概率机器人（Probabilistic Robotics）

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Used Materials

Disclaimer: 本课件大量采用了 Jana Kosecka's Autonomous Robotics 课件及其他网络课程课件，也采用了 GitHub 中开源代码，以及部分网络博客内容

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概率机器人

- 机器人中的不确定性：
  - 环境不确定性：动态环境难以预测
  - 感知不确定性：传感器物理限制
  - 行动不确定性：执行结构噪音
  - 模型误差：真实世界的近似模型
  - 算法误差：近似算法
- 概率机器人：Explicit representation of uncertainty using the calculus of probability theory
  - Perception = state estimation
  - Action = utility optimization

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样本空间和事件

- 随机试验：事先不能完全预知其结果的试验
  - 抛掷骰子是一个随机试验
- 样本空间：随机试验的所有可能结果组成的集合，记为 Ω
  - 掷骰子的样本空间 $\Omega = \{1, 2, 3, 4, 5, 6\}$
- 原子事件（样本点）：样本空间中的点，即随机试验的可能结果，记为 ω
- 事件：样本空间的子集，记为 A, B, ...
  - $A = \{1, 3, 5\}$ 表示"掷出结果为奇数"这一事件
  - Ω 本身为必然事件，$\emptyset$ 为不可能事件
- 若两事件 $A \cap B = \emptyset$，称为互斥事件（不相容事件）
- 若两事件 $A \cap B = \emptyset$ 且 $A \cup B = \Omega$，称为互补事件

概率

- 概率测度：给样本空间中的每一个事件 A 赋予一个数值（概率） $P(A) \in [0, 1]$
- 概率测度（形式化）是一个从样本空间 Ω 的幂集 2^Ω 到区间 $[0, 1]$ 的映射 $P: 2^\Omega \to [0, 1]$，且满足以下三个 Kolmogorov 公理：

  (1) $P(\Omega) = 1$;（规范性）
  (2) $P(A) \geq 0, \forall A \in 2^\Omega$;（非负性）
  (3) $P(A \cup B) = P(A) + P(B), \forall A, B \in 2^\Omega, A \cap B = \emptyset$.（有限可加性）

- $P(A)$ 称为事件 A 的概率

随机变量和概率函数

- 随机变量是定义在样本空间 Ω 上的函数，记为 X, Y, Z
- 随机变量的取值随试验结果而定，记为 x, y, z
- 随机变量 X 的所有可能取值的集合称为其值域（状态空间），记为 Ω_X
- 设 X 为一随机变量，x 是它的一个取值，在样本空间 Ω 中，所有使 X 取值为 x 的原子事件组成一个事件，记为 $\Omega_{X=x} = \{w \in \Omega \mid X(\omega) = x\}$，简记为 “$X = x$”
- 事件 “$X = x$” 的概率 $P(X = x) = P(\Omega_{X=x})$ 依赖于 X 的取值 x，让 x 在 Ω_X 上变动，$P(X = x)$ 就称为 Ω_X 的一个取值于 $[0, 1]$ 的函数，称为随机变量 X 的概率质量函数 (probability mass function)，记为 $P(X)$
- 根据概率测度的定义

$$P(X = x) \geq 0, \forall x \in \Omega_X \text{ 简记为 } P(X) \geq 0$$

$$\sum_{x \in \Omega_X} P(X = x) = 1 \text{ 简记为 } \sum_x P(X = x) = 1.$$

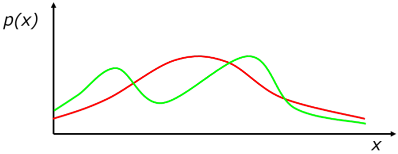
连续随机变量（Continuous Random Variables）

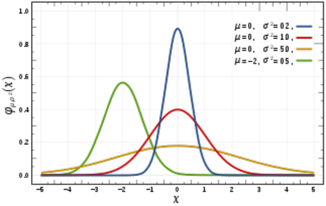
- X 是随机变量，并且取值是连续的
- $p(X = x)$ or $p(x)$ 为概率密度函数（probability density function）

$$P(x \in (a, b)) = \int_a^b p(x)\, \mathrm{d}x$$

- 例如：高斯分布（正态分布），均值为 μ，标准差为 σ

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$





联合概率分布（Joint Probability）

- 对多个随机变量 $X_1, \dots, X_n$，用联合概率分布 $P(X_1, \dots, X_n)$ 来描述各变量所有可能的状态组合的概率。
- 联合分布是定义在所有变量状态空间的笛卡尔乘积上的函数：
  - $P(X_1, \dots, X_n) : \otimes_{i=1}^{n} \Omega_{X_i} \rightarrow [0, 1]$
  - $\Sigma_{X_1, \dots, X_n} P(X_1, \dots, X_n) = 1$
- 联合分布通常表示为一张表，包含 $\Pi_{i=1}^{n} |\Omega_{X_i}|$ 个状态组合及其概率值。例，香港租房市场

	public	private	others
low	0.17	0.01	0.02
medium	0.44	0.03	0.01
upper medium	0.09	0.07	0.01
high	0	0.14	0.01

- 记 $\mathbf{X} = \{X_1, \dots, X_n\}$，$\mathbf{Y}$ 是 $\mathbf{X}$ 的真子集（$\mathbf{Y} \subset \mathbf{X}$），$\mathbf{Z} = \mathbf{X} \setminus \mathbf{Y}$。则相对于 $P(\mathbf{X})$，$\mathbf{Y}$ 的边缘分布 $P(\mathbf{Y})$ 定义为 $P(\mathbf{Y}) = \Sigma_{\mathbf{Z}} P(X_1, \dots, X_n)$，称为边缘化

条件概率分布（Conditional Probability）

- 条件概率：

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

- 条件概率分布：

$$P(X = x \mid Y = y) = \frac{P(X = x,\ Y = y)}{P(Y = y)}$$

固定 y，让 x 在 Ω_X 上变动，得到函数 $P(X \mid Y = y)$ （在给定 $Y = y$ 时变量 X 的条件概率分布）；
$P(X \mid Y) = \{P(X \mid Y = y) \mid y \in \Omega_Y\}$ （给定 Y 时变量 X 的条件概率分布）

$$P(\mathbf{X} \mid \mathbf{Y}) = \frac{P(\mathbf{X}, \mathbf{Y})}{P(\mathbf{Y})}$$

- 链规则：

$$P(X_1, X_2, \ldots, X_n) = P(X_1)P(X_2 \mid X_1) \cdots P(X_n \mid X_1, \ldots, X_{n-1})$$

条件独立（Conditional Independence）

- 事件 A 与 B 相互独立：$P(A\cap B)=P(A)P(B)$
  - 当 $P(A)>0$ 时，$P(B)=P(B\mid A)$.
- 事件 A 与 B 在给定 C 时相互条件独立：

$$P(A\cap B\mid C)=P(A\mid C)P(B\mid C)$$

  - 当 $P(B\cap C)>0$ 时，$P(A\mid C)=P(A\mid B\cap C)$.
- 两个变量 X 和 Y 相互独立，记为 $X\perp Y$:

$$P(X,Y)=P(X)P(Y)$$

  - 若 $P(Y=y)>0$，则 $P(X)=P(X\mid Y=y)$.
- 三个随机变量 X, Y 和 Z，设 $P(Z=z)>0, \forall z\in\Omega_Z$，$X$ 和 Y 在给定 Z 时相互条件独立，记为 $X\perp Y\mid Z$:

$$P(X,Y\mid Z)=P(X\mid Z)P(Y\mid Z)$$

.
  - 若 $P(Y=y,Z=z)>0$，
    则 $P(X\mid Y=y,Z=z)=P(X\mid Z=z)$.

一些规则

Discrete case

$$\sum_x P(x) = 1$$

$$P(x) = \sum_y P(x, y)$$

$$P(x) = \sum_y P(x \mid y) P(y)$$

Continuous case

$$\int p(x)\, dx = 1$$

$$p(x) = \int p(x, y)\, dy$$

$$p(x) = \int p(x \mid y)\, p(y)\, dy$$

贝叶斯公式（Bayes Formula）

- 在考虑证据 $E = e$ 之前，对事件 $H = h$ 的概率估计 $P(H = h)$ 称为先验概率；而在考虑证据之后，对 $H = h$ 的概率估计 $P(H = h \mid E = e)$ 称为后验概率
- 贝叶斯定理（贝叶斯规则、公式）

$$P(H = h \mid E = e) = \frac{P(H = h)P(E = e \mid H = h)}{P(E = e)}$$

$$P(X \mid E = e) = \frac{P(X)P(E = e \mid X)}{P(E = e)}$$

$$P(x, y) = P(x \mid y)P(y) = P(y \mid x)P(x)$$

$$\Rightarrow$$

$$P(x \mid y) = \frac{P(y \mid x)\ P(x)}{P(y)} = \frac{\text{likelihood} \cdot \text{prior}}{\text{evidence}}$$

贝叶斯公式（Bayes Formula）

$$P(x \mid y) = \frac{P(y \mid x)P(x)}{P(y)} = \frac{P(y \mid x)P(x)}{\sum_{x'} P(y \mid x')P(x')}$$

$$P(x \mid y) = \frac{P(y \mid x)P(x)}{P(y)} = \frac{P(y \mid x)P(x)}{\int P(y \mid x')P(x')\,\mathrm{d}x'}$$

- 如果 x 是一个希望由 y 推测出来的数值，则概率 $P(x)$ 称为先验概率分布（prior probability distribution）
- y 称为数据（data），也就是传感器测量值
- $P(x)$ 总结了在综合数据 y 之前已经有的关于 x 的信息
- 概率 $P(x \mid y)$ 称为在 X 上的后验概率分布（posterior probability distribution）
- 贝叶斯准则利用“逆”条件概率 $P(y \mid x)$ 和先验概率 $P(x)$ 计算后验概率 $P(x \mid y)$
- $P(y \mid x)$ 称为生成模型（generative model），表示变量 X 如何引起检测数据 Y

归一化（Normalization）

$$P(x \mid y) = \frac{P(y \mid x)\ P(x)}{P(y)} = \eta\ P(y \mid x)\, P(x)$$

$$\eta = P(y)^{-1} = \frac{1}{\sum_x P(y \mid x) P(x)}$$

Algorithm:

$$\forall x : \mathrm{aux}_{x|y} = P(y \mid x)\ P(x)$$

$$\eta = \frac{1}{\sum_x \mathrm{aux}_{x|y}}$$

$$\forall x : P(x \mid y) = \eta\ \mathrm{aux}_{x|y}$$

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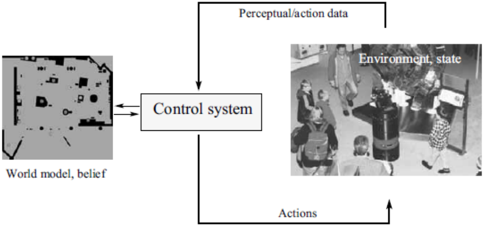
机器人环境交互



- 环境状态（state）: $x_{t_1:t_2} = x_{t_1}, x_{t_1+1}, \ldots, x_{t_2}$
- 环境传感器测量（measurement）: $z_{t_1:t_2} = z_{t_1}, z_{t_1+1}, \ldots, z_{t_2}$
- 控制行动（control action）: $u_{t_1:t_2} = u_{t_1}, u_{t_1+1}, \ldots, u_{t_2}$

