \\
& + 4637165397258385176176955537957113705171438a_{i+\frac{1}{2},j+\frac{1}{2}}^6a_{i,j} \\
& + 1226440005433813283100595708201779028363970a_{i+\frac{1}{2},j+\frac{1}{2}}^7)\tau_{max}^{15}h^9 \\
& + 1826434842624a_{i,j}^4a_{i+\frac{1}{2},j+\frac{1}{2}}^4 \\
& (5735376305254471497397007185633322552332115a_{i,j}^8 \\
& + 1709501352109512008062597478578959919914800a_{i+\frac{1}{2},j+\frac{1}{2}}^7a_{i,j} \\
& + 355614542756479161277327105199371564782709331a_{i+\frac{1}{2},j+\frac{1}{2}}^2a_{i,j}^6 \\
& + 70479522127743247319870522725960510761860785a_{i+\frac{1}{2},j+\frac{1}{2}}^3a_{i,j}^5 \\
& + 1002829664307157485423082511164031759353903459a_{i+\frac{1}{2},j+\frac{1}{2}}^4a_{i,j}^4 \\
& + 70479522127743247319870522725960510761860785a_{i+\frac{1}{2},j+\frac{1}{2}}^5a_{i,j}^3 \\
& + 355614542756479161277327105199371564782709331a_{i+\frac{1}{2},j+\frac{1}{2}}^6a_{i,j}^2 \\
& + 1709501352109512008062597478578959919914800a_{i+\frac{1}{2},j+\frac{1}{2}}^7a_{i,j} \\
& + 5735376305254471497397007185633322552332115a_{i+\frac{1}{2},j+\frac{1}{2}}^8)\tau_{max}^{16}h^8 \\
& + 3652869685248a_{i,j}^6a_{i+\frac{1}{2},j+\frac{1}{2}}^6 \\
& (203384565657636861119423563747531084300163875a_{i,j}^5 \\
& + 472300903909347413055630862958395552513750987a_{i+\frac{1}{2},j+\frac{1}{2}}^4a_{i,j}^4 \\
& + 845439059698435708279323900172583571971402815a_{i+\frac{1}{2},j+\frac{1}{2}}^2a_{i,j}^3 \\
& + 845439059698435708279323900172583571971402815a_{i+\frac{1}{2},j+\frac{1}{2}}^3a_{i,j}^2
\end{aligned}$$

$$\begin{aligned}
& + 472300903909347413055630862958395552513750987a_{i+\frac{1}{2},j+\frac{1}{2}}^4 a_{i,j} \\
& + 203384565657636861119423563747531084300163875a_{i+\frac{1}{2},j+\frac{1}{2}}^5 \tau_{max}^{17} h^7 \\
& + 463856467968a_{i,j}^6 a_{i+\frac{1}{2},j+\frac{1}{2}}^6 \\
& (1369249982086609549862770039779934986011183875a_{i,j}^6 \\
& + 278139799007723218202002921462902585975963495a_{i+\frac{1}{2},j+\frac{1}{2}}^5 a_{i,j}^5 \\
& + 20485983393544401265668693110581367857751110947a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j}^4 \\
& + 1906917249724283981846234551210738981775181093a_{i+\frac{1}{2},j+\frac{1}{2}}^3 a_{i,j}^3 \\
& + 20485983393544401265668693110581367857751110947a_{i+\frac{1}{2},j+\frac{1}{2}}^4 a_{i,j}^2 \\
& + 278139799007723218202002921462902585975963495a_{i+\frac{1}{2},j+\frac{1}{2}}^5 a_{i,j} \\
& + 1369249982086609549862770039779934986011183875a_{i+\frac{1}{2},j+\frac{1}{2}}^6 \tau_{max}^{18} h^6 \\
& + 5566277615616a_{i,j}^8 a_{i+\frac{1}{2},j+\frac{1}{2}}^8 \\
& (1664951346635098223961807292973869362140155765a_{i,j}^3 \\
& + 2526101350040567759312558807452293135318345361a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j}^2 \\
& + 2526101350040567759312558807452293135318345361a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j} \\
& + 1664951346635098223961807292973869362140155765a_{i+\frac{1}{2},j+\frac{1}{2}}^3 \tau_{max}^{19} h^5 \\
& + 11132555231232a_{i,j}^8 a_{i+\frac{1}{2},j+\frac{1}{2}}^8 \\
& (1244577120142977931757886187009203449137132565a_{i,j}^4 \\
& + 145703135473265603367263926004750654965009150a_{i+\frac{1}{2},j+\frac{1}{2}}^3 a_{i,j}^3 \\
& + 5277782995676146054115129935872458903443701703a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j}^2 \\
& + 145703135473265603367263926004750654965009150a_{i+\frac{1}{2},j+\frac{1}{2}}^3 a_{i,j} \\
& + 1244577120142977931757886187009203449137132565a_{i+\frac{1}{2},j+\frac{1}{2}}^4 \tau_{max}^{20} h^4 \\
& + 36865115691022402100861417388552566459381060053952888832000 \\
& a_{i,j}^{10} a_{i+\frac{1}{2},j+\frac{1}{2}}^{10} (a_{i,j} + a_{i+\frac{1}{2},j+\frac{1}{2}}) \tau_{max}^{21} h^3 \\
& + 49531972439734470923977749503756206080a_{i,j}^{10} a_{i+\frac{1}{2},j+\frac{1}{2}}^{10}
\end{aligned}$$

