马尔可夫决策过程

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Used Materials

Disclaimer: 本课件大量采用了 Rich Sutton's RL class, David Silver's Deep RL tutorial 和其他网络课程课件，也采用了 GitHub 中开源代码，以及部分网络博客内容

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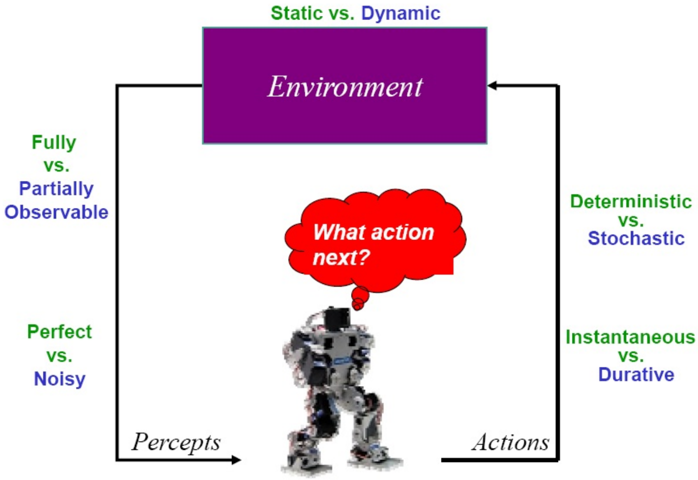
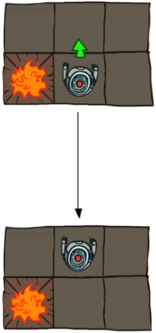
智能体决策

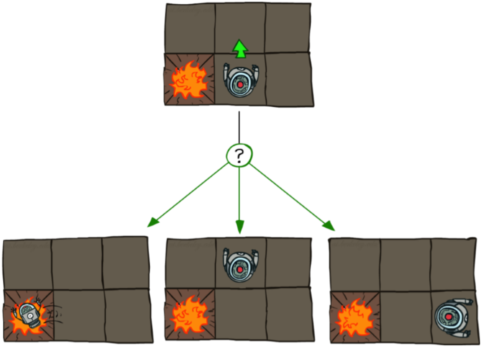

Static vs. Dynamic
Environment
Fully vs. Partially Observable
Perfect vs. Noisy
What action next?
Deterministic vs. Stochastic
Instantaneous vs. Durative
Percepts
Actions

环境的不确定性

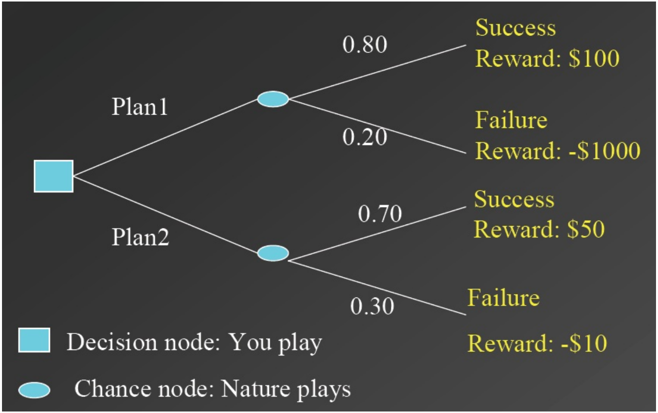
确定性的环境



不确定性的环境



决策树



- Plan1 的期望效用：
  $EU(Plan1) = 100 \times 0.80 - 1000 \times 0.20 = -120$
- Plan2 的期望效用： $EU(Plan2) = 50 \times 0.70 - 10 \times 0.30 = 32$
- 最大期望效用（MEU）：选择行动 A_i 使得期望效用最大

马尔可夫性质

- 当一个随机过程在给定现在状态及所有过去状态情况下，其未来状态的条件概率分布仅依赖于当前状态
- 在给定现在状态时，随机过程与过去状态（历史）是条件独立的

定义

A state s_t is *Markov* if and only if

$$P[s_{t+1} \mid s_t] = P[s_{t+1} \mid s_1, \ldots, s_t]$$

- 状态包含了关于历史的所有信息
- 一旦知道当前状态，过去的历史可以完全抛弃
- 当前状态足够用来预计未来
- 这一性质能够简化问题的表示和求解，但提高了对信息全局性和充分性的要求

MDPs 概述

- 马尔可夫决策过程（Markov Decision Processes, MDPs）
- 人工智能、决策分析、运筹学、控制论和经济学共同关注
- 大量实际问题可以用 MDPs 表达、研究
- MDPs 借助于概率作为刻画不确定性的基本手段
- MDPs 类型：
  - 完全可观察的 MDPs (Full Observable MDPs): FOMDPs
  - 完全不可观察的 MDPs: NOMDPs
  - 部分可观察的 MDPs (Partial Observable MDPs): POMDPs

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Markov Property

"The future is independent of the past given the present"

Definition

A state S_t is *Markov* if and only if

$$\mathbb{P}[S_{t+1} \mid S_t] = \mathbb{P}[S_{t+1} \mid S_1, ..., S_t]$$

- The state captures all relevant information from the history
- Once the state is known, the history may be thrown away
- i.e. The state is a sufficient statistic of the future

State Transition Matrix

For a Markov state s and successor state s', the *state transition probability* is defined by

$$\mathcal{P}_{ss'} = \mathbb{P}\left[S_{t+1} = s' \mid S_t = s\right]$$

State transition matrix $\mathcal{P}$ defines transition probabilities from all states s to all successor states s',

$$\mathcal{P} \quad = \textit{from} \quad \begin{array}{c} \textit{to} \\ \begin{bmatrix} \mathcal{P}_{11} & \dots & \mathcal{P}_{1n} \\ \vdots & & \\ \mathcal{P}_{n1} & \dots & \mathcal{P}_{nn} \end{bmatrix} \end{array}$$

where each row of the matrix sums to 1.

Markov Process

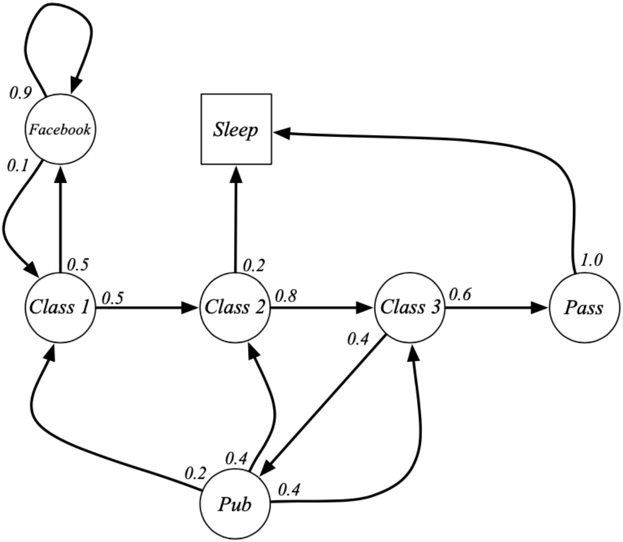
A Markov process is a memoryless random process, i.e. a sequence of random states $S_1, S_2, ...$ with the Markov property.

Definition

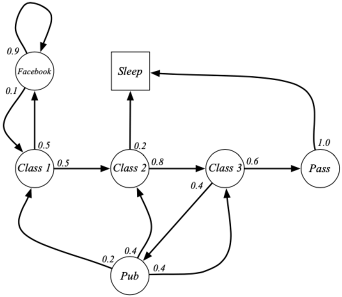
A *Markov Process* (or *Markov Chain*) is a tuple $\langle \mathcal{S}, \mathcal{P} \rangle$

- $\mathcal{S}$ is a (finite) set of states
- $\mathcal{P}$ is a state transition probability matrix,
  $\mathcal{P}_{ss'} = \mathbb{P}\left[S_{t+1} = s' \mid S_t = s\right]$

Example: Student Markov Chain



Example: Student Markov Chain Episodes

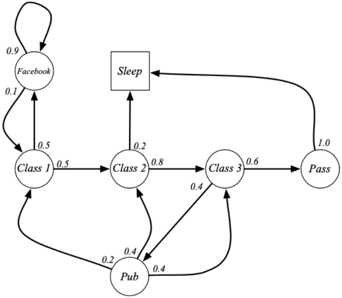


Sample episodes for Student Markov Chain starting from $S_1 = \mathrm{C1}$

$$S_1, S_2, ..., S_T$$

- C1 C2 C3 Pass Sleep
- C1 FB FB C1 C2 Sleep
- C1 C2 C3 Pub C2 C3 Pass Sleep
- C1 FB FB C1 C2 C3 Pub C1 FB FB FB C1 C2 C3 Pub C2 Sleep

Example: Student Markov Chain Transition Matrix



$$\mathcal{P} = \begin{array}{cc} & \begin{array}{ccccccc} \mathit{C1} & \mathit{C2} & \mathit{C3} & \mathit{Pass} & \mathit{Pub} & \mathit{FB} & \mathit{Sleep} \end{array} \\ \begin{array}{c} \mathit{C1} \\ \mathit{C2} \\ \mathit{C3} \\ \mathit{Pass} \\ \mathit{Pub} \\ \mathit{FB} \\ \mathit{Sleep} \end{array} & \begin{bmatrix} & 0.5 & & & & 0.5 & \\ & & 0.8 & & & & 0.2 \\ & & & 0.6 & 0.4 & & \\ & & & & & & 1.0 \\ 0.2 & 0.4 & 0.4 & & & & \\ 0.1 & & & & & 0.9 & \\ & & & & & & 1 \end{bmatrix} \end{array}$$