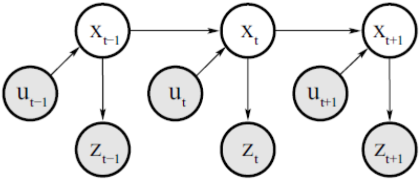
概率生成法则

- 状态概率生成法则：$P(x_t \mid x_{0:t-1}, z_{1:t-1}, u_{1:t})$
- 如果状态 x 是完整的，则：
  - 状态转移概率：$P(x_t \mid x_{0:t-1}, z_{1:t-1}, u_{1:t}) = P(x_t \mid x_{t-1}, u_t)$
  - 测量概率：$P(z_t \mid x_{0:t-1}, z_{1:t-1}, u_{1:t}) = P(z_t \mid x_t)$
- 置信度（belief）：已知条件下对状态估计的概率分布

$$bel(x_t) = P(x_t \mid z_{1:t}, u_{1:t})$$

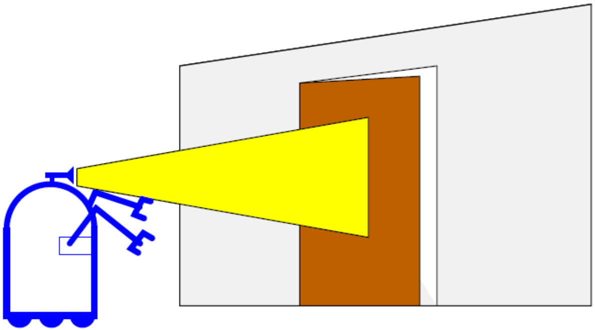
$$\overline{bel}(x_t) = P(x_t \mid z_{1:t-1}, u_{1:t}) \qquad \text{prediction}$$

- 由 $\overline{bel}(x_t)$ 计算 $bel(x_t)$ 为修正（correction）或测量更新（measurement update）



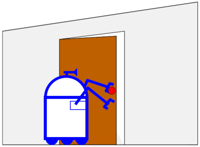
状态估计

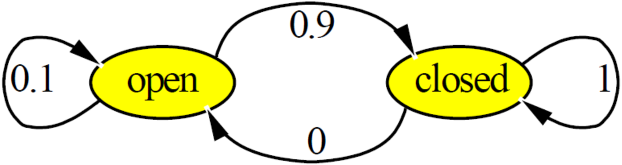
- Suppose a robot obtains measurement z
- What is $P(open \mid z)$?



状态估计

- $P(x \mid u, x')$ for $u =$ "close door"





If the door is open, the action "close door" succeeds in 90% of all cases

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Recursive Bayesian Updating

给定观察 $z_1, \ldots, z_n$，估计状态 x

$$P(x \mid z_1, \ldots, z_n) = \frac{P(z_n \mid x, z_1, \ldots, z_{n-1}) P(x \mid z_1, \ldots, z_{n-1})}{P(z_n \mid z_1, \ldots, z_{n-1})}$$

- Markov Assumption: z_n is independent of $z_1, \ldots, z_{n-1}$ if we know x

$$\begin{aligned} P(x \mid z_1, \ldots, z_n) &= \frac{P(z_n \mid x) P(x \mid z_1, \ldots, z_{n-1})}{P(z_n \mid z_1, \ldots, z_{n-1})} \\ &= \eta P(z_n \mid x) P(x \mid z_1, \ldots, z_{n-1}) \\ &= \eta_{1,\ldots,n} \left(\prod_{i=1,\ldots,n} P(z_i \mid x) \right) P(x) \end{aligned}$$

Integrating the Outcome of Actions

- Continuous case:

$$P(x \mid u) = \int P(x \mid u, x')P(x')\, \mathtt{d}x'$$

- Discrete case:

$$P(x \mid u) = \sum P(x \mid u, x')P(x')$$

Bayes Filters: Framework

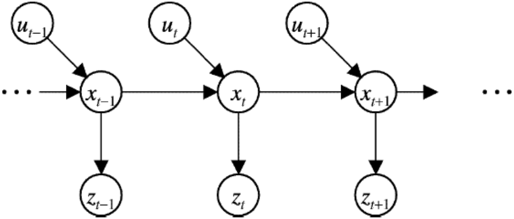
- Given:
  - Stream of observations z and action data u:

$$d_t = \{u_1, z_1, \ldots, u_t, z_t\}$$

  - Sensor model $P(z \mid x)$
  - Action model $P(x \mid u, x')$
  - Prior probability of the system state $P(x)$
- Wanted:
  - Estimate of the state X of a dynamical system
  - The posterior of the state is also called Belief:

$$bel(x_t) = P(x_t \mid u_1, z_1, \ldots, u_t, z_t)$$

Markov Assumption



$$P(z_t \mid x_{0:t}, z_{1:t-1}, u_{1:t}) = P(z_t \mid x_t)$$
$$P(x_t \mid x_{1:t-1}, z_{1:t-1}, u_{1:t}) = P(x_t \mid x_{t-1}, u_t)$$

Underlying Assumptions

- Static world
- Independent noise
- Perfect model, no approximation errors

Bayes Filters

$$
\begin{aligned}
& bel(x_t) \\
&= P(x_t \mid z_{1:t}, u_{1:t}) \\
&= \eta P(z_t \mid x_t, z_{1:t-1}, u_{1:t}) P(x_t \mid z_{1:t-1}, u_{1:t}) && \text{Bayes} \\
&= \eta P(z_t \mid x_t) P(x_t \mid z_{1:t-1}, u_{1:t}) && \text{Markov} \\
&= \eta P(z_t \mid x_t) \overline{bel}(x_t) \\
&= \eta P(z_t \mid x_t) \int P(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t}) p(x_{t-1} \mid z_{1:t-1}, u_{1:t}) \, \mathrm{d}x_{t-1} && \text{Total prob.} \\
&= \eta P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) P(x_{t-1} \mid z_{1:t-1}, u_{1:t}) \, \mathrm{d}x_{t-1} && \text{Markov} \\
&= \eta P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) P(x_{t-1} \mid z_{1:t-1}, u_{1:t-1}) \, \mathrm{d}x_{t-1} && \text{Markov} \\
&= \eta P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) bel(x_{t-1}) \, \mathrm{d}x_{t-1}
\end{aligned}
$$

Bayes Filter Algorithm

$$Bel(x_t) = \eta P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) Bel(x_{t-1}) dx_{t-1}$$

Algorithm **Bayes_filter**($Bel(x)$, d):
$\eta = 0$
If d is a perceptual data item z then
    For all x do
        $Bel'(x) = P(z \mid x) Bel(x)$
        $\eta = \eta + Bel'(x)$
    For all x do
        $Bel'(x) = \eta^{-1} Bel'(x)$
Else if d is an action data item u then
    For all x do
        $Bel'(x) = \int P(x \mid u, x') \, Bel(x') \, dx'$
Return $Bel'(x)$

Bayes Filters are Familiar

- Prediction

$$\overline{bel}(x_t) = \int P(x_t \mid u_t, x_{t-1})\, bel(x_{t-1})\, \mathrm{d}x_{t-1}$$

- Correction

$$bel(x_t) = \eta P(z_t \mid x_t)\overline{bel}(x_t)$$

$$Bel(x_t) = \eta P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) Bel(x_{t-1}) dx_{t-1}$$

- Kalman filters
- Particle filters
- Hidden Markov models
- Dynamic Bayesian networks
- Partially Observable Markov Decision Processes (POMDPs)

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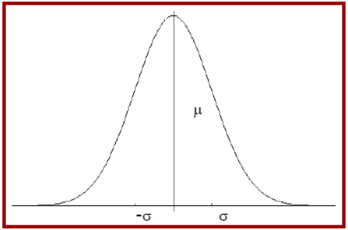
高斯分布

Gaussians

$$p(x) \sim N(\mu, \sigma^2):$$

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

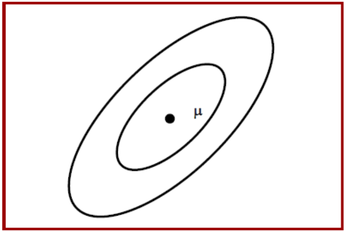
Univariate



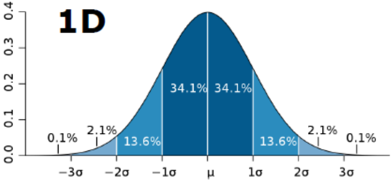
$$p(\mathbf{x}) \sim N(\mu, \Sigma):$$

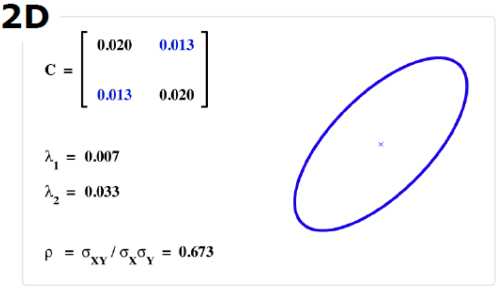
$$p(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}|\Sigma|^{1/2}} e^{-\frac{1}{2}(\mathbf{x}-\mu)^t \Sigma^{-1}(\mathbf{x}-\mu)}$$

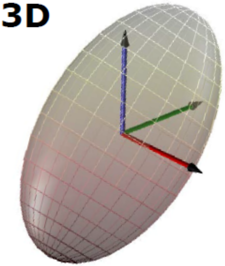
Multivariate



高斯分布


1D
34.1%
34.1%
13.6%
13.6%
2.1%
2.1%
0.1%
0.1%
−3σ
−2σ
−1σ
μ
1σ
2σ
3σ
0.0
0.1
0.2
0.3
0.4


2D
C = [0.020 0.013; 0.013 0.020]
λ1 = 0.007
λ2 = 0.033
ρ = σXY / σXσY = 0.673


3D

Properties of Gaussians

- Univariate case

$$\left.\begin{aligned} X &\sim N(\mu,\sigma^2) \\ Y &= aX+b \end{aligned}\right\} \Rightarrow \quad Y \sim N(a\mu+b, a^2\sigma^2)$$

$$\left.\begin{aligned} X_1 &\sim N(\mu_1,\sigma_1^{\ 2}) \\ X_2 &\sim N(\mu_2,\sigma_2^{\ 2}) \end{aligned}\right\} \Rightarrow p(X_1)\cdot p(X_2) \sim N\left(\frac{\sigma_2^{\ 2}}{\sigma_1^{\ 2}+\sigma_2^{\ 2}}\mu_1 + \frac{\sigma_1^{\ 2}}{\sigma_1^{\ 2}+\sigma_2^{\ 2}}\mu_2, \quad \frac{1}{\sigma_1^{\ -2}+\sigma_2^{\ -2}}\right)$$

Properties of Gaussians

- Multivariate case

$$\left.\begin{array}{r} X \sim N(\mu, \Sigma) \\ Y = AX + B \end{array}\right\} \Rightarrow \quad Y \sim N(A\mu + B, A\Sigma A^T)$$

$$\left.\begin{array}{l} X_1 \sim N(\mu_1, \Sigma_1) \\ X_2 \sim N(\mu_2, \Sigma_2) \end{array}\right\} \Rightarrow p(X_1) \cdot p(X_2) \sim N\left(\frac{\Sigma_2}{\Sigma_1 + \Sigma_2} \mu_1 + \frac{\Sigma_1}{\Sigma_1 + \Sigma_2} \mu_2, \quad \frac{1}{\Sigma_1^{-1} + \Sigma_2^{-1}} \right)$$

(where division "–" denotes matrix inversion)

- We **stay Gaussian** as long as we start with Gaussians and perform only **linear transformations**

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Discrete Kalman Filter

Estimates the state x of a discrete-time controlled process that is governed by the linear stochastic difference equation

$$x_t = A_t x_{t-1} + B_t u_t + \varepsilon_t$$

with a measurement

$$z_t = C_t x_t + \delta_t$$

Components of a Kalman Filter

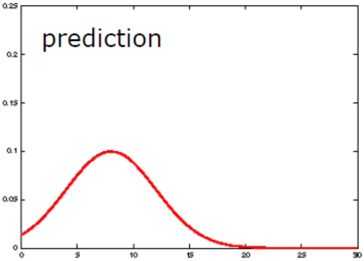
A_t — Matrix $(n \times n)$ that describes how the state evolves from t-1 to t without controls or noise.