$$\begin{aligned}
& (2587661596660313943875a_{i,j}^2 + 130692634564583157233a_{i+\frac{1}{2},j+\frac{1}{2}}a_{i,j} \\
& + 2587661596660313943875a_{i+\frac{1}{2},j+\frac{1}{2}}^2)\tau_{max}^{22}h^2 \\
& + 426590355493017768656985138297061848222462989929694793564160 \\
& a_{i,j}^{12}a_{i+\frac{1}{2},j+\frac{1}{2}}^{12}\tau_{max}^{24})/(254418278400(675675h^6 + 1623322651104a_{i,j}^2\tau_{max}^2h^4 \\
& + 49213569888000a_{i,j}^4\tau_{max}^4h^2 \\
& + 327591628066816a_{i,j}^6\tau_{max}^6)^2(675675h^6 + 1623322651104a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2h^4 \\
& + 49213569888000a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4h^2 + 327591628066816a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^6)^2) \\
& = O(h^{20})
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
& \int_{x_i}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_j} (v_I(x, y) - x^8 - u_I(x - \frac{h}{2}, y + \frac{h}{2}) + (x - \frac{h}{2})^8)^2 dx dy = O(h^{20}), \\
& \int_{x_{i-\frac{1}{2}}}^{x_i} \int_{y_j}^{y_{j+\frac{1}{2}}} (v_I(x, y) - x^8 - u_I(x + \frac{h}{2}, y - \frac{h}{2}) + (x + \frac{h}{2})^8)^2 dx dy = O(h^{20}), \\
& \int_{x_{i-\frac{1}{2}}}^{x_i} \int_{y_j}^{y_{j+\frac{1}{2}}} (v_I(x, y) - x^8 - u_I(x + \frac{h}{2}, y + \frac{h}{2}) + (x + \frac{h}{2})^8)^2 dx dy = O(h^{20}).
\end{aligned}$$

9. $k = 8, u = x^9$, by the definition of the projection,

$$\begin{aligned}
& \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} u_I dx dy = \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} x^9 dx dy \\
& \tilde{P}_h(u_I; x^k y^l; f, g, u)_{i,j} = \tilde{P}_h(x^9; x^k y^l; f, g, u)_{i,j}, \quad k = 0, 1, \dots, 8, \quad l = 0, 1, \dots, 8 \\
& \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} v_I dx dy = \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} x^9 dx dy \\
& \tilde{Q}_h(v_I; x^k y^l; f, g, u)_{i+\frac{1}{2},j+\frac{1}{2}} = \tilde{Q}_h(x^9; x^k y^l; f, g, u)_{i+\frac{1}{2},j+\frac{1}{2}}, \quad k = 0, 1, \dots, 8, \quad l = 0, 1, \dots, 8
\end{aligned}$$

then by solving the above linear system we have

$$\begin{aligned}
u_I(x, y) &= \sum_{k=0}^8 \sum_{l=0}^8 \alpha_{k,l} x^k y^l \\
v_I(x, y) &= \sum_{k=0}^8 \sum_{l=0}^8 \beta_{k,l} x^k y^l
\end{aligned} \tag{2.35}$$

By similar calculation we have,

$$\begin{aligned}
& \int_{x_i}^{x_{i+\frac{1}{2}}} \int_{y_j}^{y_{j+\frac{1}{2}}} (v_I(x, y) - x^9 - u_I(x - \frac{h}{2}, y - \frac{h}{2}) + (x - \frac{h}{2})^9)^2 dx dy \\
&= (3(a_{i,j} - a_{i+\frac{1}{2},j+\frac{1}{2}})^2 h^{22} \tau_{max}^2 (1100247162677562372226747959055445824218750 h^{28} \\
&+ 36271643488641303971494745633646826318359375 (a_{i,j} + a_{i+\frac{1}{2},j+\frac{1}{2}}) \tau_{max} h^{27} \\
&+ 47592581951289375000 (603358537378470717071922052091 a_{i,j}^2 \\
&- 18776909426726176784250157368370 a_{i+\frac{1}{2},j+\frac{1}{2}} a_{i,j} \\
&+ 603358537378470717071922052091 a_{i+\frac{1}{2},j+\frac{1}{2}}^2) \tau_{max}^2 h^{26} \\
&- 17799625649782226250000 (392265732763604422564840797 a_{i,j}^3 \\
&+ 881527885756504264593056283964 a_{i+\frac{1}{2},j+\frac{1}{2}} a_{i,j}^2 \\
&+ 881527885756504264593056283964 a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j} \\
&+ 392265732763604422564840797 a_{i+\frac{1}{2},j+\frac{1}{2}}^3) \tau_{max}^3 h^{25} \\
&+ 1562012946093600000 (2476844634069685910790681532329585 a_{i,j}^4 \\
&- 101288246902584516669721282319307240 a_{i+\frac{1}{2},j+\frac{1}{2}} a_{i,j}^3 \\
&+ 131580620158454441035102761180088544783174 a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j}^2 \\
&- 101288246902584516669721282319307240 a_{i+\frac{1}{2},j+\frac{1}{2}}^3 a_{i,j} \\
&+ 2476844634069685910790681532329585 a_{i+\frac{1}{2},j+\frac{1}{2}}^4) \tau_{max}^4 h^{24} \\
&- 86301215271671400000 (7750842255561192531347154033305 a_{i,j}^5 \\
&- 24002984726861235502615709120783935 a_{i+\frac{1}{2},j+\frac{1}{2}} a_{i,j}^4 \\
&+ 34968196700601573740765634573416756678 a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j}^3 \\
&+ 34968196700601573740765634573416756678 a_{i+\frac{1}{2},j+\frac{1}{2}}^3 a_{i,j}^2 \\
&- 24002984726861235502615709120783935 a_{i+\frac{1}{2},j+\frac{1}{2}}^4 a_{i,j} \\
&+ 7750842255561192531347154033305 a_{i+\frac{1}{2},j+\frac{1}{2}}^5) \tau_{max}^5 h^{23} \\
&+ 119010510178560000 (1381088270529089528757979201207871490 a_{i,j}^6 \\
&- 76457860683796307522246724105451833700 a_{i+\frac{1}{2},j+\frac{1}{2}} a_{i,j}^5
\end{aligned}$$