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Markov Reward Process

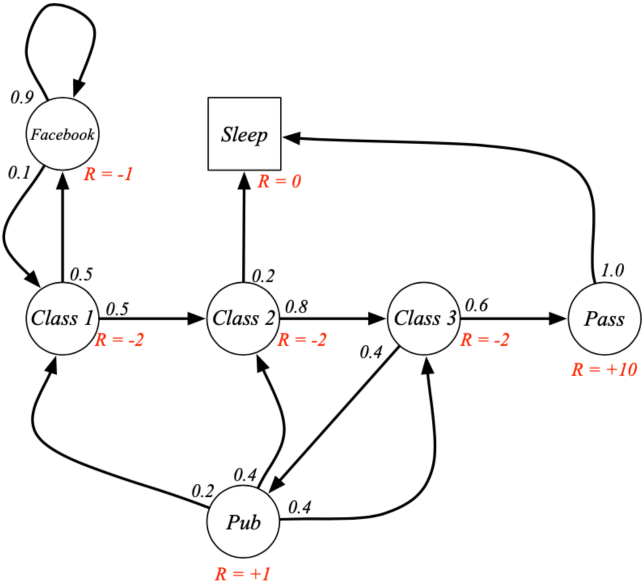
A Markov reward process is a Markov chain with values.

Definition

A *Markov Reward Process* is a tuple $\langle \mathcal{S}, \mathcal{P}, \mathcal{R}, \gamma \rangle$

- $\mathcal{S}$ is a finite set of states
- $\mathcal{P}$ is a state transition probability matrix,
  $\mathcal{P}_{ss'} = \mathbb{P}\left[S_{t+1} = s' \mid S_t = s\right]$
- $\mathcal{R}$ is a reward function, $\mathcal{R}_s = \mathbb{E}\left[R_{t+1} \mid S_t = s\right]$
- γ is a discount factor, $\gamma \in [0, 1]$

Example: Student MRP



Return

Definition

The *return* G_t is the total discounted reward from time-step t.

$$G_t = R_{t+1} + \gamma R_{t+2} + ... = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1}$$

- The *discount* $\gamma \in [0, 1]$ is the present value of future rewards
- The value of receiving reward R after $k+1$ time-steps is $\gamma^k R$.
- This values immediate reward above delayed reward.
  - γ close to 0 leads to "myopic" evaluation
  - γ close to 1 leads to "far-sighted" evaluation

Why discount?

Most Markov reward and decision processes are discounted. Why?

- Mathematically convenient to discount rewards
- Avoids infinite returns in cyclic Markov processes
- Uncertainty about the future may not be fully represented
- If the reward is financial, immediate rewards may earn more interest than delayed rewards
- Animal/human behaviour shows preference for immediate reward
- It is sometimes possible to use *undiscounted* Markov reward processes (i.e. $\gamma = 1$), e.g. if all sequences terminate.

Value Function

The value function $v(s)$ gives the long-term value of state s

Definition

The *state value function* $v(s)$ of an MRP is the expected return starting from state s

$$v(s) = \mathbb{E}\left[G_t \mid S_t = s\right]$$

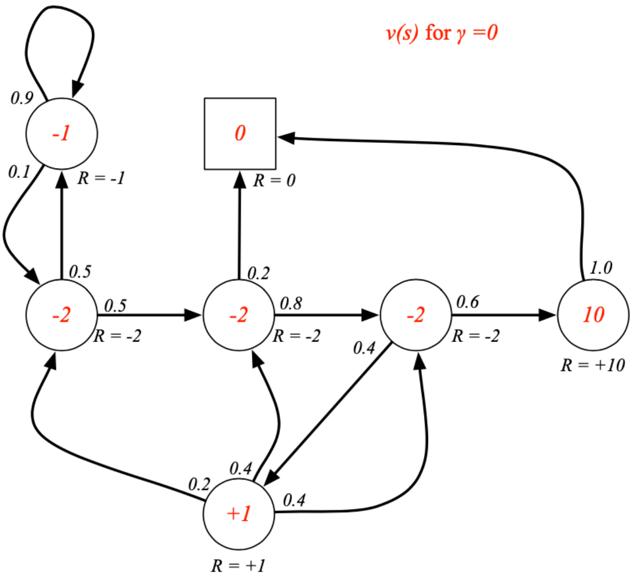
Example: Student MRP Returns

Sample returns for Student MRP:
Starting from $S_1 = C1$ with $\gamma = \frac{1}{2}$

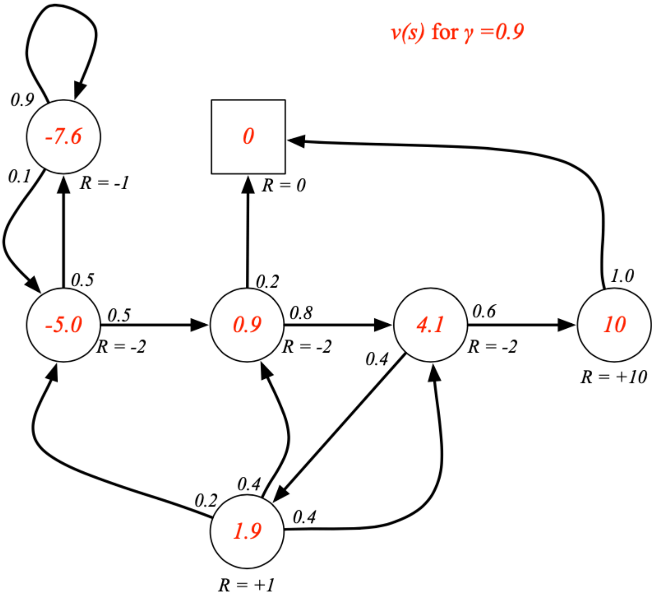
$$G_1 = R_2 + \gamma R_3 + ... + \gamma^{T-2} R_T$$

C1 C2 C3 Pass Sleep	$v_1 = -2 - 2 * \frac{1}{2} - 2 * \frac{1}{4} + 10 * \frac{1}{8}$	$=$	-2.25
C1 FB FB C1 C2 Sleep	$v_1 = -2 - 1 * \frac{1}{2} - 1 * \frac{1}{4} - 2 * \frac{1}{8} - 2 * \frac{1}{16}$	$=$	-3.125
C1 C2 C3 Pub C2 C3 Pass Sleep	$v_1 = -2 - 2 * \frac{1}{2} - 2 * \frac{1}{4} + 1 * \frac{1}{8} - 2 * \frac{1}{16} \dots$	$=$	-3.41
C1 FB FB C1 C2 C3 Pub C1 ...	$v_1 = -2 - 1 * \frac{1}{2} - 1 * \frac{1}{4} - 2 * \frac{1}{8} - 2 * \frac{1}{16} \dots$		
FB FB FB C1 C2 C3 Pub C2 Sleep		$=$	-3.20

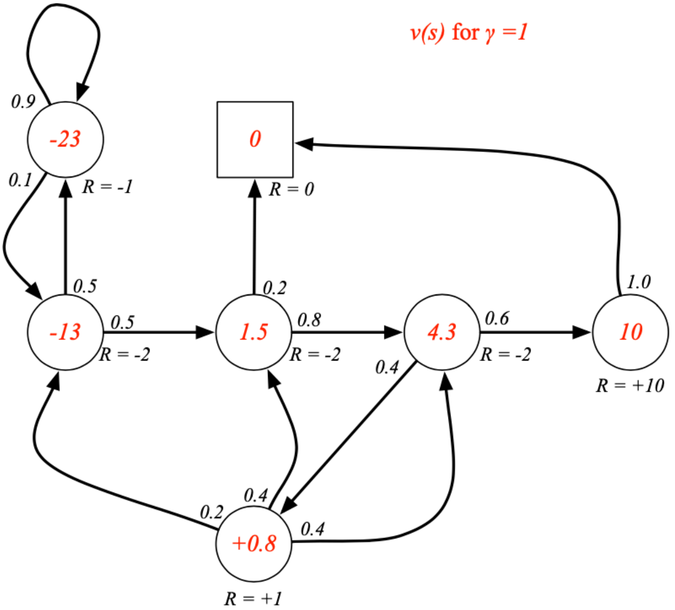
Example: State-Value Function for Student MRP (1)



Example: State-Value Function for Student MRP (2)



Example: State-Value Function for Student MRP (3)



Bellman Equation for MRPs (1)

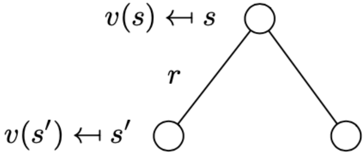
The value function can be decomposed into two parts:

- immediate reward R_{t+1}
- discounted value of successor state $\gamma v(S_{t+1})$

$$
\begin{aligned}
v(s) &= \mathbb{E}\left[G_t \mid S_t = s\right] \\
&= \mathbb{E}\left[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \ldots \mid S_t = s\right] \\
&= \mathbb{E}\left[R_{t+1} + \gamma \left(R_{t+2} + \gamma R_{t+3} + \ldots\right) \mid S_t = s\right] \\
&= \mathbb{E}\left[R_{t+1} + \gamma G_{t+1} \mid S_t = s\right] \\
&= \mathbb{E}\left[R_{t+1} + \gamma v(S_{t+1}) \mid S_t = s\right]
\end{aligned}
$$

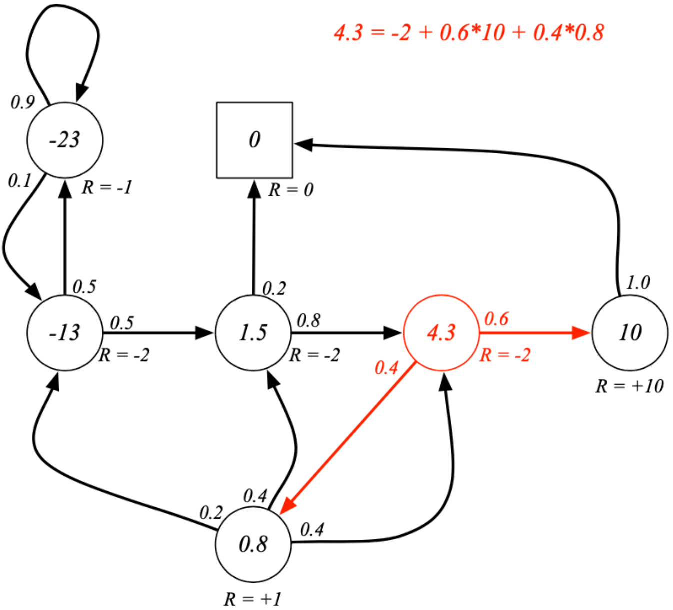
Bellman Equation for MRPs (2)

$$v(s) = \mathbb{E}\left[R_{t+1} + \gamma v(S_{t+1}) \mid S_t = s\right]$$



$$v(s) = \mathcal{R}_s + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'} v(s')$$

Example: Bellman Equation for Student MRP



Bellman Equation in Matrix Form

The Bellman equation can be expressed concisely using matrices,

$$v = \mathcal{R} + \gamma \mathcal{P} v$$

where v is a column vector with one entry per state

$$\begin{bmatrix} v(1) \\ \vdots \\ v(n) \end{bmatrix} = \begin{bmatrix} \mathcal{R}_1 \\ \vdots \\ \mathcal{R}_n \end{bmatrix} + \gamma \begin{bmatrix} \mathcal{P}_{11} & \dots & \mathcal{P}_{1n} \\ \vdots & & \\ \mathcal{P}_{11} & \dots & \mathcal{P}_{nn} \end{bmatrix} \begin{bmatrix} v(1) \\ \vdots \\ v(n) \end{bmatrix}$$

Solving the Bellman Equation

- The Bellman equation is a linear equation
- It can be solved directly:

$$
\begin{aligned}
v &= \mathcal{R} + \gamma \mathcal{P} v \\
(I - \gamma \mathcal{P})\, v &= \mathcal{R} \\
v &= (I - \gamma \mathcal{P})^{-1} \mathcal{R}
\end{aligned}
$$

- Computational complexity is $O(n^3)$ for n states
- Direct solution only possible for small MRPs
- There are many iterative methods for large MRPs, e.g.
  - Dynamic programming
  - Monte-Carlo evaluation
  - Temporal-Difference learning

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Markov Decision Process

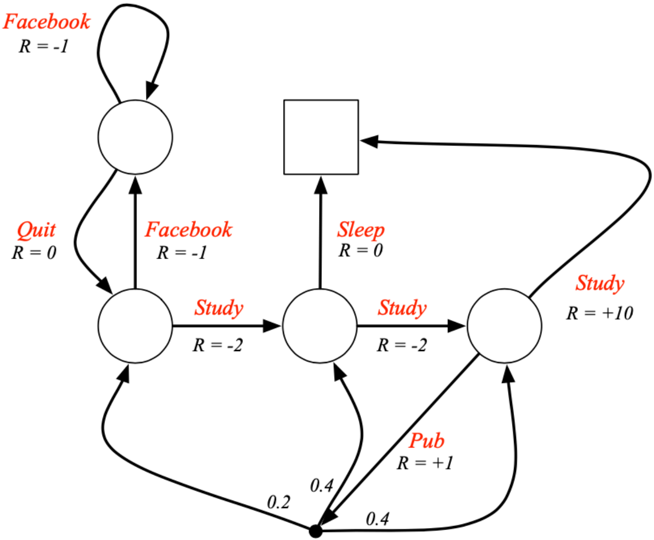
A Markov decision process (MDP) is a Markov reward process with decisions. It is an *environment* in which all states are Markov.

Definition

A *Markov Decision Process* is a tuple $\langle \mathcal{S}, \mathcal{A}, \mathcal{P}, \mathcal{R}, \gamma \rangle$

- $\mathcal{S}$ is a finite set of states
- $\mathcal{A}$ is a finite set of actions
- $\mathcal{P}$ is a state transition probability matrix,
  $\mathcal{P}_{ss'}^{a} = \mathbb{P}[S_{t+1} = s' \mid S_t = s, A_t = a]$
- $\mathcal{R}$ is a reward function, $\mathcal{R}_s^a = \mathbb{E}[R_{t+1} \mid S_t = s, A_t = a]$
- γ is a discount factor $\gamma \in [0, 1]$.

Example: Student MDP



Policies (1)

Definition

A *policy* π is a distribution over actions given states,

$$\pi(a|s) = \mathbb{P}\left[A_t = a \mid S_t = s\right]$$

- A policy fully defines the behaviour of an agent
- MDP policies depend on the current state (not the history)
- i.e. Policies are *stationary* (time-independent), $A_t \sim \pi(\cdot|S_t), \forall t > 0$

Policies (2)

- Given an MDP $\mathcal{M} = \langle \mathcal{S}, \mathcal{A}, \mathcal{P}, \mathcal{R}, \gamma \rangle$ and a policy π
- The state sequence $S_1, S_2, \ldots$ is a Markov process $\langle \mathcal{S}, \mathcal{P}^\pi \rangle$
- The state and reward sequence $S_1, R_2, S_2, \ldots$ is a Markov reward process $\langle \mathcal{S}, \mathcal{P}^\pi, \mathcal{R}^\pi, \gamma \rangle$
- where

$$\mathcal{P}^\pi_{s,s'} = \sum_{a \in \mathcal{A}} \pi(a|s) \mathcal{P}^a_{ss'}$$

$$\mathcal{R}^\pi_s = \sum_{a \in \mathcal{A}} \pi(a|s) \mathcal{R}^a_s$$

Value Function

Definition

The *state-value function* $v_\pi(s)$ of an MDP is the expected return starting from state s, and then following policy π

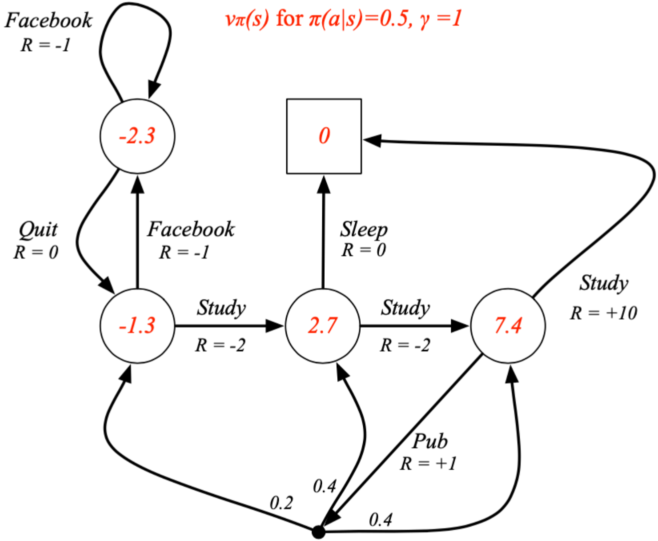
$$v_\pi(s) = \mathbb{E}_\pi \left[G_t \mid S_t = s \right]$$

Definition

The *action-value function* $q_\pi(s, a)$ is the expected return starting from state s, taking action a, and then following policy π

$$q_\pi(s, a) = \mathbb{E}_\pi \left[G_t \mid S_t = s, A_t = a \right]$$

Example: State-Value Function for Student MDP



Bellman Expectation Equation

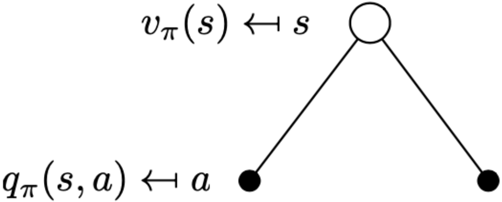
The state-value function can again be decomposed into immediate reward plus discounted value of successor state,

$$v_\pi(s) = \mathbb{E}_\pi \left[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s\right]$$

The action-value function can similarly be decomposed,

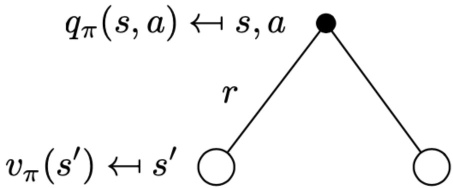
$$q_\pi(s, a) = \mathbb{E}_\pi \left[R_{t+1} + \gamma q_\pi(S_{t+1}, A_{t+1}) \mid S_t = s, A_t = a\right]$$

