Partitioning 3-uniform hypergraphs

Jie Ma
Xingxing Yu*

School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332-0160
USA

Abstract

Bollobás and Thomason conjectured that the vertices of any r-uniform hypergraph with m edges can be partitioned into r sets so that each set meets at least $rm/(2r-1)$ edges. For $r = 3$, Bollobás, Reed and Thomason proved the lower bound $(1-1/e)m/3 \approx 0.21m$, which was improved to $(5/9)m$ by Bollobás and Scott (while the conjectured bound is $0.6m$). In this paper, we show that any 3-uniform hypergraph with m edges can be partitioned into 3 sets, each of which meets at least $0.65m - O(m^{6/7})$ edges. In particular, this Bollobás-Thomason conjecture holds when $r = 3$ and m is large.

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1 Introduction

Let G be a graph or hypergraph, and let $S, T \subseteq V(G)$ with $S \cap T = \emptyset$. We write $e_G(S) := |\{e \in E(G) : e \subseteq S\}|$, $e_G(S,T) := |\{e \in E(G) : e \cap S \neq \emptyset \neq e \cap T\}|$, and $d_G(S) := |\{e \in E(G) : e \cap S \neq \emptyset\}|$. When understood, the reference to G in the subscript may be dropped.

An example of classical graph partitioning problems is the well known *Maximum Bipartite Subgraph Problem*: Given a graph G find a partition V_1, V_2 of $V(G)$ maximizing $e(V_1, V_2)$. There is an extensive body of work on this problem, from various perspectives, see [10]. Note that the Maximum Bipartite Subgraph Problem asks for a partition of an input graph that optimizes only one quantity.

Any problem that asks for partitions of graphs or hypergraphs to optimize several quantities simultaneously is said to be a *judicious* partitioning problem. The *Bottleneck Bipartition Problem* is one such example: Given a graph G find a partition V_1, V_2 of $V(G)$ minimizing $\max\{e(V_1), e(V_2)\}$, or equivalently, maximizing $\min\{d(V_1), d(V_2)\}$ (since $d(V_i) = |E(G)| - e(V_{3-i})$ for $i = 1, 2$). This problem was raised by Entringer, and is shown to be NP-hard in [12]. In [1] it is shown that the Maximum Bipartite Subgraph Problem and the Bottleneck Bipartition Problem are related. We refer the reader to [6,11] for other interesting partitioning problems.

Note that if V_1, V_2 is a partition of a graph G maximizing $e(V_1, V_2)$, then each $v \in V_i$ has at least as many neighbors in V_{3-i} as in V_i. So $e(V_1, V_2) \geq 2e(V_i)$ for $i = 1, 2$, which implies $e(V_i) \leq m/3$, where m is the number of edges in G. Hence $d(V_i) \geq m - m/3 = 2m/3$ for $i = 1, 2$. In an attempt to extend this to hypergraphs, Bollobás and Thomason made the following conjecture; see [5].

Conjecture 1.1 (*Bollobás and Thomason*) *For any integer $r \geq 3$, the vertex set of any r-uniform hypergraph with m edges admits a partition $V_1, \ldots, V_r$ such that for $i = 1, \ldots, m$,*

$$d(V_i) \geq \frac{r}{2r-1} m.$$

The conjectured bound is the best possible for complete r-uniform graphs on $2r-1$ vertices. To see this, note that such a graph has $m = \binom{2r-1}{r}$ edges, and any r-partition of such a graph has a partition set with just one vertex, which meets $\binom{2r-2}{r-1}$ edges. Bollobás, Reed and Thomason [3] proved that every 3-uniform hypergraph with m edges has a partition V_1, V_2, V_3 such that $d(V_i) \geq (1 - 1/e)m \approx 0.21m$ (here e is the base of the natural logarithm). In [5], this bound is improved to $(5/9)m$ by Bollobás and Scott. Note that the bound for $r = 3$ in Conjecture 1.1 is $0.6m$. Halesgrave [7] extended an idea of Bollobás and Scott in [5] and solved the $r = 3$ case completely. (Bollobás informed us that Halesgrave actually did it in 2006.) For large graphs, this bound may be improved. In this paper, we prove the following result, which implies Conjecture 1.1 for $r = 3$ and m large.