$$\begin{aligned}
& + 232527795648485692055919616044493067786121087a_{i+\frac{1}{2},j+\frac{1}{2}}^2a_{i,j}^4 \\
& - 137642650758131781459680385519949473952301362a_{i+\frac{1}{2},j+\frac{1}{2}}^3a_{i,j}^3 \\
& + 232527795648485692055919616044493067786121087a_{i+\frac{1}{2},j+\frac{1}{2}}^4a_{i,j}^2 \\
& - 76457860683796307522246724105451833700a_{i+\frac{1}{2},j+\frac{1}{2}}^5a_{i,j} \\
& + 1381088270529089528757979201207871490a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^6h^{22} \\
& - 2023178673035520000(9072423166234218203840328672221325a_{i,j}^7 \\
& - 109016676753568143536347011053427742455a_{i+\frac{1}{2},j+\frac{1}{2}}^6a_{i,j}^6 \\
& + 141471341084999001425393433217531978820294a_{i+\frac{1}{2},j+\frac{1}{2}}^2a_{i,j}^5 \\
& - 893618706668429901997509362349101515272669984a_{i+\frac{1}{2},j+\frac{1}{2}}^3a_{i,j}^4 \\
& - 893618706668429901997509362349101515272669984a_{i+\frac{1}{2},j+\frac{1}{2}}^4a_{i,j}^3 \\
& + 141471341084999001425393433217531978820294a_{i+\frac{1}{2},j+\frac{1}{2}}^5a_{i,j}^2 \\
& - 109016676753568143536347011053427742455a_{i+\frac{1}{2},j+\frac{1}{2}}^6a_{i,j} \\
& + 9072423166234218203840328672221325a_{i+\frac{1}{2},j+\frac{1}{2}}^7\tau_{max}^7h^{21} \\
& + 177544817049600(12998137365405447658144174189512481017500a_{i,j}^8 \\
& - 1135156802616869989367341847124668260937500a_{i+\frac{1}{2},j+\frac{1}{2}}^7a_{i,j}^7 \\
& + 6614772780157165536692795537288316997608604578870a_{i+\frac{1}{2},j+\frac{1}{2}}^2a_{i,j}^6 \\
& - 9722888558714083444869815406558779721533966291360a_{i+\frac{1}{2},j+\frac{1}{2}}^3a_{i,j}^5 \\
& + 2311879244110624884078502406341183990639503622335887a_{i+\frac{1}{2},j+\frac{1}{2}}^4a_{i,j}^4 \\
& - 9722888558714083444869815406558779721533966291360a_{i+\frac{1}{2},j+\frac{1}{2}}^5a_{i,j}^3 \\
& + 6614772780157165536692795537288316997608604578870a_{i+\frac{1}{2},j+\frac{1}{2}}^6a_{i,j}^2 \\
& - 1135156802616869989367341847124668260937500a_{i+\frac{1}{2},j+\frac{1}{2}}^7a_{i,j} \\
& + 12998137365405447658144174189512481017500a_{i+\frac{1}{2},j+\frac{1}{2}}^8\tau_{max}^8h^{20} \\
& - 215805725123788800(888457065097635122437710169588178125a_{i,j}^9 \\
& - 13958448974879714404915130494435784889375a_{i+\frac{1}{2},j+\frac{1}{2}}^8a_{i,j}^8
\end{aligned}$$

$$\begin{aligned}
& + 34904708339364654174924652814938752680359150a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j}^7 \\
& - 621180411470546269732340215917410738499047359350a_{i+\frac{1}{2},j+\frac{1}{2}}^3 a_{i,j}^6 \\
& - 6137736630581073285363696286829426419551582444623a_{i+\frac{1}{2},j+\frac{1}{2}}^4 a_{i,j}^5 \\
& - 6137736630581073285363696286829426419551582444623a_{i+\frac{1}{2},j+\frac{1}{2}}^5 a_{i,j}^4 \\
& - 621180411470546269732340215917410738499047359350a_{i+\frac{1}{2},j+\frac{1}{2}}^6 a_{i,j}^3 \\
& + 34904708339364654174924652814938752680359150a_{i+\frac{1}{2},j+\frac{1}{2}}^7 a_{i,j}^2 \\
& - 13958448974879714404915130494435784889375a_{i+\frac{1}{2},j+\frac{1}{2}}^8 a_{i,j} \\
& + 888457065097635122437710169588178125a_{i+\frac{1}{2},j+\frac{1}{2}}^9 \tau_{max}^9 h^{19} \\
& + 6763612078080(1634528636805173906473549391917644025040625a_{i,j}^{10} \\
& - 282552484263505693659935098779956210575968750a_{i+\frac{1}{2},j+\frac{1}{2}}^9 a_{i,j}^9 \\
& + 2432160368250335153225589561159951054248108071082375a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j}^8 \\
& - 8310677190402830036475253936167190697765323270636500a_{i+\frac{1}{2},j+\frac{1}{2}}^3 a_{i,j}^7 \\
& + 7356127678092772811353630086301910924963391405438894565a_{i+\frac{1}{2},j+\frac{1}{2}}^4 a_{i,j}^6 \\
& + 768997102910832792382112464333824085699163231116331694a_{i+\frac{1}{2},j+\frac{1}{2}}^5 a_{i,j}^5 \\
& + 7356127678092772811353630086301910924963391405438894565a_{i+\frac{1}{2},j+\frac{1}{2}}^6 a_{i,j}^4 \\
& - 8310677190402830036475253936167190697765323270636500a_{i+\frac{1}{2},j+\frac{1}{2}}^7 a_{i,j}^3 \\
& + 2432160368250335153225589561159951054248108071082375a_{i+\frac{1}{2},j+\frac{1}{2}}^8 a_{i,j}^2 \\
& - 282552484263505693659935098779956210575968750a_{i+\frac{1}{2},j+\frac{1}{2}}^9 a_{i,j} \\
& + 1634528636805173906473549391917644025040625a_{i+\frac{1}{2},j+\frac{1}{2}}^{10} \tau_{max}^{10} h^{18} \\
& - 229962810654720(2878123627484681181624264388736389250000a_{i,j}^{11} \\
& - 14812024869257169580296362818104402721140625a_{i+\frac{1}{2},j+\frac{1}{2}}^{10} a_{i,j}^{10} \\
& + 319879411192695470451622148491886177201553056875a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j}^9 \\
& - 8019635559103183879993033159378623968758372354886250a_{i+\frac{1}{2},j+\frac{1}{2}}^3 a_{i,j}^8 \\
& - 757642843520594158062329994358407558498610442132921650a_{i+\frac{1}{2},j+\frac{1}{2}}^4 a_{i,j}^7
\end{aligned}$$