Bellman Expectation Equation for v_π



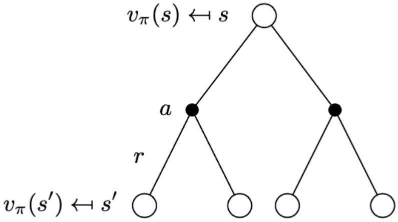
$$v_\pi(s) = \sum_{a \in \mathcal{A}} \pi(a|s) q_\pi(s,a)$$

Bellman Expectation Equation for q_π



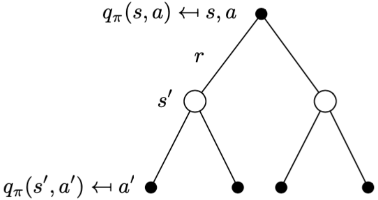
$$q_\pi(s,a) = \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_\pi(s')$$

Bellman Expectation Equation for v_π (2)



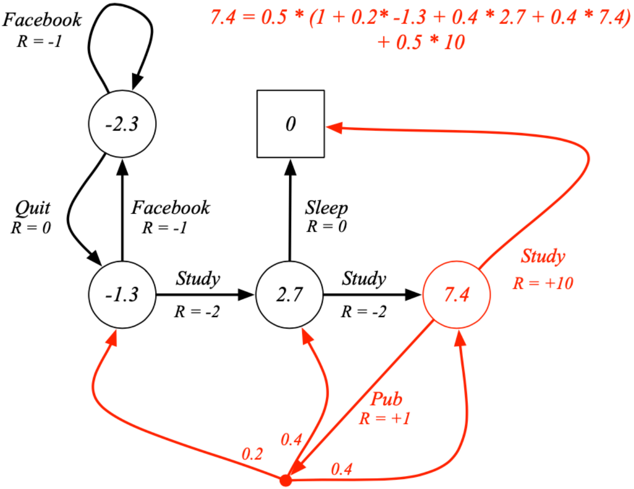
$$v_\pi(s) = \sum_{a \in \mathcal{A}} \pi(a|s) \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_\pi(s') \right)$$

Bellman Expectation Equation for q_π (2)



$$q_\pi(s,a) = \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a \sum_{a' \in \mathcal{A}} \pi(a'|s') q_\pi(s',a')$$

Example: Bellman Expectation Equation in Student MDP



Bellman Expectation Equation (Matrix Form)

The Bellman expectation equation can be expressed concisely using the induced MRP,

$$v_\pi = \mathcal{R}^\pi + \gamma \mathcal{P}^\pi v_\pi$$

with direct solution

$$v_\pi = (I - \gamma \mathcal{P}^\pi)^{-1} \mathcal{R}^\pi$$

Optimal Value Function

Definition

The *optimal state-value function* $v_*(s)$ is the maximum value function over all policies

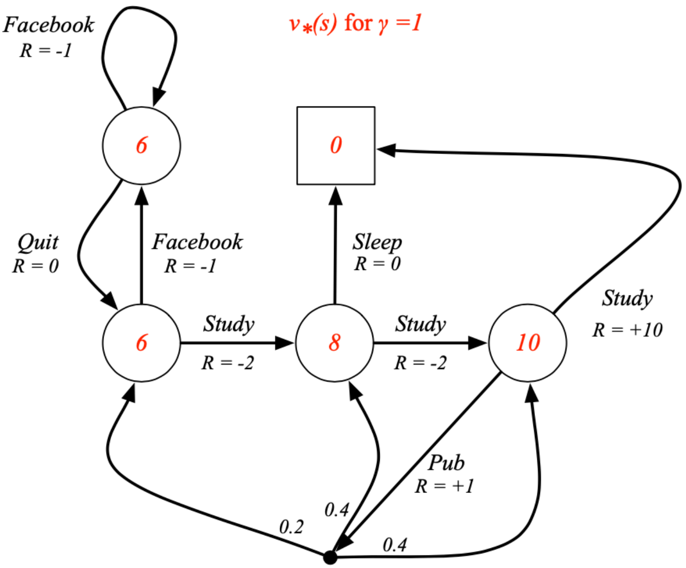
$$v_*(s) = \max_\pi v_\pi(s)$$

The *optimal action-value function* $q_*(s, a)$ is the maximum action-value function over all policies

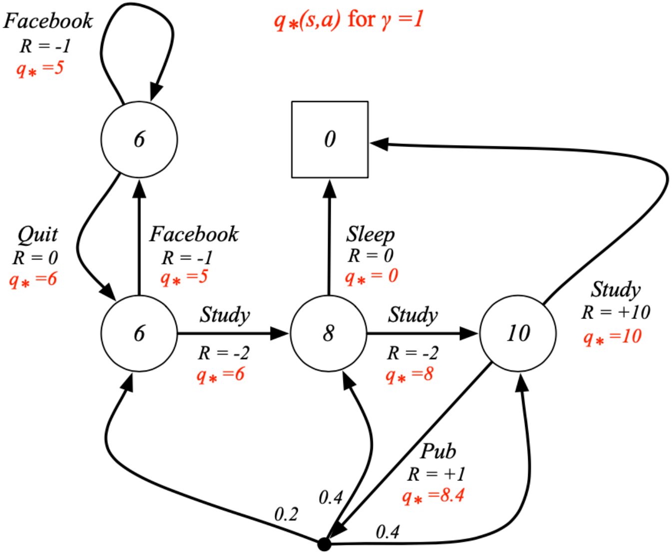
$$q_*(s, a) = \max_\pi q_\pi(s, a)$$

- The optimal value function specifies the best possible performance in the MDP.
- An MDP is "solved" when we know the optimal value fn.

Example: Optimal Value Function for Student MDP



Example: Optimal Action-Value Function for Student MDP



Optimal Policy

Define a partial ordering over policies

$$\pi \geq \pi' \text{ if } v_\pi(s) \geq v_{\pi'}(s), \forall s$$

Theorem

For any Markov Decision Process

- *There exists an optimal policy* π_* *that is better than or equal to all other policies,* $\pi_* \geq \pi, \forall \pi$
- *All optimal policies achieve the optimal value function,* $v_{\pi_*}(s) = v_*(s)$
- *All optimal policies achieve the optimal action-value function,* $q_{\pi_*}(s, a) = q_*(s, a)$

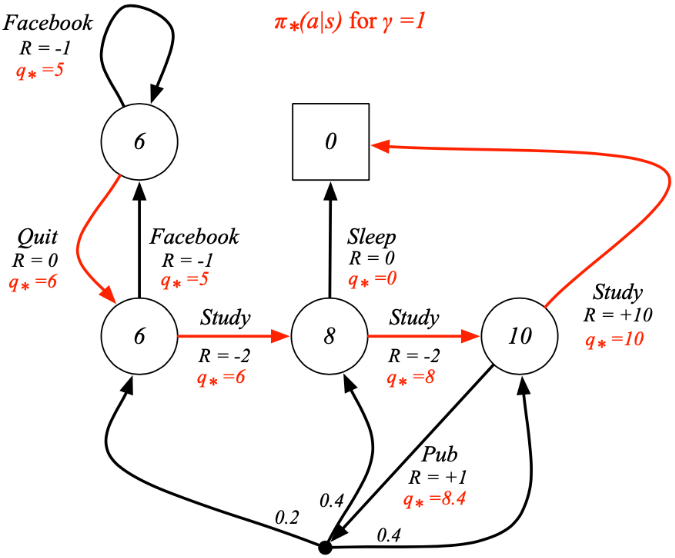
Finding an Optimal Policy

An optimal policy can be found by maximising over $q_*(s, a)$,

$$\pi_*(a|s) = \begin{cases} 1 & \text{if } a = \underset{a \in \mathcal{A}}{\operatorname{argmax}}\ q_*(s, a) \\ 0 & \textit{otherwise} \end{cases}$$

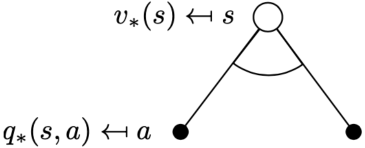
- There is always a deterministic optimal policy for any MDP
- If we know $q_*(s, a)$, we immediately have the optimal policy

Example: Optimal Policy for Student MDP



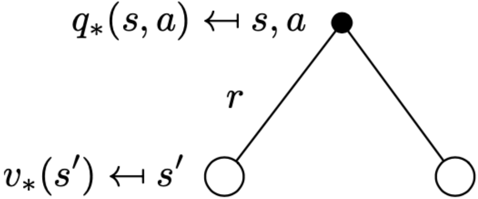
Bellman Optimality Equation for v_*

The optimal value functions are recursively related by the Bellman optimality equations:



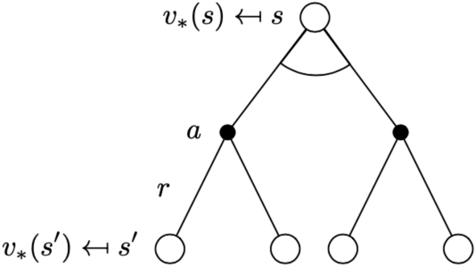
$$v_*(s) = \max_a q_*(s,a)$$

Bellman Optimality Equation for q_*



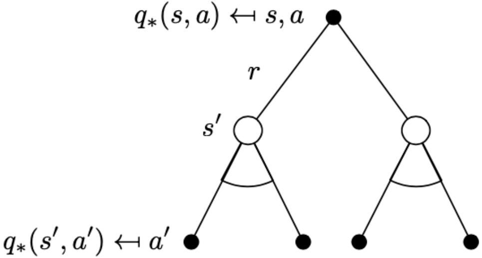
$$q_*(s, a) = \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_*(s')$$

Bellman Optimality Equation for v_* (2)



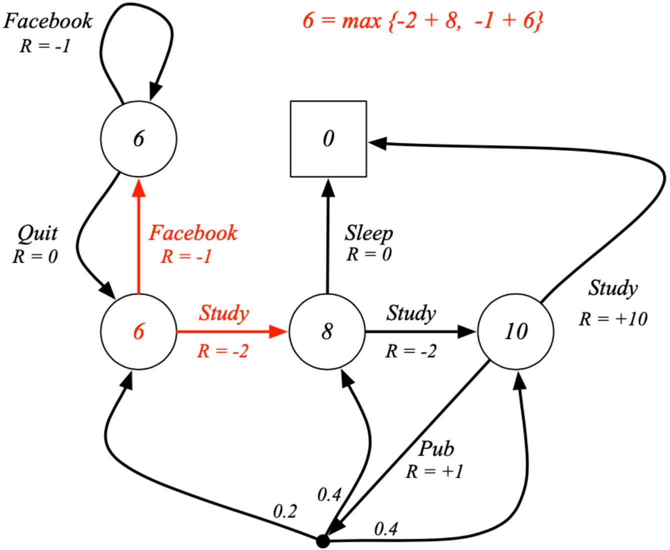
$$v_*(s) = \max_a \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_*(s')$$

Bellman Optimality Equation for q_* (2)



$$q_*(s,a) = \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a \max_{a'} q_*(s',a')$$

Example: Bellman Optimality Equation in Student MDP



Solving the Bellman Optimality Equation

- Bellman Optimality Equation is non-linear
- No closed form solution (in general)
- Many iterative solution methods
  - Value Iteration
  - Policy Iteration
  - Q-learning
  - Sarsa

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POMDPs

A Partially Observable Markov Decision Process is an MDP with hidden states. It is a hidden Markov model with actions.

Definition

A *POMDP* is a tuple $\langle \mathcal{S}, \mathcal{A}, \mathcal{O}, \mathcal{P}, \mathcal{R}, \mathcal{Z}, \gamma \rangle$

- $\mathcal{S}$ is a finite set of states
- $\mathcal{A}$ is a finite set of actions
- $\mathcal{O}$ is a finite set of observations
- $\mathcal{P}$ is a state transition probability matrix, $\mathcal{P}^a_{ss'} = \mathbb{P}[S_{t+1} = s' \mid S_t = s, A_t = a]$
- $\mathcal{R}$ is a reward function, $\mathcal{R}^a_s = \mathbb{E}[R_{t+1} \mid S_t = s, A_t = a]$
- $\mathcal{Z}$ is an observation function, $\mathcal{Z}^a_{s'o} = \mathbb{P}[O_{t+1} = o \mid S_{t+1} = s', A_t = a]$
- γ is a discount factor $\gamma \in [0, 1]$.

Belief States

Definition

A *history* H_t is a sequence of actions, observations and rewards,

$$H_t = A_0, O_1, R_1, ..., A_{t-1}, O_t, R_t$$

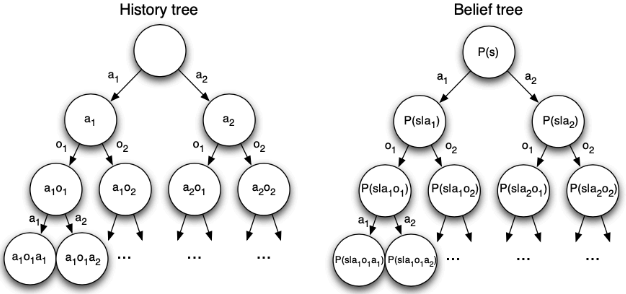
Definition

A *belief state* $b(h)$ is a probability distribution over states, conditioned on the history h

$$b(h) = (\mathbb{P}\left[S_t = s^1 \mid H_t = h\right], ..., \mathbb{P}\left[S_t = s^n \mid H_t = h\right])$$

Reductions of POMDPs

- The history H_t satisfies the Markov property
- The belief state $b(H_t)$ satisfies the Markov property



- A POMDP can be reduced to an (infinite) history tree
- A POMDP can be reduced to an (infinite) belief state tree

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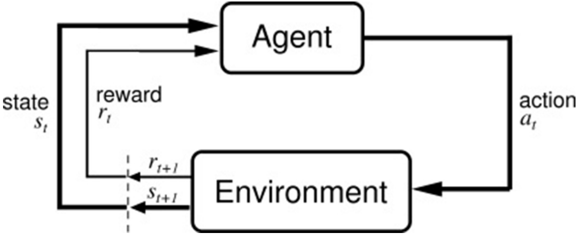
MDP 基本框架



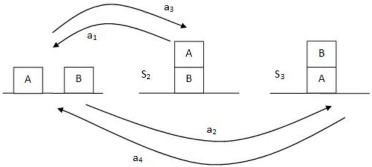
- MDPs 立足于 agent 与环境的直接交互
- 只考虑离散时间，假设 agent 与环境的交互过程可分解为一系列“阶段”，每个阶段由“感知–决策–行动”构成
- 已知环境模型

MDP 模型

- MDP 模型是一个四元组 (S, A, T, R)，其中
  - S 是一个有限集，其中每个元素 $s \in S$ 代表一个状态
  - A 是一个有限集，其中每个元素 $a \in A$ 代表一个行动
  - $T: S \times A \to \Pi(S)$ 称为状态转移函数，将每一对“状态　行动”映射为 S 上的一个概率分布，用记号 $T(s, a, s')$ 表示在状态 s 上执行 a 达到 s' 的概率
  - $R: S \times A \to \mathcal{R}$ 称为回报（reword）函数，$R(s, a)$ 表示在 s 上执行行动 a 所得到的即时回报（直接回报）
- 如何规定一个 MDP 模型是一个具体应用问题，不是 MDPs 本身研究的内容
- MDPs 研究的主题是，给定一个 MDP 模型，如何求最优策略，即期望回报最大的策略

积木世界

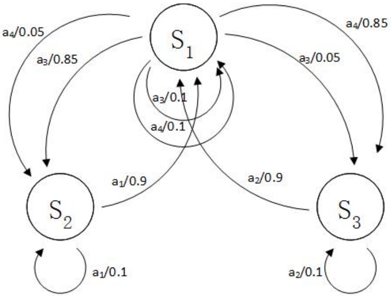
- 积木世界只有两块积木 A 和 B
- $S = \{s_1, s_2, s_3\}$
- $A = \{a_1, a_2, a_3, a_4\}$
    - $a_1 = move(A, B, F)$
    - $a_2 = move(B, A, F)$
    - $a_3 = move(A, F, B)$
    - $a_4 = move(B, F, A)$



积木世界 (con't)

- 规定 T

	a_1			a_2			a_3			a_4		
	s_1	s_2	s_3	s_1	s_2	s_3	s_1	s_2	s_3	s_1	s_2	s_3
s_1	1	0	0	1	0	0	0.1	0.85	0.05	0.1	0.05	0.85
s_2	0.9	0.1	0	0	1	0	0	1	0	0	1	0
s_3	0	0	1	0.9	0	0.1	0	0	1	0	0	1



积木世界 (con't)

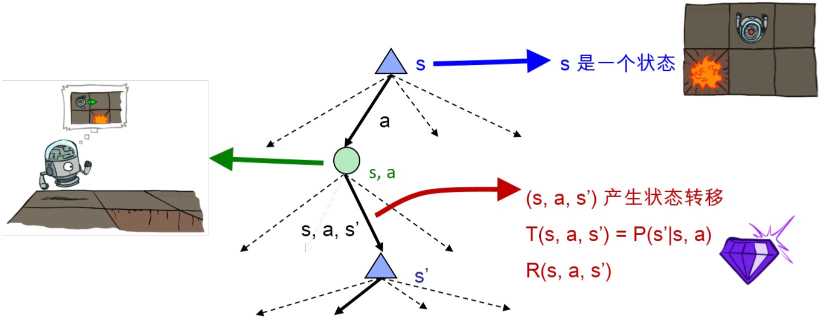
- 规定 R: “达到一个状态获得的回报”和“执行一个行动的代价”的综合
- 假设 s_2 是“最有益的”状态，s_3 是“最有害的”，s_1 是中性的，每个行动都有代价，而且“放上去”比“拿下来”代价大，R 的值仅仅反映“档次”

	a_1	a_2	a_3	a_4
s_1	-1	-1	1	-2
s_2	-2	-1	-1	-1
s_3	-1	0	-1	-1

最优策略（Optional Policy）

- “长期”分两类：
  - 无限阶段问题（例如，网络维护）
  - 有限阶段问题
- 策略分为“平稳的（stating）”与“非平稳的（non-stating）”
  - 平稳策略可表示为 $\pi : S \to A$
  - 非平稳策略 $\pi = \{\pi_t, \ldots, \pi_1\}$，每个 π_i 是一个 S 到 A 的映射。最后一次决策用 π_1，产生的行动是 $\pi_1(s)$，第一次用 π_t