Theorem 1.2 *Every 3-uniform hypergraph with m edges has a partition into sets V_1, V_2, V_3 such that for $i = 1, 2, 3$,*

$$d(V_i) \geq 0.65m - o(m).$$

We use an approach developed by Bollobás and Scott [4,6]. The idea is to partition the large degree vertices first, and then partition the remaining vertices using a random process.

The key is to find appropriate probabilities for this random process which result in the desired bounds on the expectations of $d(V_i)$. An application of Azuma-Hoeffding inequality then allows us to bound the deviations from these expectations.

We organize our paper as follows. In Section 2, we first state two lemmas, Lemmas 2.1 and 2.2, which assert that certain inequalities hold. We then use these two lemmas to prove Lemma 2.3 which, in turn, is used to prove Theorem 1.2. In Lemma 2.3, we need to bound three quantities simultaneously. In Section 3, we prove two lemmas that can be used to bound two quantities simultaneously. These lemmas will then be used in Section 4 to prove Lemmas 2.1 and 2.2. We conclude with Section 5 by mentioning two related problems.

2 Proof of Theorem 1.2

As mentioned in the introduction, we need two lemmas which provide inequalities needed for our proof. The meaning of the parameters in these lemmas will be clear from the proof of Lemma 2.3; each is related to the number of edges of a certain type. The first lemma tries to bound three quantities $f_i(p_i)$, $i = 1, 2, 3$. It says that, under certain conditions, there exist p_i such that either all three functions are bounded from above, or can be made equal. We use $\mathbb{R}^+$ to denote the set of nonnegative reals.

Lemma 2.1 *Let $b_{ij}, x_i, a_i, c \in \mathbb{R}^+$, $1 \le i \ne j \le 3$, such that $b_{ij} = b_{ji}$, $b_{ij} \ge \max\{2x_i, 2x_j\}$, and $b_{12} + b_{23} + b_{31} + x_1 + x_2 + x_3 + a_1 + a_2 + a_3 + c = 1$. For any permutation ijk of $\{1, 2, 3\}$, let*

$$f_i := (1 - p_i)(b_{jk} + x_j + x_k) + (1 - p_i)^2(a_j + a_k) + (1 - p_i)^3 c.$$

Then there exists $p_1, p_2, p_3 \in [0, 1]$ with $p_1 + p_2 + p_3 = 1$ such that

(i) $f_i \le 0.35$ for $i = 1, 2, 3$, or

(ii) $f_1 = f_2 = f_3$ and $p_i \in (0, 1)$ for $i = 1, 2, 3$.

The second lemma (when combined with Lemma 2.1) deals with the case $c = 0$ of Lemma 2.3.

Lemma 2.2 *Let $a_i, x_i, b_{ij} \in \mathbb{R}^+$, $1 \le i \ne j \le 3$, such that $b_{ij} = b_{ji}$, $b_{ij} \ge \max\{2x_i, 2x_j\}$ and $b_{12} + b_{23} + b_{31} + x_1 + x_2 + x_3 + a_1 + a_2 + a_3 = 1$. For any permutation ijk of $\{1, 2, 3\}$, let*

$$f_k := (1 - p_k)(b_{ij} + x_i + x_j) + (1 - p_k)^2(a_i + a_j)$$

Suppose there exist $p_1, p_2, p_3 \in (0, 1)$ such that $p_1 + p_2 + p_3 = 1$ and $f_1 = f_2 = f_3$. Then for such p_1, p_2, p_3, we have $f_k \le 0.35$ for $k = 1, 2, 3$.

The proofs of the above two lemmas will be the context of Sections 3 and 4. We can now prove the main lemma.