$$\begin{aligned}
& - 1052041039947118762333138560053367878357723741168180491a_{i+\frac{1}{2},j+\frac{1}{2}}^5a_{i,j}^6 \\
& - 1052041039947118762333138560053367878357723741168180491a_{i+\frac{1}{2},j+\frac{1}{2}}^6a_{i,j}^5 \\
& - 757642843520594158062329994358407558498610442132921650a_{i+\frac{1}{2},j+\frac{1}{2}}^7a_{i,j}^4 \\
& - 8019635559103183879993033159378623968758372354886250a_{i+\frac{1}{2},j+\frac{1}{2}}^8a_{i,j}^3 \\
& + 319879411192695470451622148491886177201553056875a_{i+\frac{1}{2},j+\frac{1}{2}}^9a_{i,j}^2 \\
& - 14812024869257169580296362818104402721140625a_{i+\frac{1}{2},j+\frac{1}{2}}^{10}a_{i,j} \\
& + 2878123627484681181624264388736389250000a_{i+\frac{1}{2},j+\frac{1}{2}}^{11})\tau_{max}^{11}h^{17} \\
& + 28858078199808(294268492226031378667969966077916478281250a_{i,j}^{12} \\
& - 236161506481849546070233976106274177112500000a_{i+\frac{1}{2},j+\frac{1}{2}}^{11}a_{i,j}^{10} \\
& + 2713350071991957938204503838551143251438027831688750a_{i+\frac{1}{2},j+\frac{1}{2}}^2a_{i,j}^{10} \\
& - 23530211804947575547179775426357352139634642830010000a_{i+\frac{1}{2},j+\frac{1}{2}}^3a_{i,j}^9 \\
& + 63725509242974404237389260236210954684251780492952630125a_{i+\frac{1}{2},j+\frac{1}{2}}^4a_{i,j}^8 \\
& + 8744721099627909795672887753416351889485809029059518800a_{i+\frac{1}{2},j+\frac{1}{2}}^5a_{i,j}^7 \\
& + 209273760833645673576341575313464830902246817989905451396a_{i+\frac{1}{2},j+\frac{1}{2}}^6a_{i,j}^6 \\
& + 8744721099627909795672887753416351889485809029059518800a_{i+\frac{1}{2},j+\frac{1}{2}}^7a_{i,j}^5 \\
& + 63725509242974404237389260236210954684251780492952630125a_{i+\frac{1}{2},j+\frac{1}{2}}^8a_{i,j}^4 \\
& - 23530211804947575547179775426357352139634642830010000a_{i+\frac{1}{2},j+\frac{1}{2}}^9a_{i,j}^3 \\
& + 2713350071991957938204503838551143251438027831688750a_{i+\frac{1}{2},j+\frac{1}{2}}^{10}a_{i,j}^2 \\
& - 236161506481849546070233976106274177112500000a_{i+\frac{1}{2},j+\frac{1}{2}}^{11}a_{i,j} \\
& + 294268492226031378667969966077916478281250a_{i+\frac{1}{2},j+\frac{1}{2}}^{12})\tau_{max}^{12}h^{16} \\
& + 6132341617459200(31138870323868746449369591656183703125a_{i,j}^{13} \\
& - 11734483363541331378666623294564819685771875a_{i+\frac{1}{2},j+\frac{1}{2}}^{12}a_{i,j}^{12} \\
& - 33967905504115394998356045976621383135254595250a_{i+\frac{1}{2},j+\frac{1}{2}}^2a_{i,j}^{11} \\
& + 660614725980244601792215805789889135943431963986550a_{i+\frac{1}{2},j+\frac{1}{2}}^3a_{i,j}^{10}
\end{aligned}$$