MDP 的求解



问题：给定一个MDP模型，如何求解出最优策略？

值函数

- 任给策略 π，计算预期（长期）回报，用值函数 (value function) 度量预期回报
- 令 $V_{\pi,t}(s)$ 表示从 s 开始，执行策略 π 共 t 个阶段所获得的预期累计回报
  - 当 $t=1$ 时
  $$V_{\pi,1}(s) = R(s, \pi_1(s))$$
  - 当 $t=2$ 时
  $$V_{\pi,2}(s) = R(s, \pi_2(s)) + \Sigma_{s'} T(s, \pi_2(s), s') \cdot R(s', \pi_1(s'))$$
  - 多步情况
  $$V_{\pi,t}(s) = R(s, \pi_t(s)) + \Sigma_{s'} T(s, \pi_t(s), s') \cdot V_{\pi,t-1}(s')$$

值函数（con't）

- 有时采用“折扣准则”：折扣因子 $\gamma\ (<1)$ （在无限阶段中更有必要用，保证收敛）

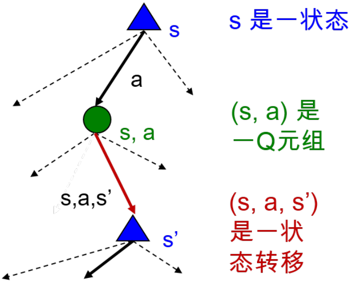
$$V_{\pi,t}(s) = R(s,\pi_t(s)) + \gamma\Sigma_{s'} T(s,\pi_t(s),s') \cdot V_{\pi,t-1}(s')$$

- 给定一个 MDP 模型和一个策略 π，由此可得 $V_{\pi,t}$，所有 π 中，$V_{\pi,t}$ 最大的 π 称为最优策略，记为 π^*，对应的值函数记为 V_t^*，反过来，如果知道了 V_t^*，也可得到 π^*

$$\pi_t^*(s) = argmax_a \left[R(s,a) + \gamma\Sigma_{s'} T(s,a,s') V_{t-1}^*(s')\right]$$

最优依据

- 定义1：状态s的**最优值函数** $V^*(s)$为，以s为初始状态，执行最优策略获得的期望收益值
- 定义2：元组(s, a)的**最优Q函数** $Q^*(s, a)$为，以s为初始状态，执行动作a后，按最优策略执行获得的期望收益值
- 定义3：**最优策略**π^*为执行过程中最大化期望收益值的策略



贝尔曼等式

- 利用贝尔曼等式，可以用递归的方式来计算状态的**期望收益值**：

$$V^*(s) = \max_a Q^*(s,a)$$

$$Q^*(s,a) = \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^*(s')\right]$$

Richard Bellman
(1920 - 1984)

$$V^*(s) = \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^*(s')\right]$$

最优值函数的计算

值迭代算法

- 值迭代（value iteration）算法求 V_t^* 序列，借助于辅助 $Q_t^a(s)$，其直观含义为：在 s 执行 a，然后执行 $t-1$ 步的最优策略所产生的预期回报

对所有 $s \in S$, $V_0(s) := 0$; $t := 0$;

loop

$t := t + 1$;

loop 对所有 $s \in S$

loop 对所有 $a \in A$

$$Q_t^a(s) := R(s,a) + \gamma \Sigma_{s'} T(s,a,s') V_{t-1}(s')$$

end loop

$V_t(s) := max_a\, Q_t^a(s)$ (Bellman equation)

end loop

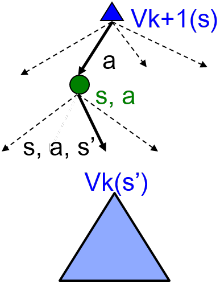
until $|V_t(s) - V_{t-1}(s)| < \epsilon$ 对所有 $s \in S$ 成立

值迭代算法 (con't)

1. 对所有状态s，初始化 $V_0(s) = 0$
2. 给定一组$V_k(s)$的值，对所有的状态进行下列迭代更新：

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_k(s')\right]$$

3. 不断重复步骤2，直到收敛



本质上是一种动态规划的方法

值迭代算法的一些结论

- 每一步值迭代的计算复杂度是 $O(|A||S|^2)$
- 根据不动点理论，算法将最终收敛到最优值
  - 收敛到最优值所需的迭代次数为：

    poly(|S|, |A|, 1/(1-γ))

策略迭代算法

- 策略迭代算法
  1. 先给定初始策略 π，求解线性方程，计算出 π 相应的值函数 V
  $$V(s) = R(s, \pi(s)) + \gamma \sum_{s'} T(s, a, s') V(s').$$
  2. 基于计算出的值函数 V，按下式更新策略
  $$\pi(s) \leftarrow argmax_a \left\{ R(s, a) + \gamma \sum_{s'} T(s, a, s') V(s') \right\}$$
  3. 不断重复步骤 1，2，直到收敛
- 策略迭代算法最终收敛到最优值
  - 收敛到最优值所需的迭代不大于 $|A|^{|S|}$（确定性策略的总数目）

讨论

- 一个具体的 MDP 模型刻画了一个具体实际问题，即提供了该问题的一个数学模型
- MDPs 问题归结为最优策略的求解问题，三个要点：
  - 将 agents 的行动归结为策略的执行
  - 将 agents 的决策问题归结为求最优策略
  - 以“效用最大化”作为“最优”的基本标准
- 经典规划与 MDPs 的联系（异同）
  - 行动效果确定 vs. 不确定
  - 经典规划实际上假定环境是完全不可观察
  - 经典规划是以达到目标状态为“成功”标准，没有其他区别；而 MDPs 以效用最大化为“成功”标准，它的框架允许精华的，也可以通过优化途径求 π^*
- 一个经典规划问题可以描述为一个特殊的 NOMDP 问题，每个目标状态 s 上，定义一个“驻留行动”使得 $T(s, a, s) = 1$

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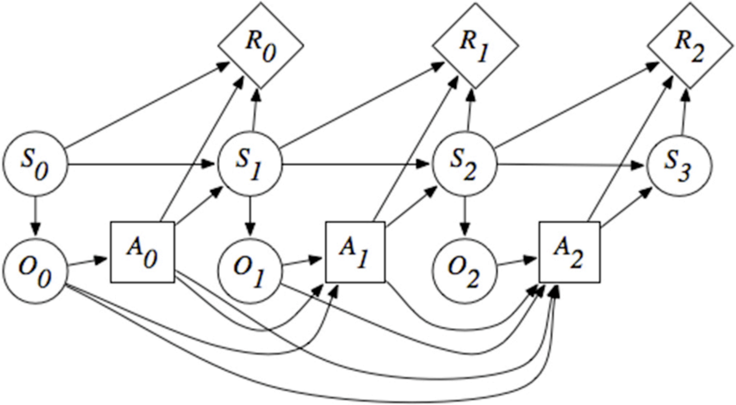
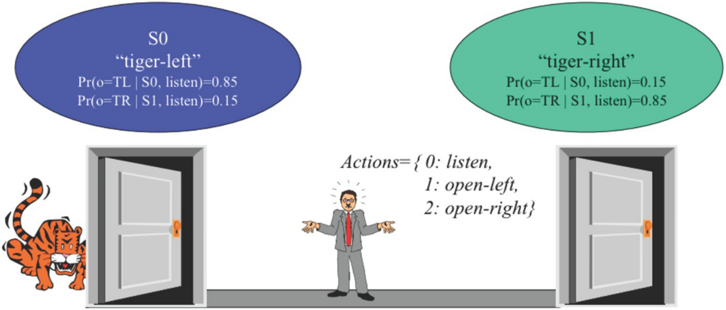
POMDPs 基本定义

- MDPs 刻画了行动的不确定性：事先不知道，事后知道；观察是确定的，知道行动的效果
- 一般情况下，行动和观察都是不确定的
- 一个 POMDP 模型是一个六元组 (S, A, T, R, Ω, O)，其中 S, A, T, R 的定义与 MDP 模型相同；Ω 是一个有限集，其中元素称为"观察"；$O: S \times A \to \Pi(\Omega)$ 称为观察函数，有时记为 $O(s', a, o)$，a 代表执行的行动，s' 是 a 达到的结果状态，o 在上述条件下出现的观察，$O(s', a, o)$ 是执行 a 达到 s' 观察到 o 的概率 $P_r(o \mid s', a)$
- 在 POMDP 问题中，"环境"被看成一个"黑箱"，一个状态是黑箱某个时期的内部情况，观察是这个黑箱的"输出"

POMDPs



老虎问题



Reward Function

- *Penalty for wrong opening: -100*
- *Reward for correct opening: +10*
- *Cost for listening action: -1*

Observations

- *to hear the tiger on the left (TL)*
- *to hear the tiger on the right(TR)*

信念状态

- 令 $B = \Pi(S)$ 为 S 上所有状态分布的集合，$b \in B$ 称为一个“信念状态”（belief state）
- 假设 $S = \{s_1, \ldots, s_n\}$，则 $\forall i$：$0 \leq b(s_i) \leq 1$，$\Sigma_{i=1}^{n} b(s_i) = 1$
  $b = (b(s_1), \ldots, b(s_n))$，$b(s_i)$ 表示环境处于 s_i 状态的概率
- 当 $|S|$ 有穷时，$\pi(S)$ 是无穷的