Lemma 2.3 *Let $b_{ij}, x_i, a_i, c \in \mathbb{R}^+$, $1 \le i \ne j \le 3$, such that $b_{ij} = b_{ji}$, $b_{ij} \ge \max\{2x_i, 2x_j\}$ and $b_{12} + b_{23} + b_{31} + x_1 + x_2 + x_3 + a_1 + a_2 + a_3 + c = 1$. Then there exist $p_1, p_2, p_3 \in [0, 1]$ with $p_1 + p_2 + p_3 = 1$ such that for any $\{i, j, k\} = \{1, 2, 3\}$,*

$$f_i := (1 - p_i)(b_{jk} + x_j + x_k) + (1 - p_i)^2(a_j + a_k) + (1 - p_i)^3 c \le 0.35.$$

Proof. By Lemma 2.1, we may assume that there exist $p_1, p_2, p_3 \in (0,1)$ with $p_1 + p_2 + p_3 = 1$ such that $f_1 = f_2 = f_3$. Let $\mathscr{D}$ be the set of points $(a_1, a_2, a_3, x_1, x_2, x_3, b_{12}, b_{23}, b_{31}, c, p_1, p_2, p_3) \in [0,1]^{13}$ satisfying

$$\begin{aligned}
&b_{ij} \geq \max\{2x_i, 2x_j\}, \\
&b_{12} + b_{23} + b_{31} + x_1 + x_2 + x_3 + a_1 + a_2 + a_3 + c = 1, \\
&p_1 + p_2 + p_3 = 1, \\
&p_i \in [0,1] \text{ for } i = 1,2,3, \text{ and} \\
&f_1 = f_2 = f_3.
\end{aligned}$$

Note that $\mathscr{D} \neq \emptyset$ and $\mathscr{D}$ is a compact subset of $[0,1]^{13}$. So $f_1(\mathbf{v})$ has an absolute maximum over $\mathscr{D}$. Let $\mathscr{M}$ denote all $\mathbf{v} \in \mathscr{D}$ for which $f_1(\mathbf{v})$ is the maximum of f_1 over $\mathscr{D}$. It suffices to show that there is some $\mathbf{v} \in \mathscr{M}$ such that $f_i(\mathbf{v}) \leq 0.35$ for $i = 1,2,3$. Let

$$\mathbf{v} := (a_1, a_2, a_3, x_1, x_2, x_3, b_{12}, b_{23}, b_{31}, c, p_1, p_2, p_3) \in \mathscr{M}.$$

We claim that $\mathbf{v}$ may be chosen so that $c = 0$. For, suppose $c \neq 0$. Define

$$\mathbf{v}' := (a_1 + p_1 c, a_2 + p_2 c, a_3 + p_3 c, x_1, x_2, x_3, b_{12}, b_{23}, b_{31}, 0, p_1, p_2, p_3).$$

It is easy to check that $\mathbf{v}' \in \mathscr{D}$ and $f_i(\mathbf{v}') = f_i(\mathbf{v})$ for $i = 1,2,3$. Since $\mathbf{v} \in \mathscr{M}$, we have $\mathbf{v}' \in \mathscr{M}$. Now it follows from Lemma 2.2 that for any $i = 1,2,3$, $f_i(\mathbf{v}) = f_i(\mathbf{v}') \leq 0.35$. ∎

We also need the following lemma, which is easy to prove. Let G be a graph (multiple edges allowed) and let $w : E(G) \to \mathbb{R}^+$. For any $S \subseteq V(G)$, we write $w(S) = \sum_{e \subseteq S} w(e)$. For any $S, T \subseteq V(G)$ with $S \cap T = \emptyset$, we use (S,T) to denote the set of edges st with $s \in S$ and $t \in T$; and write $w(S,T) = \sum_{e \in (S,T)} w(e)$.

Lemma 2.4 *Let G be a graph and let $w : E(G) \to \mathbb{R}^+$, and let $V(G) = V_1 \cup \ldots \cup V_k$ be a k-partition minimizing $\sum_{i=1}^k w(V_i)$. Then for any $1 \leq i \neq j \leq k$*

$$w(V_i, V_j) \geq \max\{2w(V_i), 2w(V_j)\}.$$

Proof. For any $v \in V_i$ and for any $j \in \{1, \ldots, k\} \setminus \{i\}$, we have

$$\sum_{\{uv \in E(G): u \in V_i - v\}} w(uv) \leq \sum_{\{uv \in E(G): u \in V_j\}} w(uv).$$

Summing over $v \in V_i$, we get $2w(V_i) \leq w(V_i, V_j)$. ∎

Finally we need the Azuma-Hoeffding inequality [2, 8] to bound deviations. We use the version given in [4].