$$\begin{aligned}
& + 656921004685650654596836595699758385868303459796998655a_{i+\frac{1}{2},j+\frac{1}{2}}^4 a_{i,j}^9 \\
& + 828238349517872069712092208307559854269603908735034523a_{i+\frac{1}{2},j+\frac{1}{2}}^5 a_{i,j}^8 \\
& + 2907846615912738424611579188259045667564783472133520852a_{i+\frac{1}{2},j+\frac{1}{2}}^6 a_{i,j}^7 \\
& + 2907846615912738424611579188259045667564783472133520852a_{i+\frac{1}{2},j+\frac{1}{2}}^7 a_{i,j}^6 \\
& + 828238349517872069712092208307559854269603908735034523a_{i+\frac{1}{2},j+\frac{1}{2}}^8 a_{i,j}^5 \\
& + 656921004685650654596836595699758385868303459796998655a_{i+\frac{1}{2},j+\frac{1}{2}}^9 a_{i,j}^4 \\
& + 660614725980244601792215805789889135943431963986550a_{i+\frac{1}{2},j+\frac{1}{2}}^{10} a_{i,j}^3 \\
& - 33967905504115394998356045976621383135254595250a_{i+\frac{1}{2},j+\frac{1}{2}}^{11} a_{i,j}^2 \\
& - 11734483363541331378666623294564819685771875a_{i+\frac{1}{2},j+\frac{1}{2}}^{12} a_{i,j} \\
& + 31138870323868746449369591656183703125a_{i+\frac{1}{2},j+\frac{1}{2}}^{13} \tau_{max}^{13} h^{15} \\
& + 2690729902080(657772539900895770133421475843368737375000a_{i,j}^{14} \\
& - 935792378625641846254166367499934604773750000a_{i+\frac{1}{2},j+\frac{1}{2}}^{13} a_{i,j}^{13} \\
& + 21300219977070328338528245279899276449119296331851250a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j}^{12} \\
& - 1055002499130830585822194606146619189707195916981872500a_{i+\frac{1}{2},j+\frac{1}{2}}^3 a_{i,j}^{11} \\
& + 8813375535711277110376291351085383021901842286499079376825a_{i+\frac{1}{2},j+\frac{1}{2}}^4 a_{i,j}^{10} \\
& + 1289864642882741809338863672653955489273400847636955815570a_{i+\frac{1}{2},j+\frac{1}{2}}^5 a_{i,j}^9 \\
& + 87444575568666764600564927595166198846135850421382242058732a_{i+\frac{1}{2},j+\frac{1}{2}}^6 a_{i,j}^8 \\
& + 6101710921402747694131746838483255439633765384124783957150a_{i+\frac{1}{2},j+\frac{1}{2}}^7 a_{i,j}^7 \\
& + 87444575568666764600564927595166198846135850421382242058732a_{i+\frac{1}{2},j+\frac{1}{2}}^8 a_{i,j}^6 \\
& + 1289864642882741809338863672653955489273400847636955815570a_{i+\frac{1}{2},j+\frac{1}{2}}^9 a_{i,j}^5 \\
& + 8813375535711277110376291351085383021901842286499079376825a_{i+\frac{1}{2},j+\frac{1}{2}}^{10} a_{i,j}^4 \\
& - 1055002499130830585822194606146619189707195916981872500a_{i+\frac{1}{2},j+\frac{1}{2}}^{11} a_{i,j}^3 \\
& + 21300219977070328338528245279899276449119296331851250a_{i+\frac{1}{2},j+\frac{1}{2}}^{12} a_{i,j}^2 \\
& - 935792378625641846254166367499934604773750000a_{i+\frac{1}{2},j+\frac{1}{2}}^{13} a_{i,j}
\end{aligned}$$

$$\begin{aligned}
& + 657772539900895770133421475843368737375000a_{i+\frac{1}{2},j+\frac{1}{2}}^{14})\tau_{max}^{14}h^{14} \\
& - 13082328783912960a_{i,j}a_{i+\frac{1}{2},j+\frac{1}{2}} \\
& (2316518916288108165584169195913809747875000a_{i,j}^{13} \\
& - 23009335833927844596786769820307182449456956250a_{i+\frac{1}{2},j+\frac{1}{2}}^{12} \\
& + 2234076898252230587581171210332767321591221886615000a_{i+\frac{1}{2},j+\frac{1}{2}}^2a_{i,j}^{11} \\
& - 2569262692700314105474432502299419529611490298684243500a_{i+\frac{1}{2},j+\frac{1}{2}}^3a_{i,j}^{10} \\
& - 2610100289178123801574956327658860133849487175164481335a_{i+\frac{1}{2},j+\frac{1}{2}}^4a_{i,j}^9 \\
& - 31210213643687675237037019874204852537483967904732637840a_{i+\frac{1}{2},j+\frac{1}{2}}^5a_{i,j}^8 \\
& - 30882306708740486599466467604869866303812925806329378094a_{i+\frac{1}{2},j+\frac{1}{2}}^6a_{i,j}^7 \\
& - 30882306708740486599466467604869866303812925806329378094a_{i+\frac{1}{2},j+\frac{1}{2}}^7a_{i,j}^6 \\
& - 31210213643687675237037019874204852537483967904732637840a_{i+\frac{1}{2},j+\frac{1}{2}}^8a_{i,j}^5 \\
& - 2610100289178123801574956327658860133849487175164481335a_{i+\frac{1}{2},j+\frac{1}{2}}^9a_{i,j}^4 \\
& - 2569262692700314105474432502299419529611490298684243500a_{i+\frac{1}{2},j+\frac{1}{2}}^{10}a_{i,j}^3 \\
& + 2234076898252230587581171210332767321591221886615000a_{i+\frac{1}{2},j+\frac{1}{2}}^{11}a_{i,j}^2 \\
& - 23009335833927844596786769820307182449456956250a_{i+\frac{1}{2},j+\frac{1}{2}}^{12}a_{i,j} \\
& + 2316518916288108165584169195913809747875000a_{i+\frac{1}{2},j+\frac{1}{2}}^{13})\tau_{max}^{15}h^{13} \\
& + 28701118955520a_{i,j}^2a_{i+\frac{1}{2},j+\frac{1}{2}}^2 \\
& (398300550488038340685997391635330992385175359210000a_{i,j}^{12} \\
& - 37180359759375534401449830283255044573383668595730000a_{i+\frac{1}{2},j+\frac{1}{2}}^{11} \\
& + 3573832402210526483256799138604423351086859453677682305375a_{i+\frac{1}{2},j+\frac{1}{2}}^2a_{i,j}^{10} \\
& + 517281761558761920542340699747782447916007900186874589300a_{i+\frac{1}{2},j+\frac{1}{2}}^3a_{i,j}^9 \\
& + 108752310025750705460582991819978419235883663099732700371042a_{i+\frac{1}{2},j+\frac{1}{2}}^4a_{i,j}^8 \\
& + 866784114356566136886481081798822559742292006303275152980a_{i+\frac{1}{2},j+\frac{1}{2}}^5a_{i,j}^7 \\
& + 334416288697967943954453597407311821589951489108850085896415a_{i+\frac{1}{2},j+\frac{1}{2}}^6a_{i,j}^6
\end{aligned}$$