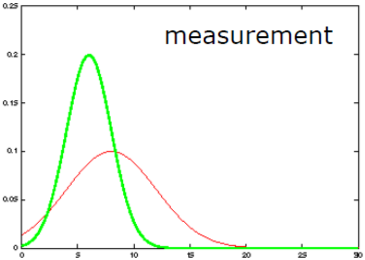
B_t — Matrix $(n \times l)$ that describes how the control u_t changes the state from t-1 to t.

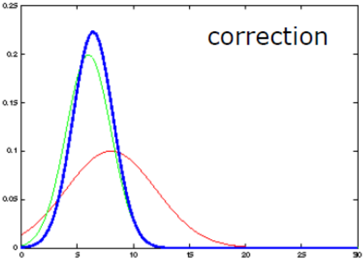
C_t — Matrix $(k \times n)$ that describes how to map the state x_t to an observation z_t.

ε_t, δ_t — Random variables representing the process and measurement noise that are assumed to be independent and normally distributed with covariance Q_t and R_t respectively.

Kalman Filter Updates in 1D

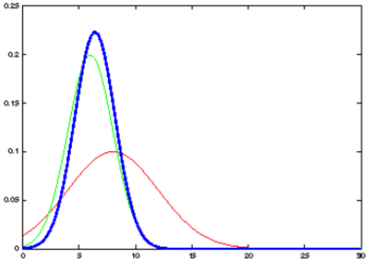






It's a weighted mean!

Kalman Filter Updates in 1D



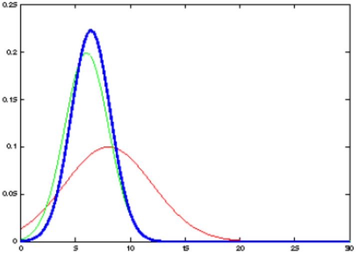
How to get the blue one?

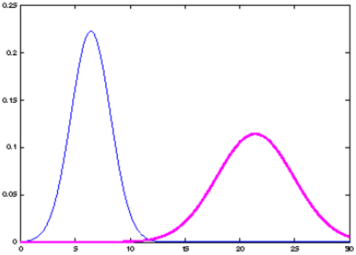
Kalman correction step

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - \bar{\mu}_t) \\ \sigma_t^2 = (1 - K_t)\bar{\sigma}_t^2 \end{cases} \quad \text{with} \quad K_t = \frac{\bar{\sigma}_t^2}{\bar{\sigma}_t^2 + \bar{\sigma}_{obs,t}^2}$$

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - C_t\bar{\mu}_t) \\ \Sigma_t = (I - K_t C_t)\bar{\Sigma}_t \end{cases} \quad \text{with} \quad K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + R_t)^{-1}$$

Kalman Filter Updates in 1D





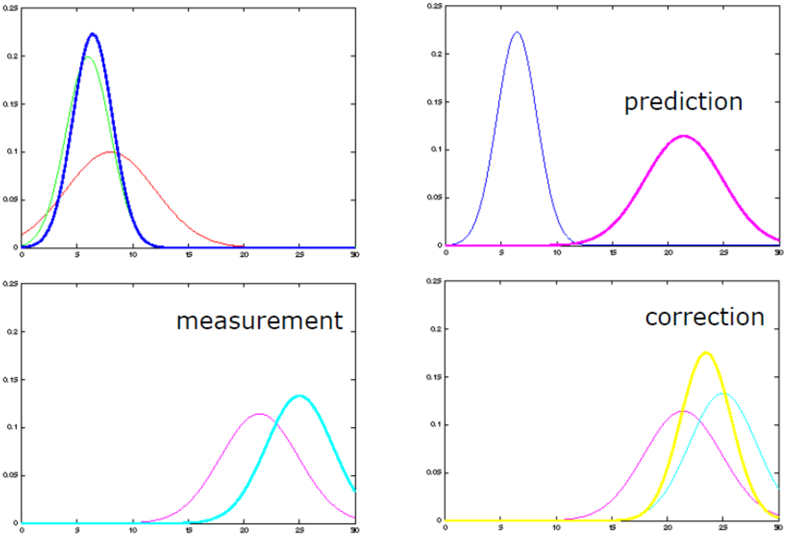
$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = a_t \mu_{t-1} + b_t u_t \\ \bar{\sigma}_t^2 = a_t^2 \sigma_t^2 + \sigma_{act,t}^2 \end{cases}$$

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t u_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + Q_t \end{cases}$$

How to get the magenta one?

State prediction step

Kalman Filter Updates



Linear Gaussian Systems: Initialization

Initial belief is normally distributed:

$$bel(x_0) = N(x_0; \mu_0, \Sigma_0)$$

Linear Gaussian Systems: Dynamics

Dynamics are linear functions of the state and the control plus additive noise:

$$x_t = A_t x_{t-1} + B_t u_t + \varepsilon_t$$

$$p(x_t \mid u_t, x_{t-1}) = N\left(x_t; A_t x_{t-1} + B_t u_t, Q_t\right)$$

$$\overline{bel}(x_t) = \int \underbrace{p(x_t \mid u_t, x_{t-1})}_{\sim N\left(x_t; A_t x_{t-1} + B_t u_t, Q_t\right)} \; \underbrace{bel(x_{t-1})}_{\sim N\left(x_{t-1}; \mu_{t-1}, \Sigma_{t-1}\right)} \, dx_{t-1}$$

Linear Gaussian Systems: Dynamics

$$\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1}) \qquad bel(x_{t-1})\, dx_{t-1}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\sim N(x_t; A_t x_{t-1} + B_t u_t, Q_t) \quad \sim N(x_{t-1}; \mu_{t-1}, \Sigma_{t-1})$$

$$\Downarrow$$

$$\overline{bel}(x_t) = \eta \int \exp\left\{-\frac{1}{2}(x_t - A_t x_{t-1} - B_t u_t)^T Q_t^{-1} (x_t - A_t x_{t-1} - B_t u_t)\right\}$$

$$\exp\left\{-\frac{1}{2}(x_{t-1} - \mu_{t-1})^T \Sigma_{t-1}^{-1} (x_{t-1} - \mu_{t-1})\right\} dx_{t-1}$$

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t u_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + Q_t \end{cases}$$

Linear Gaussian Systems: Observations

Observations are a linear function of the state plus additive noise:

$$z_t = C_t x_t + \delta_t$$

$$p(z_t \mid x_t) = N(z_t; C_t x_t, R_t)$$

$$bel(x_t) = \eta \quad p(z_t \mid x_t) \quad \overline{bel}(x_t)$$

$$p(z_t \mid x_t) \Downarrow \; \sim N(z_t; C_t x_t, R_t) \qquad \overline{bel}(x_t) \Downarrow \; \sim N(x_t; \bar{\mu}_t, \bar{\Sigma}_t)$$

Linear Gaussian Systems: Observations

$$bel(x_t) = \eta \quad p(z_t \mid x_t) \qquad \overline{bel}(x_t)$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\sim N(z_t; C_t x_t, R_t) \qquad \sim N\left(x_t; \bar{\mu}_t, \bar{\Sigma}_t\right)$$

$$\Downarrow$$

$$bel(x_t) = \eta \exp\left\{-\frac{1}{2}(z_t - C_t x_t)^T R_t^{-1}(z_t - C_t x_t)\right\} \exp\left\{-\frac{1}{2}(x_t - \bar{\mu}_t)^T \bar{\Sigma}_t^{-1}(x_t - \bar{\mu}_t)\right\}$$

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - C_t\bar{\mu}_t) \\ \Sigma_t = (I - K_t C_t)\bar{\Sigma}_t \end{cases} \qquad \text{with} \quad K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + R_t)^{-1}$$

Kalman Filter Algorithm

1. Algorithm **Kalman_filter**($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$):

2. Prediction:
3. $\bar{\mu}_t = A_t \mu_{t-1} + B_t u_t$
4. $\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + Q_t$

5. Correction:
6. $K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + R_t)^{-1}$
7. $\mu_t = \bar{\mu}_t + K_t (z_t - C_t \bar{\mu}_t)$
8. $\Sigma_t = (I - K_t C_t) \bar{\Sigma}_t$

9. Return μ_t, Σ_t

The Prediction-Correction-Cycle

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = a_t \mu_{t-1} + b_t u_t \\ \bar{\sigma}_t^2 = a_t^2 \sigma_t^2 + \sigma_{act,t}^2 \end{cases}$$

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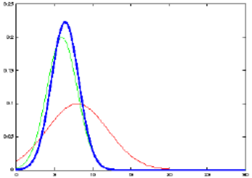
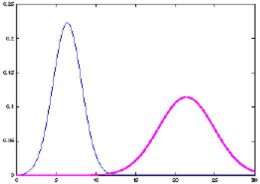
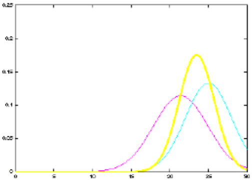




The Prediction-Correction-Cycle



$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - \bar{\mu}_t) \\ \sigma_t^2 = (1 - K_t)\bar{\sigma}_t^2 \end{cases}, K_t = \frac{\bar{\sigma}_t^2}{\bar{\sigma}_t^2 + \sigma_{obs,t}^2}$$

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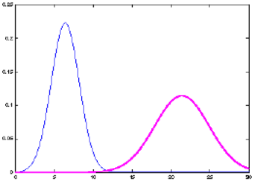
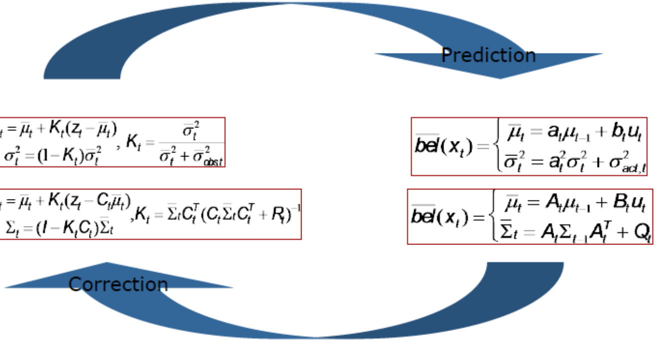


The Prediction-Correction-Cycle



$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = a_t \mu_{t-1} + b_t u_t \\ \bar{\sigma}_t^2 = a_t^2 \sigma_t^2 + \sigma_{act,t}^2 \end{cases}$$