Lemma 2.5 *Let $Z_1, \ldots, Z_n$ be independent random variables taking values in $\{1, \ldots, k\}$, let $Z := (Z_1, \ldots, Z_n)$, and let $f : \{1, \ldots, k\}^n \to \mathbb{N}$ such that $|f(Y) - f(Y')| \leq c_i$ for any $Y, Y' \in \{1, \ldots, k\}^n$ which differ only in the ith coordinate. Then for any $z > 0$,*

$$\mathbb{P}\left(f(Z) \geq \mathbb{E}(f(Z)) + z\right) \leq \exp\left(\frac{-z^2}{2\sum_{i=1}^k c_i^2}\right),$$

$$\mathbb{P}\left(f(Z) \le \mathbb{E}(f(Z)) - z\right) \le \exp\left(\frac{-z^2}{2\sum_{i=1}^{k} c_i^2}\right).$$

Now Theorem 1.2 is a consequence of the following result.

Theorem 2.6 *Let G be a 3-uniform hypergraph with m edges. Then there is a partition $V(G) = V_1 \cup V_2 \cup V_3$ such that, for $i = 1, 2, 3$,*

$$d(V_i) \ge 0.65m - O(m^{6/7}).$$

Proof. We may assume that G is connected; as otherwise, we may simply consider the individual components. Hence every vertex of G has positive degree.

Let $V(G) = \{v_1, \ldots, v_n\}$ such that $d(v_1) \ge d(v_2) \ge \ldots \ge d(v_n)$. Let $U_1 := \{v_1, \ldots, v_t\}$ and $U_2 := V(G) \setminus U_1$, with $t = \lfloor m^\alpha \rfloor$ and $0 < \alpha < 1/3$. Since $m \le \binom{n}{3}$ and $t < m^{1/3}$, we have $t \le n-2$ for $n \ge 3$ (by a simple calculation). Moreover,

$$m^\alpha d(v_{t+1}) \le (1+t)d(v_{t+1}) \le \sum_{i=1}^{t+1} d(v) < \sum_{v \in V(G)} d(v) = 3m;$$

so $d(v_{t+1}) < 3m^{1-\alpha}$. Hence

$$\sum_{i=t+1}^{n} d(v_i)^2 < 3m^{1-\alpha} \sum_{i=1}^{n} d(v_i) = 9m^{2-\alpha}.$$

For any partition $U_1 = X_1 \cup X_2 \cup X_3$ and for $1 \le i \ne j \le 3$, define

$$\begin{aligned}
x_i &= |\{e \in E(G) : |e \cap X_i| = 2, |e \cap U_2| = 1\}|,\\
a_i &= |\{e \in E(G) : |e \cap X_i| = 1, |e \cap U_2| = 2\}|,\\
b_{ij} &= |\{e \in E(G) : |e \cap X_i| = |e \cap X_j| = |e \cap U_2| = 1\}|,\\
c &= |\{e \in E(G) : |e \cap U_2| = 3\}|.
\end{aligned}$$

Then $m = e(U_1) + b_{12} + b_{23} + b_{13} + x_1 + x_2 + x_3 + a_1 + a_2 + a_3 + c$.

By Lemma 2.4, we may choose the partition $U_1 = X_1 \cup X_2 \cup X_3$ such that for $1 \le i \ne j \le 3$,

$$b_{ij} \ge \max\{2x_i, 2x_j\}.$$

For $1 \le i \le 3$, assign color i to the vertices in X_i. We extend the coloring to U_2 as follows: each vertex in U_2 is independently colored i with probability p_i for $1 \le i \le 3$, where $p_1 + p_2 + p_3 = 1$ and p_i will be determined by an application of Lemma 2.3.