$$\begin{aligned}
& + 8667841143565661368886481081798822559742292006303275152980a_{i+\frac{1}{2},j+\frac{1}{2}}^7 a_{i,j}^5 \\
& + 108752310025750705460582991819978419235883663099732700371042a_{i+\frac{1}{2},j+\frac{1}{2}}^8 a_{i,j}^4 \\
& + 517281761558761920542340699747782447916007900186874589300a_{i+\frac{1}{2},j+\frac{1}{2}}^9 a_{i,j}^3 \\
& + 3573832402210526483256799138604423351086859453677682305375a_{i+\frac{1}{2},j+\frac{1}{2}}^{10} a_{i,j}^2 \\
& - 37180359759375534401449830283255044573383668595730000a_{i+\frac{1}{2},j+\frac{1}{2}}^{11} a_{i,j} \\
& + 398300550488038340685997391635330992385175359210000a_{i+\frac{1}{2},j+\frac{1}{2}}^{12} \tau_{max}^{16} h^{12} \\
& - 2439595111219200a_{i,j}^3 a_{i+\frac{1}{2},j+\frac{1}{2}}^3 \\
& (5216829605376663971736869510023198676502432824073000a_{i,j}^{11} \\
& - 38707754973779034490714085515864523123278422485573649750a_{i+\frac{1}{2},j+\frac{1}{2}}^{10} a_{i,j}^{10} \\
& - 23487662557951040805765305902058031966287280339025143620a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j}^9 \\
& - 1546401127625692389171782922169708978725595100239117516145a_{i+\frac{1}{2},j+\frac{1}{2}}^3 a_{i,j}^8 \\
& - 1382701115905465059538867484794329337189500181444526941897a_{i+\frac{1}{2},j+\frac{1}{2}}^4 a_{i,j}^7 \\
& - 4778708299270025542183899586959224165904566721735817959751a_{i+\frac{1}{2},j+\frac{1}{2}}^5 a_{i,j}^6 \\
& - 4778708299270025542183899586959224165904566721735817959751a_{i+\frac{1}{2},j+\frac{1}{2}}^6 a_{i,j}^5 \\
& - 1382701115905465059538867484794329337189500181444526941897a_{i+\frac{1}{2},j+\frac{1}{2}}^7 a_{i,j}^4 \\
& - 1546401127625692389171782922169708978725595100239117516145a_{i+\frac{1}{2},j+\frac{1}{2}}^8 a_{i,j}^3 \\
& - 23487662557951040805765305902058031966287280339025143620a_{i+\frac{1}{2},j+\frac{1}{2}}^9 a_{i,j}^2 \\
& - 38707754973779034490714085515864523123278422485573649750a_{i+\frac{1}{2},j+\frac{1}{2}}^{10} a_{i,j} \\
& + 5216829605376663971736869510023198676502432824073000a_{i+\frac{1}{2},j+\frac{1}{2}}^{11} \tau_{max}^{17} h^{11} \\
& + 191340793036800a_{i,j}^4 a_{i+\frac{1}{2},j+\frac{1}{2}}^4 \\
& (191632468709253965344733237878070249880033496630104999900a_{i,j}^{10} \\
& + 8116371865709560306059587699709842737421761537367178840a_{i+\frac{1}{2},j+\frac{1}{2}}^9 a_{i,j}^9 \\
& + 71697229107767367741227498609551487321421952618993554381338a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j}^8 \\
& + 5675568589635422387689518297559965163646556548627006376620a_{i+\frac{1}{2},j+\frac{1}{2}}^3 a_{i,j}^7
\end{aligned}$$