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$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - C_t \bar{\mu}_t) \\ \Sigma_t = (I - K_t C_t)\bar{\Sigma}_t \end{cases}, K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + R_t)^{-1}$$

Kalman Filter Summary

- Only two parameters describe belief about the state of the system
- **Highly efficient:** Polynomial in the measurement dimensionality k and state dimensionality n:

$$O(k^{2.376} + n^2)$$

- **Optimal for linear Gaussian systems**!
- However: Most robotics systems are **nonlinear**!
- Can only model unimodal beliefs

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Nonlinear Dynamic Systems

- Most realistic robotic problems involve nonlinear functions

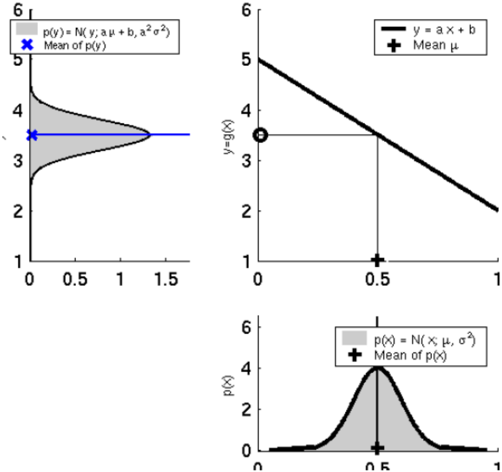
$$x_t = A_t x_{t-1} + B_t u_t + \varepsilon_t$$

$$z_t = C_t x_t + \delta_t$$

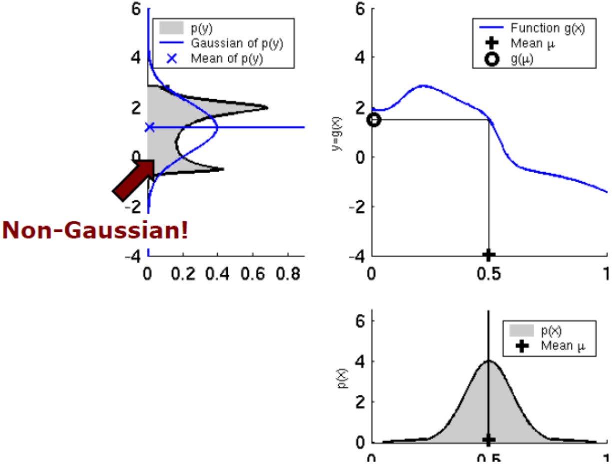
$$x_t = g(u_t, x_{t-1})$$

$$z_t = h(x_t)$$

Linearity Assumption Revisited



Non-Linear Function



Non-Gaussian Distributions

- The non-linear functions lead to non-Gaussian distributions
- Kalman filter is not applicable anymore!

What can be done to resolve this?

Local linearization!

EKF Linearization: First Order Taylor Expansion

- Prediction:

$$g(u_t, x_{t-1}) \approx g(u_t, \mu_{t-1}) + \frac{\partial g(u_t, \mu_{t-1})}{\partial x_{t-1}} (x_{t-1} - \mu_{t-1})$$

$$g(u_t, x_{t-1}) \approx g(u_t, \mu_{t-1}) + G_t \, (x_{t-1} - \mu_{t-1})$$

- Correction:

$$h(x_t) \approx h(\bar{\mu}_t) + \frac{\partial h(\bar{\mu}_t)}{\partial x_t} (x_t - \bar{\mu}_t)$$

$$h(x_t) \approx h(\bar{\mu}_t) + H_t \, (x_t - \bar{\mu}_t)$$

Jacobian matrices

Reminder: Jacobian Matrix

- It is a **non-square matrix** $n \times m$ in general
- Given a vector-valued function

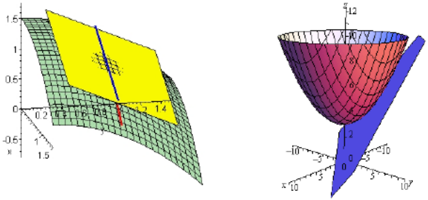
$$f(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \end{bmatrix}$$

- The **Jacobian matrix** is defined as

$$\mathbf{F_x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Reminder: Jacobian Matrix

- It is the orientation of the tangent plane to the vector-valued function at a given point



- Generalizes the gradient of a scalar valued function

EKF Linearization: First Order Taylor Expansion

- Prediction:

$$g(u_t, x_{t-1}) \approx g(u_t, \mu_{t-1}) + \frac{\partial g(u_t, \mu_{t-1})}{\partial x_{t-1}} (x_{t-1} - \mu_{t-1})$$

$$g(u_t, x_{t-1}) \approx g(u_t, \mu_{t-1}) + G_t \, (x_{t-1} - \mu_{t-1})$$

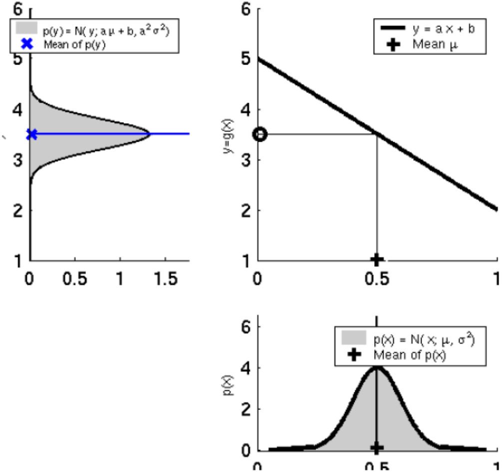
- Correction:

$$h(x_t) \approx h(\bar{\mu}_t) + \frac{\partial h(\bar{\mu}_t)}{\partial x_t} (x_t - \bar{\mu}_t)$$

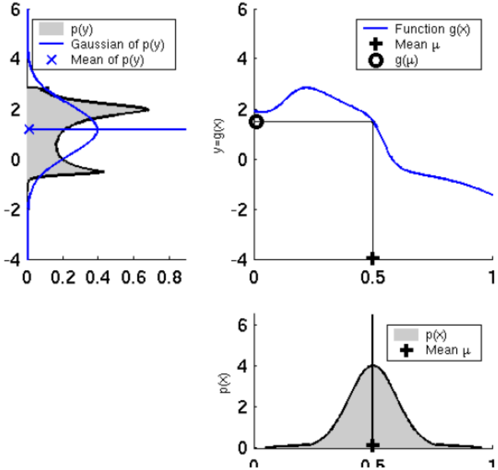
$$h(x_t) \approx h(\bar{\mu}_t) + H_t \, (x_t - \bar{\mu}_t)$$

Linear function!

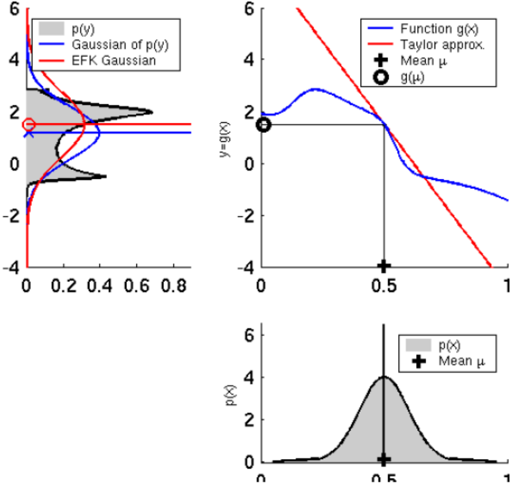
Linearity Assumption Revisited



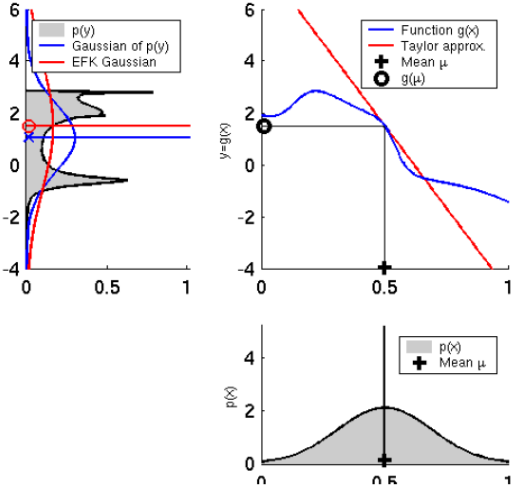
Non-Linear Function



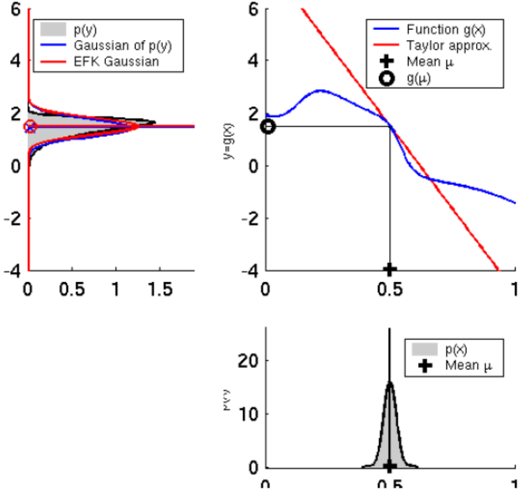
EKF Linearization (1)



EKF Linearization (2)



EKF Linearization (3)



EKF Algorithm

1. **Extended_Kalman_filter**($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$):

2. Prediction:

3. $\bar{\mu}_t = g(u_t, \mu_{t-1})$ $\longleftarrow$ $\bar{\mu}_t = A_t \mu_{t-1} + B_t u_t$

4. $\bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + Q_t$ $\longleftarrow$ $\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + Q_t$

5. Correction:

6. $K_t = \bar{\Sigma}_t H_t^T (H_t \bar{\Sigma}_t H_t^T + R_t)^{-1}$ $\longleftarrow$ $K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + R_t)^{-1}$

7. $\mu_t = \bar{\mu}_t + K_t (z_t - h(\bar{\mu}_t))$ $\longleftarrow$ $\mu_t = \bar{\mu}_t + K_t (z_t - C_t \bar{\mu}_t)$

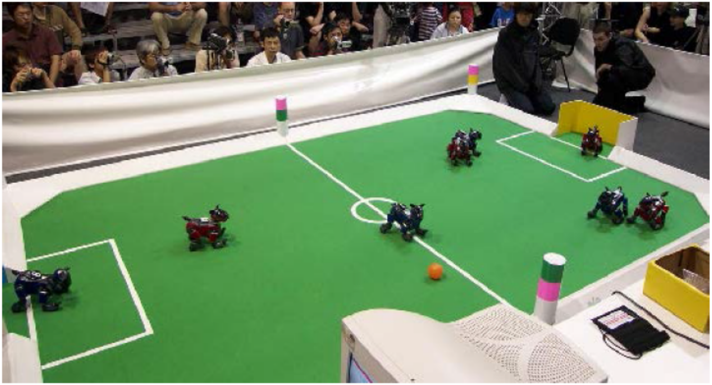
8. $\Sigma_t = (I - K_t H_t) \bar{\Sigma}_t$ $\longleftarrow$ $\Sigma_t = (I - K_t C_t) \bar{\Sigma}_t$

9. Return μ_t, Σ_t

$$H_t = \frac{\partial h(\bar{\mu}_t)}{\partial x_t} \qquad G_t = \frac{\partial g(u_t, \mu_{t-1})}{\partial x_{t-1}}$$