信念状态 (con't)

- 给定一个 POMDP 模型，一个 $b \in B$，一个 $a \in A$ 和一个 $o \in \Omega$，可以计算（估计）在 b 下执行 a 得到 o 时，环境处于状态 $s' \in S$ 的概率：

$$
\begin{aligned}
b'(s') = P_r(s' \mid o, a, b) &= \frac{P_r(o \mid s', a, b)\, P_r(s' \mid a, b)}{P_r(o \mid a, b)} \\
&= \frac{P_r(o \mid s', a)\, \Sigma_s P_r(s' \mid a, b, s) P_r(s \mid a, b)}{P_r(o \mid a, b)} \\
&= \frac{O(s', a, o)\, \Sigma_s T(s, a, s') b(s)}{P_r(o \mid a, b)}
\end{aligned}
$$

b' 是 o, a, b 的函数，记为 $b' = SE(b, a, o)$，称为“状态估计函数”

信念状态 (con't)

- 对 $b'(s')$ 中分母推导：

$$
\begin{aligned}
P_r(o \mid a, b) &= \Sigma_{s'} O(s', a, o) P_r(s' \mid a, b) \\
&= \Sigma_{s'} O(s', a, o) \Sigma_s T(s, a, s') b(s)
\end{aligned}
$$

  是给定 a, b 下出现 o 的“平均概率”
- $b'(s')$ 中分子是 a, b, s' 下出现 o 的概率
- $P_r(o \mid a, b)$ 是保证 $\Sigma_{s'} b'(s') = 1$ 的“正规化因子”
- $\Sigma_s T(s, a, s') b(s) = P_r(s' \mid a, b)$ 是不考虑观察而得到的下一状态为 s' 的概率，而 $b'(s')$ 是考虑了观察 $O(s', a, o)$ 所得出的下一状态的概率。两者关系：

$$
b'(s') = \frac{O(s', a, o)}{P_r(o \mid a, b)} \cdot P_r(s' \mid a, b)
$$

- 若 $O(s', a, o) > P_r(o \mid a, b)$，则 $b'(s') > P_r(s' \mid a, b)$。在 s' 下出现 o 的概率，高于出现 o 的平均概率

信念状态 (con't)

- b' 包含了观察和模型的全部信息：
  1. 上一时刻的全部信息 b;
  2. 模型信息 T;
  3. 观察信息 $O(s', a, o)$, $s' \in S$
- 因此，在 POMDP 问题中，B 的作用相当于 MDP 中的 S

积木世界续

- 考虑积木世界的 POMDP 模型，其中 S, A, T, R 与前例相同，$\Omega = \{o_1, o_2\}$，假定 agent 的观察能力与行动无关，而且不能区分 s_2 与 s_3
- 定义 O: $\forall a$

$$O(s_1, a, o_1) = 1,\ O(s_2, a, o_1) = 0,\ O(s_3, a, o_1) = 0$$
$$O(s_1, a, o_2) = 0,\ O(s_2, a, o_2) = 1,\ O(s_3, a, o_2) = 1$$

- 设初始信念状态 $b = (0.9, 0, 0.1)$，执行 $a_3 = move(A, F, B)$，观察为 o_2

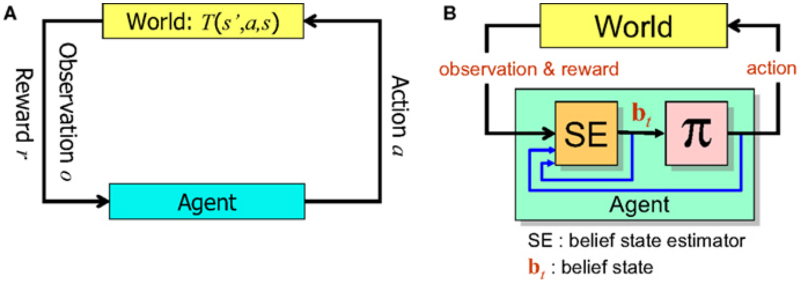
积木世界续 (con't)

计算下一时刻信念状态:

$$
\begin{aligned}
P_r(o_2 \mid a_3, b) &= \Sigma_{s'} O(s', a_3, o_2) P_r(s' \mid a_3, b) \\
&= P_r(s_2 \mid a_3, b) + P_r(s_3 \mid a_3, b) \\
&= \Sigma_s T(s, a_3, s_2) b(s) + \Sigma_s T(s, a_3, s_3) b(s) \\
&= (0.9 \times 0.85 + 0.1 \times 0) + (0.9 \times 0.05 + 1 \times 0.1) \\
&= 0.765 + 0.145 = 0.81
\end{aligned}
$$

$$
\begin{aligned}
b'(s_1) &= 0 \\
b'(s_2) &= \frac{0.765}{0.81} = 0.841 \\
b'(s_3) &= \frac{0.145}{0.81} = 0.159
\end{aligned}
$$

MDP vs. POMDP



A
World: $T(s',a,s)$
Reward r
Observation o
Action a
Agent
B
World
observation & reward
action
SE
$\mathbf{b}_t$
π
Agent
SE : belief state estimator
$\mathbf{b}_t$: belief state

POMDP to MDP

- B 在 POMDPs 中相当于 MDPs 中 S 的作用，设法将一个 POMDP 问题转化为一个变形的 MDP 问题，使得相应的 MDP 模型为 (B, A, τ, ρ)，$\tau : B \times A \to \Pi(B)$，$\rho : B \times A \to \mathcal{R}$

$$\tau(b, a, b') =_{df} P_r(b' \mid b, a) = \Sigma_o P_r(b' \mid a, b, o) P_r(o \mid a, b)$$

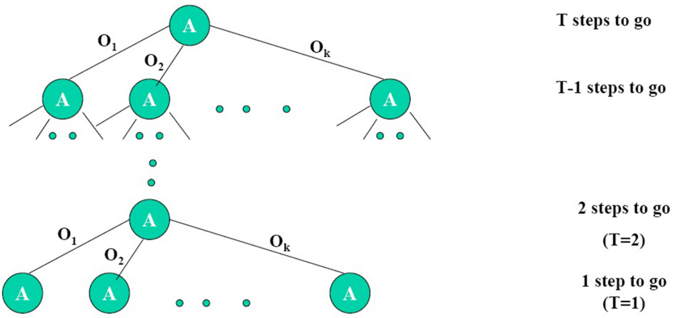
$$P_r(b' \mid a, b, o) = \begin{cases} 1, & \text{if } SE(b, a, o) = b' \\ 0, & \text{otherwise} \end{cases}$$

$$\rho(b, a) =_{df} \Sigma_s b(s) R(s, a)$$

- 由于 B 是无穷的，很难直接用值迭代算法求解

POMDPs 的策略表示

- POMDPs 的策略表示
  - $B \to A$（单步），多步的用 $\pi = (\pi_t, \ldots, \pi_1)$，但 B 是无穷的
  - 策略树：一个 t 层策略树（policy tree）共有 t 层节点，根节点上标记代表第一个行动，它的分枝代表执行该行动后得到的观察，...，是一种“有穷表示”



POMDPs 的值函数

- 采用策略树表示，令 p 是一个 t 步策略树，$V_p(s)$ 是在 s 上执行 p 的预期效用（扔采用折扣准则）:

$$
\begin{aligned}
V_p(s) &= R(s, a(p)) + \gamma \times \text{后 } t-1 \text{ 步的预期效用} \\
&= R(s, a(p)) + \gamma \Sigma_{s'} P_r(s' \mid s, a(p)) \Sigma_{o_i} P_r(o_i \mid s', a(p)) V_{o_i(P)}(s') \\
&= R(s, a(p)) + \gamma \Sigma_{s'} T(s, a(p), s') \Sigma_{o_i} O(s', a(p), o_i) V_{o_i(P)}(s')
\end{aligned}
$$

  $o_i(p)$ 是 p 的由分枝 o_i 引出的 $t-1$ 步子树
- 对任意 $b \in B$，定义 $V_p(b) = \Sigma_s b(s) V_p(s)$ 为在 b 上执行 p 的预期效用
- 令 P_t 为所有 t 步策略树的集合，P_t 有穷，b 上最优 t 阶段效用定义为

$$V_t(b) = max_{p \in P_t} V_p(b)$$

  使 $V_p(b) = V_t(b)$ 的 p 是 b 上的最佳策略

POMDPs 的值函数 (con't)

- 考虑在整个 B 上的最佳策略，一般要用策略树集来表示（因为，B 无穷、连续分布）
- $P \subseteq P_t$，满足 $\forall b \in B, \exists p \in P$ 使得 $V_p(b) \geq V_{p'}(b), \forall p' \in P_t$
- $|P_t|$ 是有穷的，

$$|P_t| = |A|^{\frac{|\Omega|^t - 1}{|\Omega| - 1}} = |A|^{|\Omega|^{t-1}}$$

- 若 $|A| = |\Omega| = t = 10$, $|P_t| \approx 10^{10^9}$，而深蓝才 10^{22}.