$$\begin{aligned}
& + 678874571969404486593651361170436732182185553225373227273135a_{i+\frac{1}{2},j+\frac{1}{2}}^4 a_{i,j}^6 \\
& + 19415404434414986619967581740528472380543827907484817568330a_{i+\frac{1}{2},j+\frac{1}{2}}^5 a_{i,j}^5 \\
& + 678874571969404486593651361170436732182185553225373227273135a_{i+\frac{1}{2},j+\frac{1}{2}}^6 a_{i,j}^4 \\
& + 5675568589635422387689518297559965163646556548627006376620a_{i+\frac{1}{2},j+\frac{1}{2}}^7 a_{i,j}^3 \\
& + 71697229107767367741227498609551487321421952618993554381338a_{i+\frac{1}{2},j+\frac{1}{2}}^8 a_{i,j}^2 \\
& + 8116371865709560306059587699709842737421761537367178840a_{i+\frac{1}{2},j+\frac{1}{2}}^9 a_{i,j} \\
& + 191632468709253965344733237878070249880033496630104999900a_{i+\frac{1}{2},j+\frac{1}{2}}^{10} \tau_{max}^{18} h^{10} \\
& - 6505586963251200a_{i,j}^5 a_{i+\frac{1}{2},j+\frac{1}{2}}^5 \\
& (7563227775089851309390916262351036341355712749517571940a_{i,j}^9 \\
& - 1894070184847691274286147261026921980758932210292314197170a_{i+\frac{1}{2},j+\frac{1}{2}}^8 a_{i,j}^8 \\
& - 1357157414949644435765512984388378233144690375592421558228a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j}^7 \\
& - 20213820140081346715907495724816240952055856286119940483478a_{i+\frac{1}{2},j+\frac{1}{2}}^3 a_{i,j}^6 \\
& - 19060018445278558110564613913870923066811341796082681421417a_{i+\frac{1}{2},j+\frac{1}{2}}^4 a_{i,j}^5 \\
& - 19060018445278558110564613913870923066811341796082681421417a_{i+\frac{1}{2},j+\frac{1}{2}}^5 a_{i,j}^4 \\
& - 20213820140081346715907495724816240952055856286119940483478a_{i+\frac{1}{2},j+\frac{1}{2}}^6 a_{i,j}^3 \\
& - 1357157414949644435765512984388378233144690375592421558228a_{i+\frac{1}{2},j+\frac{1}{2}}^7 a_{i,j}^2 \\
& - 1894070184847691274286147261026921980758932210292314197170a_{i+\frac{1}{2},j+\frac{1}{2}}^8 a_{i,j} \\
& + 7563227775089851309390916262351036341355712749517571940a_{i+\frac{1}{2},j+\frac{1}{2}}^9 \tau_{max}^{19} h^9 \\
& + 4081936918118400a_{i,j}^6 a_{i+\frac{1}{2},j+\frac{1}{2}}^6 \\
& (1203527824719757084737294693245305249155503269095733533992a_{i,j}^8 \\
& - 32007392047825031083599698065137591100139159789058029440a_{i+\frac{1}{2},j+\frac{1}{2}}^7 a_{i,j}^7 \\
& + 141305873585123356093489161870497239419385588116437348159905a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j}^6 \\
& + 3432170557744974325621663946640372968979056882320258136600a_{i+\frac{1}{2},j+\frac{1}{2}}^3 a_{i,j}^5 \\
& + 434177154524320603888523162597427546054438914788557665283103a_{i+\frac{1}{2},j+\frac{1}{2}}^4 a_{i,j}^4
\end{aligned}$$

$$\begin{aligned}
& + 3432170557744974325621663946640372968979056882320258136600a_{i+\frac{1}{2},j+\frac{1}{2}}^5 a_{i,j}^3 \\
& + 141305873585123356093489161870497239419385588116437348159905a_{i+\frac{1}{2},j+\frac{1}{2}}^6 a_{i,j}^2 \\
& - 32007392047825031083599698065137591100139159789058029440a_{i+\frac{1}{2},j+\frac{1}{2}}^7 a_{i,j} \\
& + 1203527824719757084737294693245305249155503269095733533992a_{i+\frac{1}{2},j+\frac{1}{2}}^8 \tau_{max}^{20} h^8 \\
& - 346964638040064000a_{i,j}^7 a_{i+\frac{1}{2},j+\frac{1}{2}}^7 \\
& (17041150655545150324249372730994204430805308898697369160a_{i,j}^7 \\
& - 1448756723322317642969137006701205107298486867431840063050a_{i+\frac{1}{2},j+\frac{1}{2}}^6 a_{i,j}^6 \\
& - 1209932516900506352119038564861078883913741360316360114652a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j}^5 \\
& - 4591142009662562218661157109342537794887196412421142278041a_{i+\frac{1}{2},j+\frac{1}{2}}^3 a_{i,j}^4 \\
& - 4591142009662562218661157109342537794887196412421142278041a_{i+\frac{1}{2},j+\frac{1}{2}}^4 a_{i,j}^3 \\
& - 1209932516900506352119038564861078883913741360316360114652a_{i+\frac{1}{2},j+\frac{1}{2}}^5 a_{i,j}^2 \\
& - 1448756723322317642969137006701205107298486867431840063050a_{i+\frac{1}{2},j+\frac{1}{2}}^6 a_{i,j} \\
& + 17041150655545150324249372730994204430805308898697369160a_{i+\frac{1}{2},j+\frac{1}{2}}^7 \tau_{max}^{21} h^7 \\
& + 4947802324992000a_{i,j}^8 a_{i+\frac{1}{2},j+\frac{1}{2}}^8 \\
& (41815281802728355449000442728158149563514789136281327994372a_{i,j}^6 \\
& - 3521262674996309902623997935056318204886660844829942964360a_{i+\frac{1}{2},j+\frac{1}{2}}^5 a_{i,j}^5 \\
& + 1597146344868207686202475698182219152691284677984635969958305a_{i+\frac{1}{2},j+\frac{1}{2}}^2 a_{i,j}^4 \\
& + 3319251232414997554730059395556932700081199710353978221190a_{i+\frac{1}{2},j+\frac{1}{2}}^3 a_{i,j}^3 \\
& + 1597146344868207686202475698182219152691284677984635969958305a_{i+\frac{1}{2},j+\frac{1}{2}}^4 a_{i,j}^2 \\
& - 3521262674996309902623997935056318204886660844829942964360a_{i+\frac{1}{2},j+\frac{1}{2}}^5 a_{i,j} \\
& + 41815281802728355449000442728158149563514789136281327994372a_{i+\frac{1}{2},j+\frac{1}{2}}^6 \tau_{max}^{22} h^6 \\
& - 9820315020625503584256000a_{i,j}^9 a_{i+\frac{1}{2},j+\frac{1}{2}}^9 \\
& (22814556272812811845517404036661702594338050585004a_{i,j}^5 \\
& - 668297996560155590676317152735549526774837532959801a_{i+\frac{1}{2},j+\frac{1}{2}}^4 a_{i,j}^4
\end{aligned}$$