Example: EKF Localization

- EKF localization with landmarks (point features)



1. **EKF_localization** ($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t, m$):

Prediction:

3. $$G_t = \frac{\partial g(u_t, \mu_{t-1})}{\partial \mu_{t-1}} = \begin{pmatrix} \frac{\partial x'}{\partial \mu_{t-1,x}} & \frac{\partial x'}{\partial \mu_{t-1,y}} & \frac{\partial x'}{\partial \mu_{t-1,\theta}} \\ \frac{\partial y'}{\partial \mu_{t-1,x}} & \frac{\partial y'}{\partial \mu_{t-1,y}} & \frac{\partial y'}{\partial \mu_{t-1,\theta}} \\ \frac{\partial \theta'}{\partial \mu_{t-1,x}} & \frac{\partial \theta'}{\partial \mu_{t-1,y}} & \frac{\partial \theta'}{\partial \mu_{t-1,\theta}} \end{pmatrix}$$ Jacobian of g w.r.t location

5. $$V_t = \frac{\partial g(u_t, \mu_{t-1})}{\partial u_t} = \begin{pmatrix} \frac{\partial x'}{\partial v_t} & \frac{\partial x'}{\partial \omega_t} \\ \frac{\partial y'}{\partial v_t} & \frac{\partial y'}{\partial \omega_t} \\ \frac{\partial \theta'}{\partial v_t} & \frac{\partial \theta'}{\partial \omega_t} \end{pmatrix}$$ Jacobian of g w.r.t control

1. $$Q_t = \begin{pmatrix} (\alpha_1 |v_t| + \alpha_2 |\omega_t|)^2 & 0 \\ 0 & (\alpha_3 |v_t| + \alpha_4 |\omega_t|)^2 \end{pmatrix}$$ Motion noise

2. $\bar{\mu}_t = g(u_t, \mu_{t-1})$ Predicted mean
3. $\bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + V_t Q_t V_t^T$ Predicted covariance (V maps Q into state space)

1. **EKF_localization** ($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t, m$):

Correction:

3. $$\hat{z}_t = \begin{pmatrix} \sqrt{(m_x - \bar{\mu}_{t,x})^2 + (m_y - \bar{\mu}_{t,y})^2} \\ \operatorname{atan2}(m_y - \bar{\mu}_{t,y}, m_x - \bar{\mu}_{t,x}) - \bar{\mu}_{t,\theta} \end{pmatrix}$$ Predicted measurement mean (depends on observation type)

5. $$H_t = \frac{\partial h(\bar{\mu}_t, m)}{\partial x_t} = \begin{pmatrix} \frac{\partial r_t}{\partial \bar{\mu}_{t,x}} & \frac{\partial r_t}{\partial \bar{\mu}_{t,y}} & \frac{\partial r_t}{\partial \bar{\mu}_{t,\theta}} \\ \frac{\partial \phi_t}{\partial \bar{\mu}_{t,x}} & \frac{\partial \phi_t}{\partial \bar{\mu}_{t,y}} & \frac{\partial \phi_t}{\partial \bar{\mu}_{t,\theta}} \end{pmatrix}$$ Jacobian of *h* w.r.t location

6. $$R_t = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_r^2 \end{pmatrix}$$

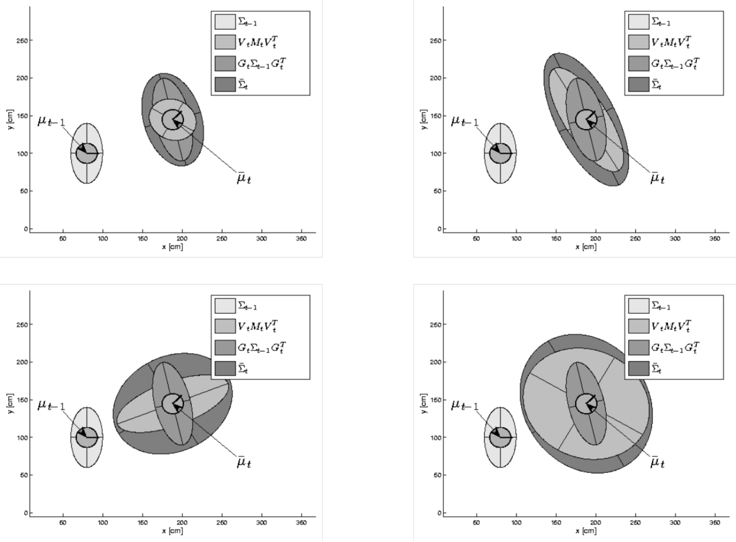
7. $S_t = H_t \bar{\Sigma}_t H_t^T + R_t$ Innovation covariance

8. $K_t = \bar{\Sigma}_t H_t^T S_t^{-1}$ Kalman gain

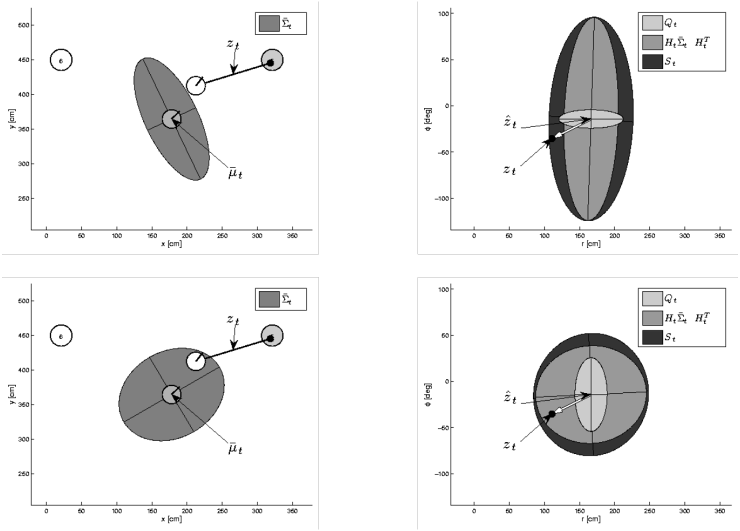
9. $\mu_t = \bar{\mu}_t + K_t(z_t - \hat{z}_t)$ Updated mean

10. $\Sigma_t = (I - K_t H_t)\bar{\Sigma}_t$ Updated covariance

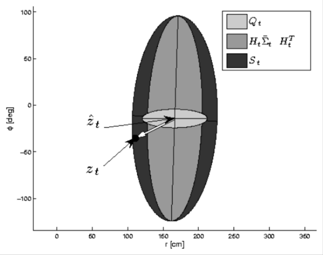
EKF Prediction Step

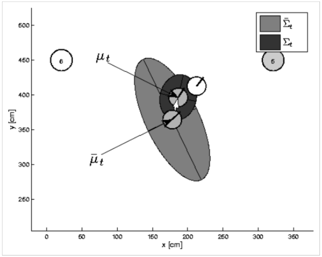


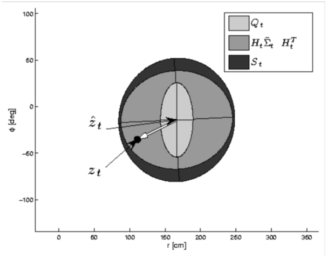
EKF Observation Prediction Step

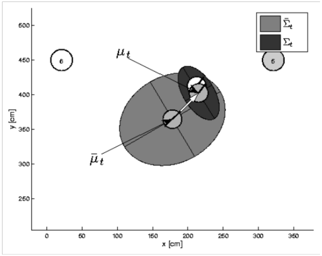


EKF Correction Step

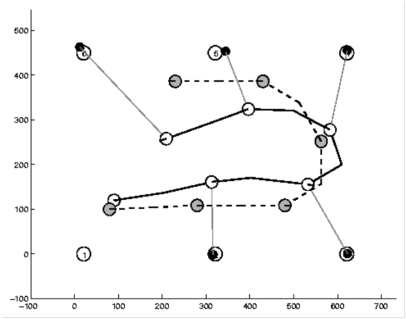


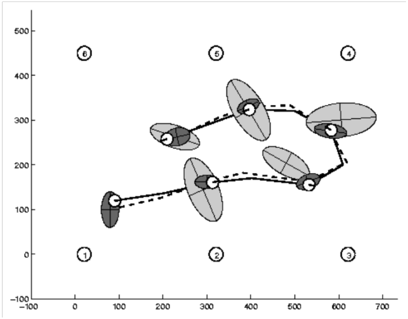




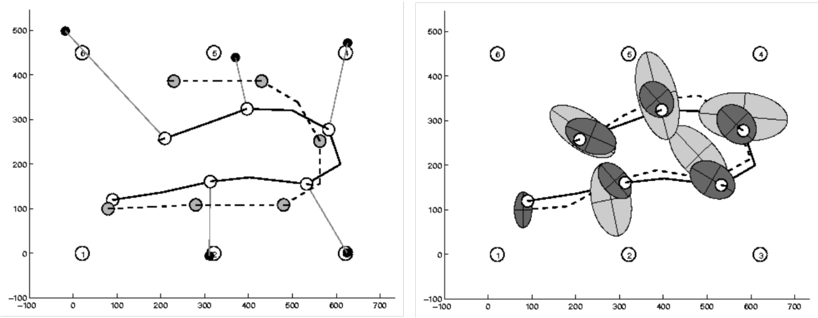


Estimation Sequence (1)

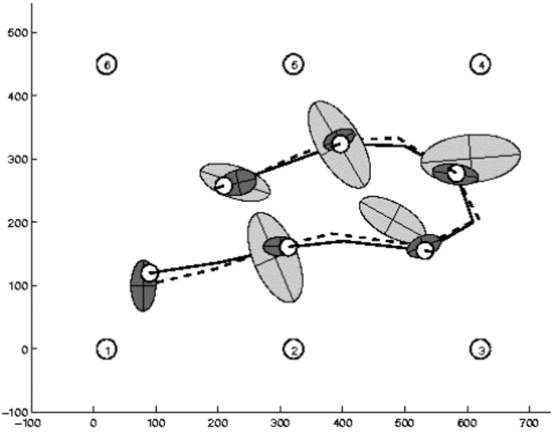




Estimation Sequence (2)



Comparison to GroundTruth



EKF Summary

- Ad-hoc solution to deal with non-linearities
- Performs local linearization in each step
- Works well in practice for moderate non-linearities
- Example: landmark localization
- There exist better ways for dealing with non-linearities such as the unscented Kalman filter called UKF

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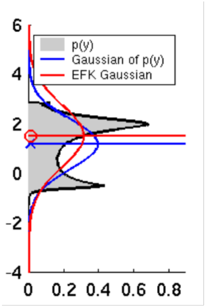
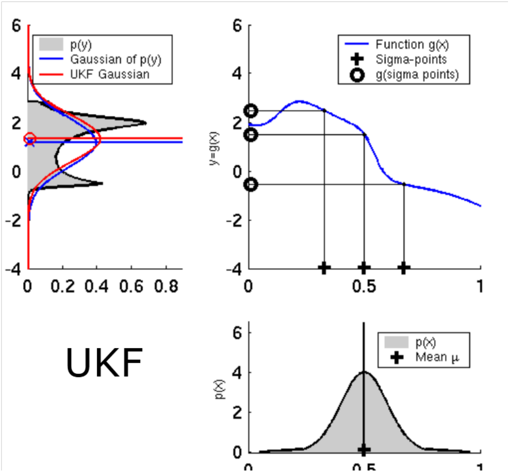
Linearization via Unscented Transform