For $i = 1, 2, 3$, let V_i be the vertices with color i, and let

$$y_i = |\{e \in E(G) : e \subseteq U_1 \text{ and } e \cap X_i \ne \emptyset\}.$$

Then, for any permutation ijk of $\{1, 2, 3\}$,

$$\mathbb{E}(d(V_i)) = b_{ij} + b_{ik} + x_i + a_i + p_i(b_{jk} + x_j + x_k) + (1 - (1-p_i)^2)(a_j + a_k) + (1 - (1-p_i)^3)c + y_i.$$

Thus

$$f_i := m - \mathbb{E}(d(V_i)) - e(U_1) + y_i = (1-p_i)(b_{jk} + x_j + x_k) + (1-p_i)^2(a_j + a_k) + (1-p_i)^3 c,$$

and

$$\alpha := m - e(U_1) = b_{12} + b_{23} + b_{31} + a_1 + a_2 + a_3 + x_1 + x_2 + x_3 + c.$$

By applying Lemma 2.3 (with $b_{ij}/\alpha, a_i/\alpha, x_i/\alpha, c/\alpha$ as b_{ij}, a_i, x_i, c, respectively), there exist $p_i \in [0,1]$ with $p_1 + p_2 + p_3 = 1$ such that for $1 \le i \le 3$, $f_i/\alpha \le 0.35$. So

$$f_i \le 0.35(m - e(U_1)).$$

Hence

$$\mathbb{E}(d(V_i)) = m - f_i - e(U_1) + y_i \ge 0.65m - 0.65e(U_1) + y_i.$$

Changing the color of any v_j, $t+1 \le j \le n$, affects $d(V_i)$ by at most $d(v_j)$. So by Lemma 2.5, we have for $i = 1, 2, 3$,

$$\mathbb{P}(d(V_i) < \mathbb{E}(d(V_i)) - z) \le \exp\left(\frac{-z^2}{2\sum_{j=t+1}^{n} d(v_j)^2}\right) < \exp\left(\frac{-z^2}{18m^{2-\alpha}}\right).$$

Taking $z = \sqrt{18 \ln 3}\, m^{1-\alpha/2}$, we have for $i = 1, 2, 3$,

$$\mathbb{P}(d(V_i) < \mathbb{E}(d(V_i)) - z) < 1/3.$$

Therefore, there exists a partition $V(G) = V_1 \cup V_2 \cup V_3$ such that for $i = 1, 2, 3$,

$$d(V_i) \ge \mathbb{E}(d(V_i)) - z \ge 0.65m - 0.65e(U_1) + y_i - z \ge 0.65m - 0.65e(U_1) - z.$$

Since $|U_1| = t \le m^{\alpha}$, $e(U_1) = O(m^{3\alpha})$. So

$$0.65e(U_1) + z = O(m^{3\alpha}) + \sqrt{18 \ln 2}\, m^{1-\alpha/2}.$$

Choosing $\alpha = \frac{2}{7}$ to minimize $\max\{3\alpha, 1 - \alpha/2\}$, we have the desired bound. ∎

3 Bounding two quantities

In this section, we prove two lemmas to be used in our proofs of Lemmas 2.1 and 2.2. The first is a slight variation of the main lemma in [4]. The difference is that here we relax the constraint $z \ge \max\{2x, 2y\}$ in [4] to $z \ge x + y$; as a consequence we have a weaker bound. Our proof mimics that in [4], where a more general result is proved.

Lemma 3.1 *Let $a, b, x, y, z, e \in \mathbb{R}^+$ such that $z \ge x + y$ and $a + b + x + y + z + e = 1$. Then there exists $p \in (0,1)$ such that*

$$p^2 a + px + p^3 e \le 1/7, \ \textit{and}\ (1-p)^2 b + (1-p)y + (1-p)^3 e \le 1/7.$$

Proof. For convenience, let

$$f_1 := p^2a + px + p^3e, \text{ and } f_2 := (1-p)^2b + (1-p)y + (1-p)^3e.$$

Note that f_1 and f_2 are continuous functions of p on $[0,1]$. We may assume that

(1) $a + x + e > 0$ and $b + y + e > 0$.

Otherwise, by symmetry, we may assume $a+x+e = 0$. Then $a = x = e = 0$ and $f_1 = 0 < 1/7$. Since f_2 is a continuous function of p, there exist $0 < \epsilon < 1$ such that $|f_2(\epsilon) - f_2(1)| < 1/7$. Thus, because $f_2(1) = 0$, we have $f_2(\epsilon) < 1/7$. So letting $p = \epsilon$, the assertion of the lemma holds. Thus we may assume (1).