$$\begin{aligned}
& - 603669597926841584614775500456871785196468044576094a_{i+\frac{1}{2},j+\frac{1}{2}}^2a_{i,j}^3 \\
& - 603669597926841584614775500456871785196468044576094a_{i+\frac{1}{2},j+\frac{1}{2}}^3a_{i,j}^2 \\
& - 668297996560155590676317152735549526774837532959801a_{i+\frac{1}{2},j+\frac{1}{2}}^4a_{i,j} \\
& + 22814556272812811845517404036661702594338050585004a_{i+\frac{1}{2},j+\frac{1}{2}}^5)\tau_{max}^{23}h^5 \\
& + 158329674399744000a_{i,j}^{10}a_{i+\frac{1}{2},j+\frac{1}{2}}^{10} \\
& (17948094745355790130444601146873057041226358101845959097106a_{i,j}^4 \\
& - 1892909327941571061240358583626021836510074192394703594020a_{i+\frac{1}{2},j+\frac{1}{2}}^3a_{i,j}^3 \\
& + 223070092924324361474885042695147848967177268115932704945375a_{i+\frac{1}{2},j+\frac{1}{2}}^2a_{i,j}^2 \\
& - 1892909327941571061240358583626021836510074192394703594020a_{i+\frac{1}{2},j+\frac{1}{2}}^3a_{i,j} \\
& + 17948094745355790130444601146873057041226358101845959097106a_{i+\frac{1}{2},j+\frac{1}{2}}^4)\tau_{max}^{24}h^4 \\
& - 7935608097475154411520000a_{i,j}^{11}a_{i+\frac{1}{2},j+\frac{1}{2}}^{11} \\
& (363331815402149950652486299926872555176673859991678a_{i,j}^3 \\
& - 3220712031035671122093763244347187829310069793792281a_{i+\frac{1}{2},j+\frac{1}{2}}^2a_{i,j}^2 \\
& - 3220712031035671122093763244347187829310069793792281a_{i+\frac{1}{2},j+\frac{1}{2}}^2a_{i,j} \\
& + 363331815402149950652486299926872555176673859991678a_{i+\frac{1}{2},j+\frac{1}{2}}^3)\tau_{max}^{25}h^3 \\
& + 3303022287195888570398146560000a_{i,j}^{12}a_{i+\frac{1}{2},j+\frac{1}{2}}^{12} \\
& (3870943533213155837623585590140549971658824411a_{i,j}^2 \\
& - 424244558726974861595284900718368831038583530a_{i+\frac{1}{2},j+\frac{1}{2}}^2a_{i,j} \\
& + 3870943533213155837623585590140549971658824411a_{i+\frac{1}{2},j+\frac{1}{2}}^2)\tau_{max}^{26}h^2 \\
& - 11851203731627063757893102598331140802206341446727 \\
& 910665292307164100034560000a_{i,j}^{13}a_{i+\frac{1}{2},j+\frac{1}{2}}^{13}(a_{i,j} + a_{i+\frac{1}{2},j+\frac{1}{2}})\tau_{max}^{27}h \\
& + 51066079539017860205584995868899497158418215749145 \\
& 47542078723480595988480000a_{i,j}^{14}a_{i+\frac{1}{2},j+\frac{1}{2}}^{14}\tau_{max}^{28})) \\
& /(20575423897600(10135125h^8 + 4380486227076960a_{i,j}^2\tau_{max}^2h^6
\end{aligned}$$

$$\begin{aligned}
& + 294877585914354432a_{i,j}^4\tau_{max}^4h^4 \\
& + 2587519954603806720a_{i,j}^6\tau_{max}^6h^2 + 1025393073507860480a_{i,j}^8 \\
& \tau_{max}^8)^2(10135125h^8 + 4380486227076960a_{i+\frac{1}{2},j+\frac{1}{2}}^2\tau_{max}^2h^6 \\
& + 294877585914354432a_{i+\frac{1}{2},j+\frac{1}{2}}^4\tau_{max}^4h^4 \\
& + 2587519954603806720a_{i+\frac{1}{2},j+\frac{1}{2}}^6\tau_{max}^6h^2 \\
& + 1025393073507860480a_{i+\frac{1}{2},j+\frac{1}{2}}^8\tau_{max}^8)^2) = O(h^{22})
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
& \int_{x_i}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_j} (v_I(x, y) - x^9 - u_I(x - \frac{h}{2}, y + \frac{h}{2}) + (x - \frac{h}{2})^9)^2 dx dy = O(h^{22}), \\
& \int_{x_{i-\frac{1}{2}}}^{x_i} \int_{y_j}^{y_{j+\frac{1}{2}}} (v_I(x, y) - x^9 - u_I(x + \frac{h}{2}, y - \frac{h}{2}) + (x + \frac{h}{2})^9)^2 dx dy = O(h^{22}), \\
& \int_{x_{i-\frac{1}{2}}}^{x_i} \int_{y_j}^{y_{j+\frac{1}{2}}} (v_I(x, y) - x^9 - u_I(x + \frac{h}{2}, y + \frac{h}{2}) + (x + \frac{h}{2})^9)^2 dx dy = O(h^{22}).
\end{aligned}$$