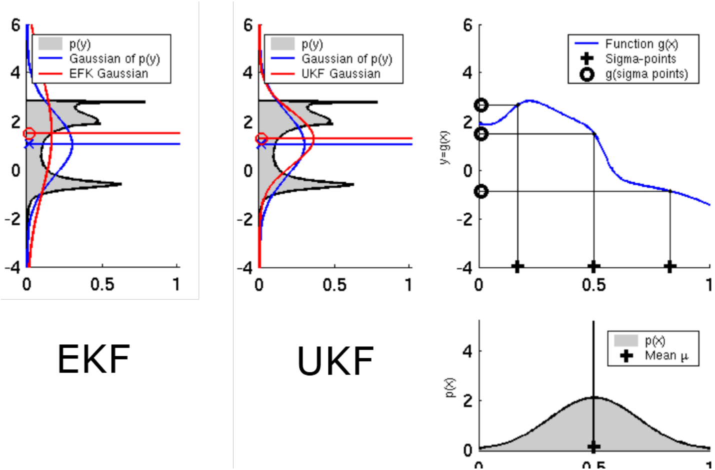


EKF

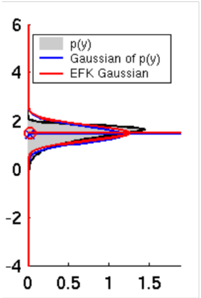


UKF

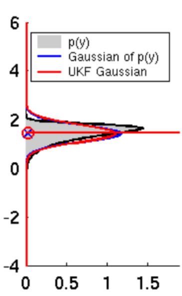
UKF Sigma-Point Estimate (2)



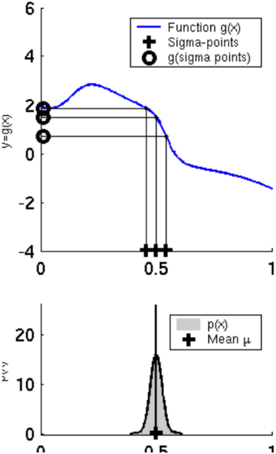
UKF Sigma-Point Estimate (3)



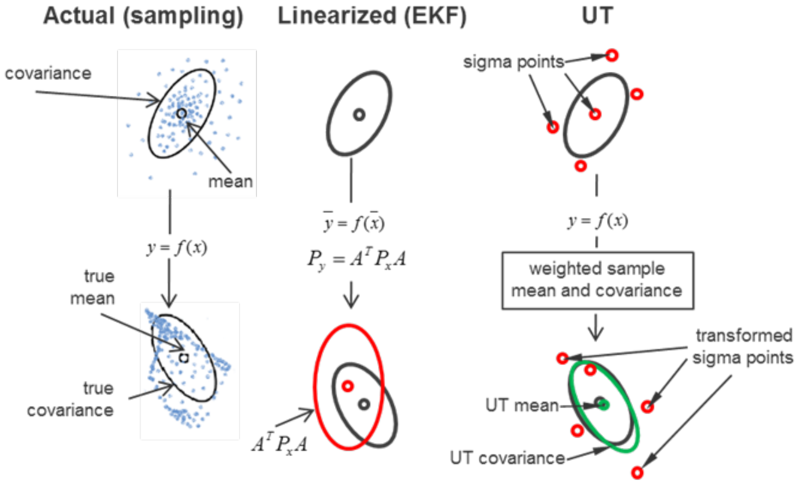
EKF



UKF



EKF vs. UKF



Actual (sampling)
Linearized (EKF)
UT
covariance
mean
sigma points
$y = f(x)$
$\bar{y} = f(\bar{x})$
$P_y = A^T P_x A$
true
mean
true
covariance
weighted sample
mean and covariance
$A^T P_x A$
transformed
sigma points
UT mean
UT covariance

UKF_localization ($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t, m$):

Prediction:

$$M_t = \begin{pmatrix} (\alpha_1 |v_t| + \alpha_2 |\omega_t|)^2 & 0 \\ 0 & (\alpha_3 |v_t| + \alpha_4 |\omega_t|)^2 \end{pmatrix}$$ Motion noise

$$Q_t = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_r^2 \end{pmatrix}$$ Measurement noise

$$\mu_{t-1}^a = \begin{pmatrix} \mu_{t-1}^T & (0\ 0)^T & (0\ 0)^T \end{pmatrix}$$ Augmented state mean

$$\Sigma_{t-1}^a = \begin{pmatrix} \Sigma_{t-1} & 0 & 0 \\ 0 & M_t & 0 \\ 0 & 0 & Q_t \end{pmatrix}$$ Augmented covariance

$$\chi_{t-1}^a = \begin{pmatrix} \mu_{t-1}^a & \mu_{t-1}^a + \gamma\sqrt{\Sigma_{t-1}^a} & \mu_{t-1}^a - \gamma\sqrt{\Sigma_{t-1}^a} \end{pmatrix}$$ Sigma points

$$\bar{\chi}_t^x = g\left(u_t + \chi_t^u, \chi_{t-1}^x\right)$$ Prediction of sigma points

$$\bar{\mu}_t = \sum_{i=0}^{2L} w_m^i \, \chi_{i,t}^x$$ Predicted mean

$$\bar{\Sigma}_t = \sum_{i=0}^{2L} w_c^i \left(\chi_{i,t}^x - \bar{\mu}_t\right)\left(\chi_{i,t}^x - \bar{\mu}_t\right)^T$$ Predicted covariance

UKF_localization (μ_{t-1}, Σ_{t-1}, u_t, z_t, m):

Correction:

$$\bar{Z}_t = h(\chi_t^x) + \chi_t^z$$ Measurement sigma points

$$\hat{z}_t = \sum_{i=0}^{2L} w_m^i \bar{Z}_{i,t}$$ Predicted measurement mean

$$S_t = \sum_{i=0}^{2L} w_c^i (\bar{Z}_{i,t} - \hat{z}_t)(\bar{Z}_{i,t} - \hat{z}_t)^T$$ Pred. measurement covariance

$$\Sigma_t^{x,z} = \sum_{i=0}^{2L} w_c^i (\bar{\chi}_{i,t}^x - \bar{\mu}_t)(\bar{Z}_{i,t} - \hat{z}_t)^T$$ Cross-covariance

$$K_t = \Sigma_t^{x,z} S_t^{-1}$$ Kalman gain

$$\mu_t = \bar{\mu}_t + K_t(z_t - \hat{z}_t)$$ Updated mean

$$\Sigma_t = \bar{\Sigma}_t - K_t S_t K_t^T$$ Updated covariance

1. **EKF_localization** ($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t, m$):

Correction:

3. $\hat{z}_t = \begin{pmatrix} \sqrt{(m_x - \bar{\mu}_{t,x})^2 + (m_y - \bar{\mu}_{t,y})^2} \\ \text{atan}2(m_y - \bar{\mu}_{t,y}, m_x - \bar{\mu}_{t,x}) - \bar{\mu}_{t,\theta} \end{pmatrix}$ Predicted measurement mean

5. $H_t = \frac{\partial h(\bar{\mu}_t, m)}{\partial x_t} = \begin{pmatrix} \frac{\partial r_t}{\partial \bar{\mu}_{t,x}} & \frac{\partial r_t}{\partial \bar{\mu}_{t,y}} & \frac{\partial r_t}{\partial \bar{\mu}_{t,\theta}} \\ \frac{\partial \varphi_t}{\partial \bar{\mu}_{t,x}} & \frac{\partial \varphi_t}{\partial \bar{\mu}_{t,y}} & \frac{\partial \varphi_t}{\partial \bar{\mu}_{t,\theta}} \end{pmatrix}$ Jacobian of *h* w.r.t location

6. $Q_t = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_r^2 \end{pmatrix}$

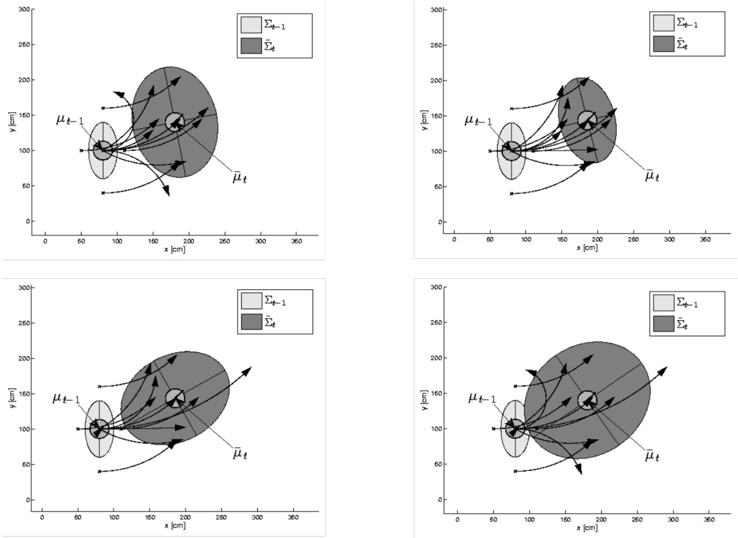
7. $S_t = H_t \bar{\Sigma}_t H_t^T + Q_t$ Pred. measurement covariance

8. $K_t = \bar{\Sigma}_t H_t^T S_t^{-1}$ Kalman gain

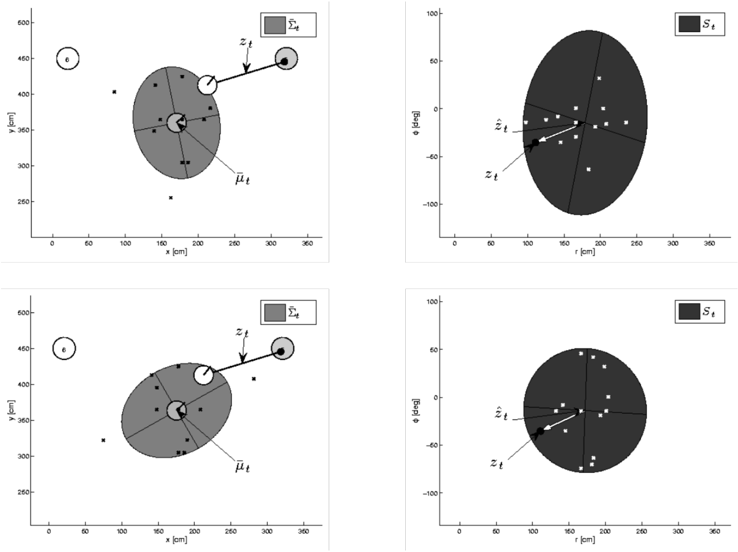
9. $\mu_t = \bar{\mu}_t + K_t (z_t - \hat{z}_t)$ Updated mean