POMDPs 的值函数 (con't)

- S 是固定的、有穷的，B 是无穷的
- 设 $S = \{s_1, \ldots, s_n\}$，$b = \{b(s_1), \ldots, b(s_n)\}$，$0 \leq b(s_i) \leq 1$，$\Sigma_i b(s_i) = 1$，“变化的”不是 s，而是 b

$$V_p(b) = \Sigma_s b(s) V_p(s) = b(s_1) V_p(s_1) + \cdots + b(s_n) V_p(s_n)$$

$V_p(b)$ 是 $b(s_1), \ldots, b(s_n)$ 的一个线性函数。$V_p(s_i)$ 与 b 无关

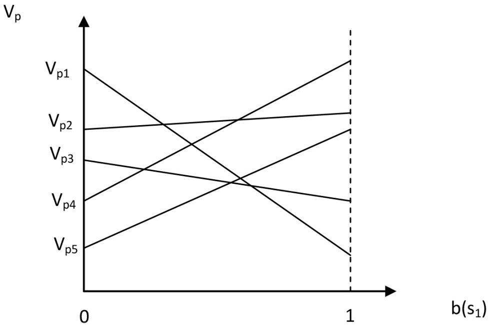
- 例：考虑 $|S| = 2$

$$\begin{aligned} V_p(b) &= b(s_1) V_p(s_1) + b(s_2) V_p(s_2) \\ &= V_p(s_1) b(s_1) + V_p(s_2)(1 - b(s_1)) \\ &= (V_p(s_1) - V_p(s_2)) b(s_1) + V_p(s_2) \end{aligned}$$

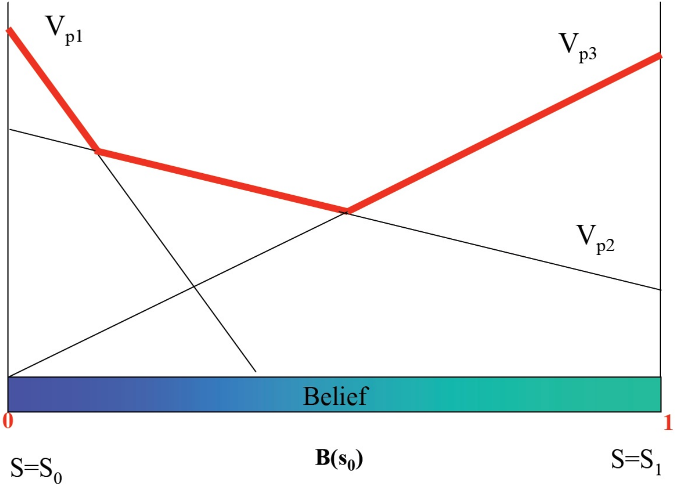
对不同的 p 有不同的 $V_p(s_1)$，$V_p(s_2)$，对应不同的线段

最大期望效用函数

- 由所有 $p \in P_t$ 产生的 $V_p(b)$ 的“上表面”可以得到 $V_t(b)$，整个 B 上最优策略集合，以及每个最优策略的“最优区域”，用线性规划可确定
- 对任意 $|S|$，情况类似，$V_t(b)$ 是一个“分块线性的凸函数”



最大期望效用函数 (con't)



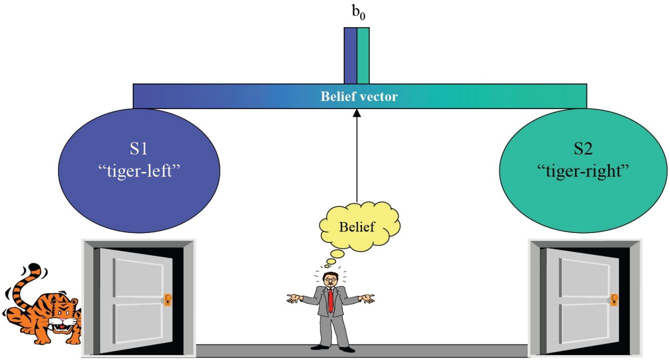
最大期望效用函数 (con't)

- 由 $V_p(b)$ 构成“上表面”的所有 p 组成的集合（代表 $V_t(b)$），称为最大期望效用函数/最佳值函数 $V_t(b)$ 的“最简表示”，记为 PV_t，每个 $p \in PV_t$ 是 B 的某个区域上的最优策略树，p 的“最优区域”是

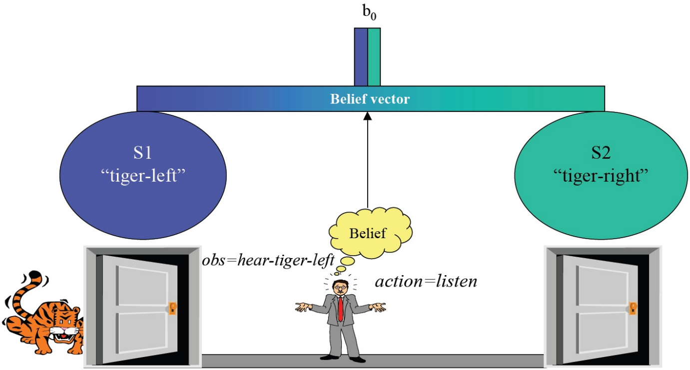
$$\{b \in B \mid \forall p' \in PV_t \setminus \{p\}, \Sigma_s b(s) V_p(s) > \Sigma_s b(s) V_{p'}(s)\}$$

  可以用线性规划方便求出
- PV_t 的任何真子集都无法表达 $V_t(b)$，称为不完备的，与“上表面”有共同点的所有 $p \in P_t$ 的集合记为 P^+V_t，称为“次简表示”
  - $PV_t \subseteq P^+V_t$;
  - $\forall p \in P^+V_t, \exists b \in B, \forall p' \in P_t$: $V_p(b) \geq V_{p'}(b)$;
  - 使用 PV_t/P^+V_t 的动机是，通常它们远远小于 P_t

老虎问题 (con't)

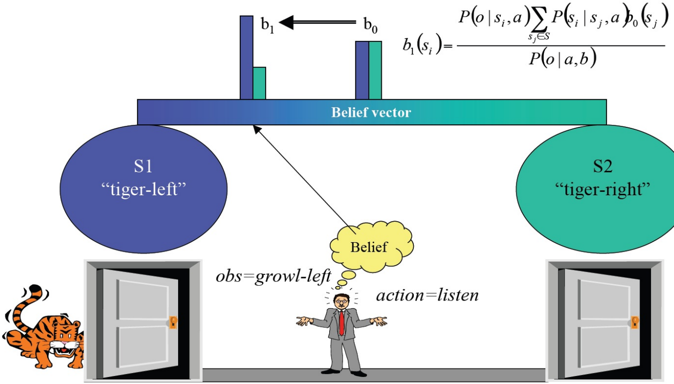


老虎问题 (con't)



b0
Belief vector
S1
"tiger-left"
S2
"tiger-right"
Belief
obs=hear-tiger-left
action=listen

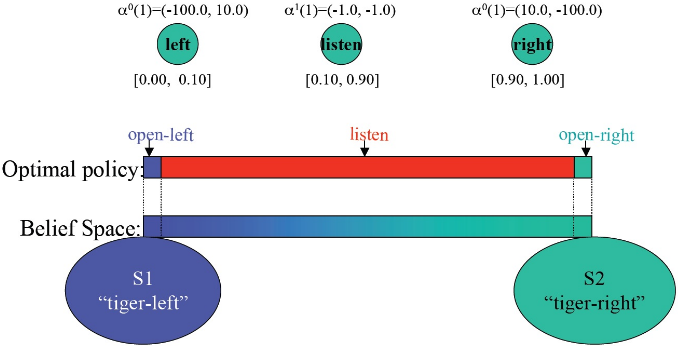
老虎问题 (con't)



b_1
b_0
$b_1(s_i) = \frac{P(o \mid s_i, a) \sum_{s_j \in S} P(s_i \mid s_j, a) b_0(s_j)}{P(o \mid a, b)}$
Belief vector
S1
"tiger-left"
S2
"tiger-right"
Belief
obs=growl-left
action=listen

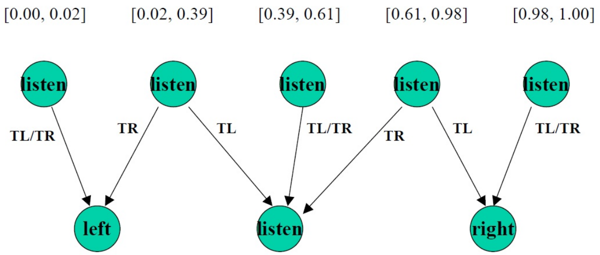
老虎问题 (con't)

- Optimal Policy for t=1



老虎问题 (con’t)

- For t=2



小结

- MDP 刻画了行动的不确定性，考虑概率和效用结合的最大期望效用
- MDP 模型为四元组 (S, A, T, R)
- MDP 可以用值迭代算法计算最优策略，计算复杂性为 P-complete
- POMDP 在 MDP 基础上刻画观察的不确定性，同样计算最大期望效用
- POMDP 模型为六元组 (S, A, T, R, Ω, O)
- POMDP 可以看做一个基于无穷信念状态的 MDP
- POMDP 的策略可以通过策略树表示
- POMDP 的最优策略的计算复杂性为 PSPACE-complete