10. $\Sigma_t = (I - K_t H_t) \bar{\Sigma}_t$ Updated covariance

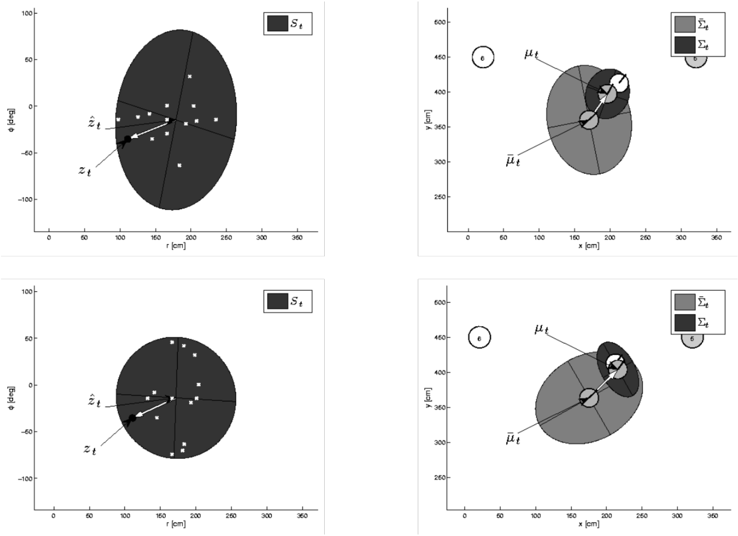
UKF Prediction Step



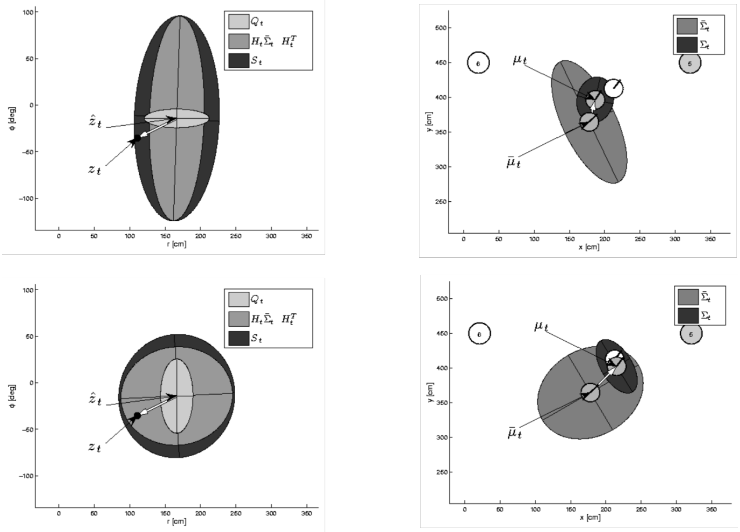
UKF Observation Prediction Step



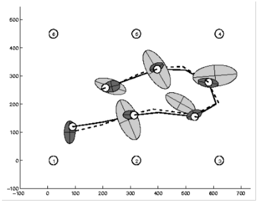
UKF Correction Step



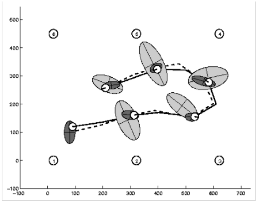
EKF Correction Step



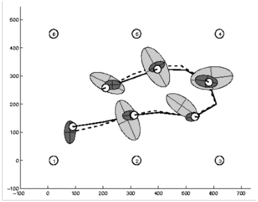
Estimation Sequence



EKF

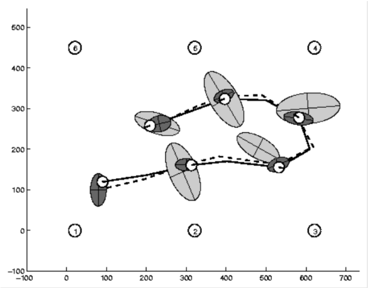


PF

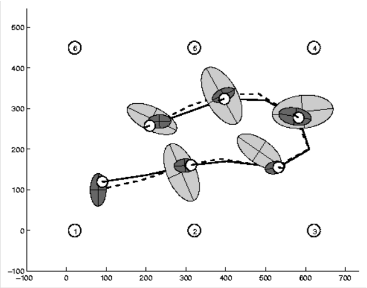


UKF

Estimation Sequence

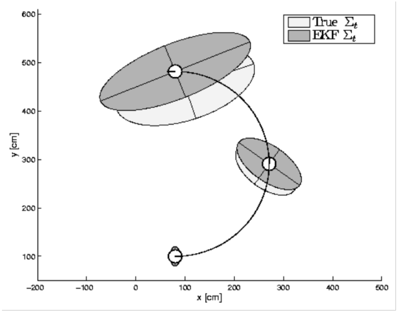


EKF

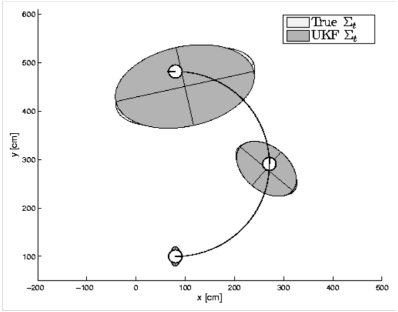


UKF

Prediction Quality



EKF



UKF

UKF Summary

- Highly efficient: Same complexity as EKF, with a constant factor slower in typical practical applications
- Better linearization than EKF: Accurate in first two terms of Taylor expansion (EKF only first term)
- Derivative-free: No Jacobians needed
- Still not optimal!

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非参数滤波

- 不同于高斯滤波，非参数滤波不依赖确定的后验函数，通过有限数量的值来近似后验，每一个值大致与状态空间的一个区域有关
  - 离散化：对状态空间进行分解，每一个值与状态空间的一个紧凑子区域的后验密度的累积概率相关
  - 采样：随机采样后验分布来近似状态空间
- 离散贝叶斯滤波，在连续空间下又称为直方图滤波(Histogram Filter)：将状态空间分解为有限多个区域，并用直方图表示后验，一个直方图分配给一个区域一个单一的累积概率
- 粒子滤波：用有限多个样本表示后验
- 非参数滤波的表达能力是以增加计算复杂性为代价的

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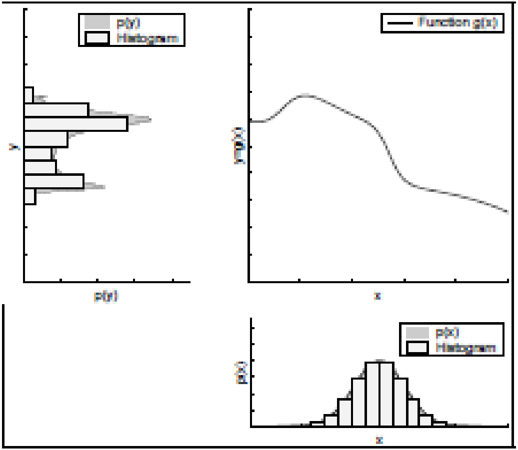
Discrete Bayes Filter Algorithm

1. Algorithm **Discrete_Bayes_filter**(*Bel(x),d*):
2. η=0
3. If *d* is a perceptual data item *z* then
4.   For all *x* do
5. $Bel'(x) = P(z \mid x) Bel(x)$
6. $\eta = \eta + Bel'(x)$
7.   For all *x* do
8. $Bel'(x) = \eta^{-1} Bel'(x)$
9. Else if *d* is an action data item *u* then
10.   For all *x* do
11. $Bel'(x) = \sum_{x'} P(x \mid u, x') \, Bel(x')$
12. Return *Bel'(x)*

连续情况



直方图滤波

- 直方图滤波将连续状态空间分解成有限区域:

$$\mathrm{dom}(X_t) = x_{1,t} \cup x_{2,t} \cup \cdots \cup x_{k,t}$$

其中 X_t 为描述机器人状态在时刻 t 的随机变量。函数 $\mathrm{dom}(X_t)$ 为状态空间

- 离散贝叶斯滤波为每一个区域 $x_{k,t}$ 分配一个概率 $p_{k,t}$
- 后验成为一个分段常数概率密度函数，它在区域 $x_{k,t}$ 中的每一个状态 x_t 分配了相同的概率

$$p(x_t) = \frac{p_{k,t}}{|x_{k,t}|}$$

其中 $|x_{k,t}|$ 为区域 $x_{k,t}$ 的绝对值

- 利用 $x_{k,t}$ 的平均状态进行探究

$$\hat{x}_{k,t} = |x_{k,t}|^{-1} \int_{x_{k,t}} x_t \, \mathrm{d}x_t$$

$$P(z_t \mid x_{k,t}) \approx P(z_t \mid \hat{x}_{k,t})$$

$$p(x_{k,t} \mid u_t, x_{i,t-1}) \approx \eta \, |x_{k,t}| \, P(\hat{x}_{k,t} \mid u_t, \hat{x}_{i,t-1})$$

Discrete Bayes Filter Summary

- Discrete filters are an alternative way for implementing Bayes Filters
- They are based on histograms for representing the density.
- They have huge memory and processing requirements
- Can easily recover from localization errors
- Their accuracy depends on the resolution of the grid.
- Special approximations need to be made to make this approach having dynamic memory and computational requirements.

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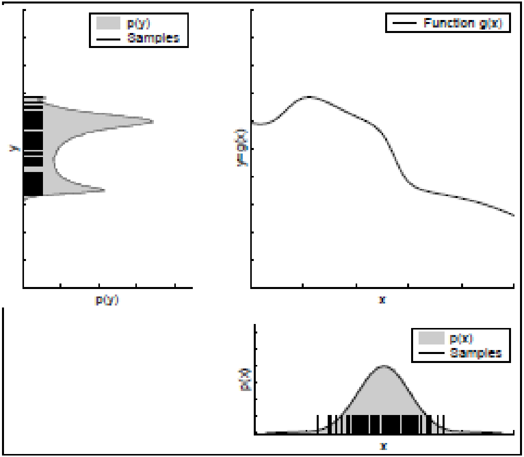
粒子表示法



Mathematical Description

- Set of weighted samples

$$S = \left\{ \left\langle s^{[i]}, w^{[i]} \right\rangle \mid i = 1, \ldots, N \right\}$$

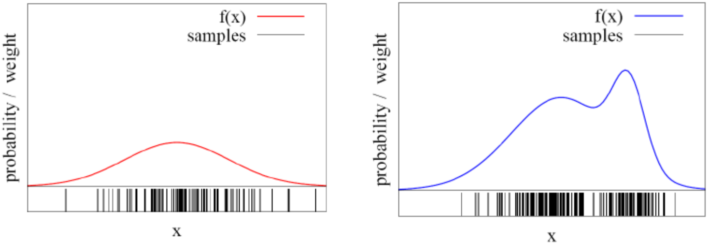
State hypothesis — Importance weight

- The samples represent the posterior

$$p(x) = \sum_{i=1}^{N} w_i \cdot \delta_{s^{[i]}}(x)$$

Function Approximation

- Particle sets can be used to approximate functions

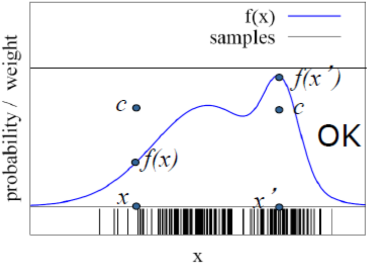


- The more particles fall into an interval, the higher the probability of that interval

- How to draw samples from a function/distribution?

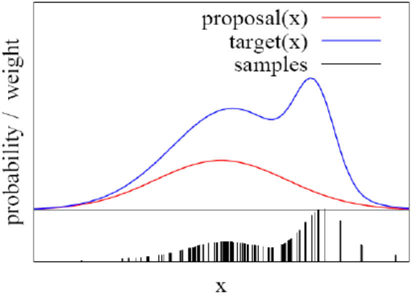
Rejection Sampling

- Let us assume that $f(x)<1$ for all x
- Sample x from a uniform distribution
- Sample c from [0,1]
- if $f(x) > c$ keep the sample
  otherwise reject the sample



Importance Sampling Principle

- We can even use a different distribution g to generate samples from f
- By introducing an importance weight w, we can account for the "differences between g and f"
- $w = f/g$
- f is called target
- g is called proposal
- Pre-condition:
  $f(x)>0 \rightarrow g(x)>0$



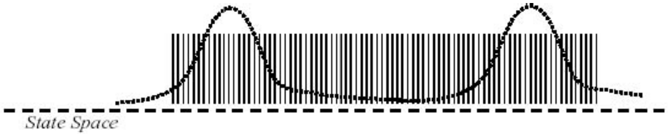
Importance Sampling

Target distribution f: $p(x \mid z_1, z_2, \ldots, z_n) = \frac{\prod_k p(z_k \mid x) \quad p(x)}{p(z_1, z_2, \ldots, z_n)}$

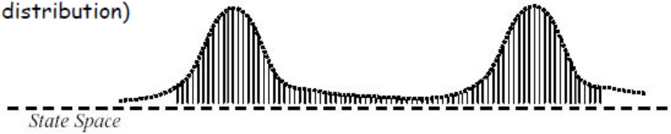
Sampling distribution g: $p(x \mid z_l) = \frac{p(z_l \mid x) p(x)}{p(z_l)}$

Importance weights w: $\frac{f}{g} = \frac{p(x \mid z_1, z_2, \ldots, z_n)}{p(x \mid z_l)} = \frac{p(z_l) \prod_{k \neq l} p(z_k \mid x)}{p(z_1, z_2, \ldots, z_n)}$

Importance Sampling with Resampling



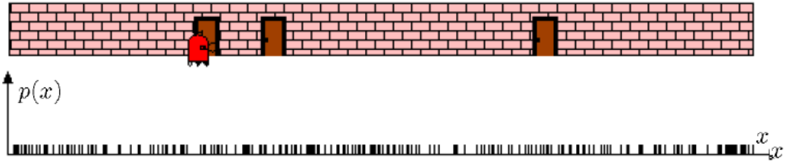
Sample from prior belief q(x) (for instance, the uniform distribution)



Compute importance weights, w(x) = p(x) /q(x)

Resample particles according to importance weights to get p(x)
Samples with high weights chosen many times; density reflects pdf

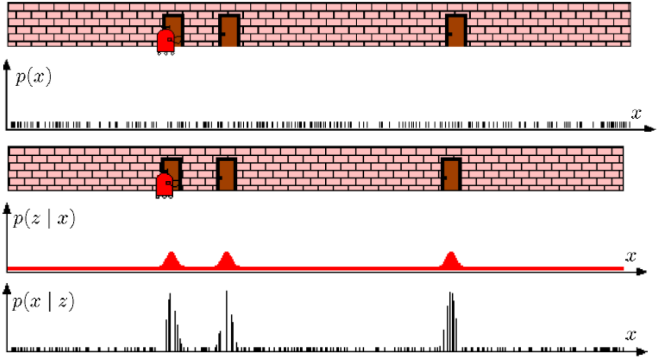
Particle Filters



Sensor Information: Importance Sampling

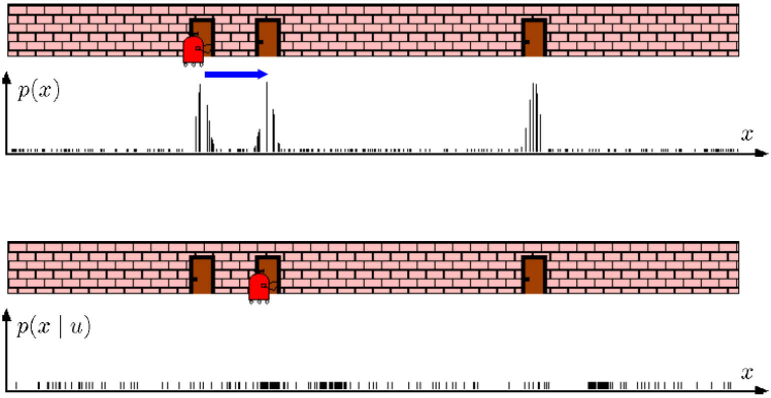
$$Bel(x) \leftarrow \alpha\ p(z \mid x)\ Bel^{-}(x)$$

$$w \leftarrow \frac{\alpha\ p(z \mid x)\ Bel^{-}(x)}{Bel^{-}(x)} = \alpha\ p(z \mid x)$$



Robot Motion

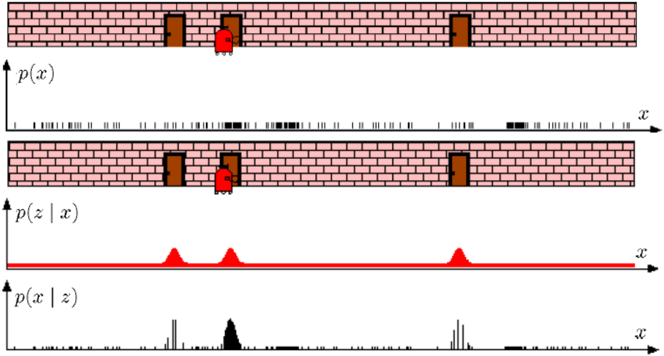
$$Bel^-(x) \leftarrow \int p(x \mid u, x')\, Bel(x')\, \mathrm{d}x'$$



Sensor Information: Importance Sampling

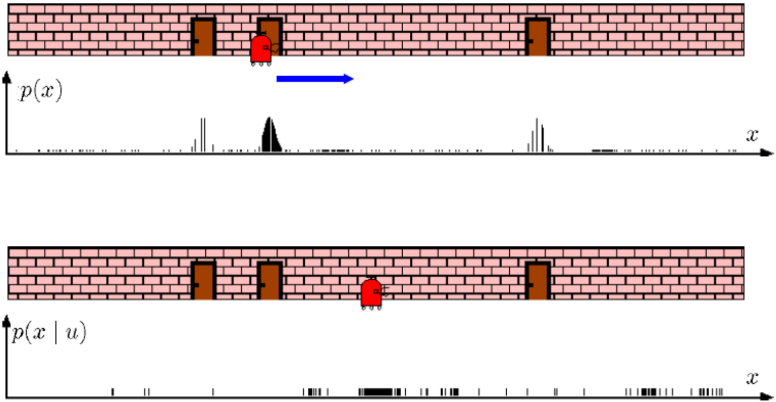
$$Bel(x) \leftarrow \alpha\, p(z \mid x)\, Bel^{-}(x)$$

$$w \leftarrow \frac{\alpha\, p(z \mid x)\, Bel^{-}(x)}{Bel^{-}(x)} = \alpha\, p(z \mid x)$$



Robot Motion

$$Bel^{-}(x) \leftarrow \int p(x \mid u, x') \, Bel(x') \, \mathrm{d}x'$$



Particle Filter Algorithm

- Sample the next generation for particles using the proposal distribution
- Compute the importance weights :

  weight = target distribution / proposal distribution

- Resampling: "Replace unlikely samples by more likely ones"

Particle Filter Algorithm

1. Algorithm **particle_filter**(S_{t-1}, u_{t-1} z_t):
2. $S_t = \emptyset, \quad \eta = 0$
3. **For** $i = 1 \dots n$ — ***Generate new samples***
4. Sample index $j(i)$ from the discrete distribution given by w_{t-1}
5. Sample x_t^i from $p(x_t \mid x_{t-1}, u_{t-1})$ using $x_{t-1}^{j(i)}$ and u_{t-1}
6. $w_t^i = p(z_t \mid x_t^i)$ — ***Compute importance weight***
7. $\eta = \eta + w_t^i$ — ***Update normalization factor***
8. $S_t = S_t \cup \{< x_t^i, w_t^i >\}$ — ***Insert***
9. **For** $i = 1 \dots n$
10. $w_t^i = w_t^i / \eta$ — ***Normalize weights***

Particle Filter Algorithm

$$Bel(x_t) = \eta\, p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, u_t)\, Bel(x_{t-1})\, dx_{t-1}$$

draw x^i_{t-1} from $Bel(x_{t-1})$

draw x^i_t from $p(x_t \mid x^i_{t-1}, u_t)$

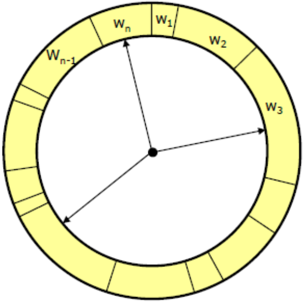
Importance factor for x^i_t:

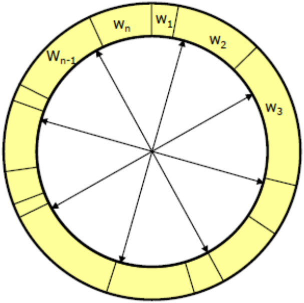
$$w^i_t = \frac{\text{target distribution}}{\text{proposal distribution}} = \frac{\eta\, p(z_t \mid x_t)\, p(x_t \mid x_{t-1}, u_t)\, Bel\,(x_{t-1})}{p(x_t \mid x_{t-1}, u_t)\, Bel\,(x_{t-1})} \propto p(z_t \mid x_t)$$

Resampling

- **Given**: Set S of weighted samples.
- **Wanted** : Random sample, where the probability of drawing x_i is given by w_i.
- Typically done n times with replacement to generate new sample set S'.

Resampling





- Roulette wheel
- Binary search, n log n

- Stochastic universal sampling
- Systematic resampling
- Linear time complexity
- Easy to implement, low variance

Resampling Algorithm

Algorithm **systematic_resampling**(*S*,*n*):

$S' = \emptyset, c_1 = w^1$

For $i = 2 \ldots n$ — ***Generate cdf***

$\quad c_i = c_{i-1} + w^i$

$u_1 \sim U[0, n^{-1}], i = 1$ — ***Initialize threshold***

For $j = 1 \ldots n$ — ***Draw samples …***

$\quad$ **While** ($u_j > c_i$) — ***Skip until next threshold reached***

$\quad\quad i = i + 1$

$\quad S' = S' \cup \{ < x^i, n^{-1} > \}$ — ***Insert***

$\quad u_{j+1} = u_j + n^{-1}$ — ***Increment threshold***

Return S'

Also called **stochastic universal sampling**

Particle Filters Summary

- Particle filters are an implementation of recursive Bayesian filtering
- They represent the posterior by a set of weighted samples
- They can model non-Gaussian distributions
- Proposal to draw new samples
- Weight to account for the differences between the proposal and the target
- Monte Carlo filter, Survival of the fittest, Condensation, Bootstrap filter

小结

- 概率机器人：采用概率方法显示的处理机器人中的不确定性
- 贝叶斯滤波是动态环境下状态估计的主要概率工具
- 当观察模型和动态模型都是线性的，并且不确定性为高斯分布，则贝叶斯滤波可以简化为卡尔曼滤波
- 对于非线性模型的情况，可以采用扩展卡尔曼滤波 (EKF) 或无迹卡尔曼滤波 (UKF)，通常 UKF 的效果更好
- 对于高度非线性和非高斯分布的情况，可以采用粒子滤波，实际效果比 KF, EKF, UKF 效果好很多，但计算代价也更大