By (1), $f_1(1) = a+x+e > 0$ and $f_2(0) = b+y+e > 0$. Therefore, since $f_1(0) = 0 = f_2(1)$ and because $f_1(p)$ (respectively, $f_2(p)$) is increasing (respectively, decreasing) and continuous on $[0,1]$, we have

(2) for any a, b, x, y, z, e satisfying (1), there exists a unique $p \in (0,1)$ such that $f_1 = f_2$.

We call $\mathbf{v} := (a, b, x, y, z, e, p) \in [0,1]^7$ a *satisfying point* if $a, b, x, y, z, e, p \in \mathbb{R}^+$, $a+b+x+y+z+e = 1$, $z \geq x+y$, $p \in [0,1]$, and $f_1 = f_2$. (In fact, $p \in (0,1)$ by (2).) Let $\mathscr{D}$ denote the set of all satisfying points. Note that $\mathscr{D}$ is a compact subset of $[0,1]^7$. A point in $\mathscr{D}$ is said to be a *maximal point* if the value of f_1 at that point is the maximum of f_1 over $\mathscr{D}$. Let $\mathscr{M}$ be the set of maximal points, which is nonempty since $\mathscr{D} \neq \emptyset$ (by (1) and (2)) and $\mathscr{D}$ is compact.

It then suffices to show that $f_1(\mathbf{v}) \leq 1/7$ for any $\mathbf{v} \in \mathscr{M}$. We do so by looking for a special maximal point. First, we show that

(3) there exists $(a, b, x, y, z, e, p) \in \mathscr{M}$ such that $e = 0$, $z = x + y$, and $ab = 0$.

Let $\mathbf{v} := (a, b, x, y, z, e, p) \in \mathscr{M}$. If $e > 0$, then let $\mathbf{v}' := (a + pe, b + (1-p)e, x, y, z, 0, p)$. It is easy to check that $\mathbf{v}' \in \mathscr{D}$ and $f_i(\mathbf{v}') = f_i(\mathbf{v})$ for $i = 1, 2$. Hence $\mathbf{v}' \in \mathscr{M}$, since $\mathbf{v} \in \mathscr{M}$ and $f_1(\mathbf{v}') = f_1(\mathbf{v})$. So we may assume $e = 0$.

We may assume $z = x + y$. For, otherwise, assume $z > x + y$. Let $\mathbf{v}' := (a + z - -x - y, b, x, y, x+y, 0, p')$ with $p' \in [0,1]$, which satisfies (1). So by (2), we may choose $p' \in (0,1)$ so that $f_1(\mathbf{v}') = f_2(\mathbf{v}')$; then $\mathbf{v}' \in \mathscr{D}$. If $p' < p$, then $f_2(\mathbf{v}') > f_2(\mathbf{v})$, contradicting the assumption that $\mathbf{v} \in \mathscr{M}$. So $p' \geq p$. Then

$$\begin{aligned}
f_1(\mathbf{v}') - f_1(\mathbf{v}) &\geq p^2(z - x - y) > 0, \quad \text{and} \\
f_2(\mathbf{v}') - f_2(\mathbf{v}) &= b((1-p')^2 - (1-p)^2) + y((1-p') - (1-p)) \\
&= -(p'-p)((2-p-p')b + y) \\
&\leq 0.
\end{aligned}$$

Hence $f_1(\mathbf{v}') > f_1(\mathbf{v}) = f_2(\mathbf{v}) \geq f_2(\mathbf{v}')$, a contradiction.

Now suppose $a > 0$ and $b > 0$. Let $\varepsilon = \min\{pa, (1-p)b\}$, and let

$$\mathbf{v}' = (a', b', x', y', z', e', p') := (a - \frac{\varepsilon}{p}, b - \frac{\varepsilon}{1-p}, x + \varepsilon, y + \varepsilon, z + 2\varepsilon, 0, p).$$

It is easy to see that $e' = 0$, $z' = x' + y'$, $a'b' = 0$, and $f_i(\mathbf{v}') = f_i(\mathbf{v})$ for $i = 1, 2$ (and hence $f_1(\mathbf{v}') = f_2(\mathbf{v}')$). Since $a + b + x + y + z = 1$,

$$a' + b' + x' + y' + z' = 1 + 4\varepsilon - \left(\frac{\varepsilon}{p} + \frac{\varepsilon}{1-p}\right).$$

Since $p(1-p) \le 1/4$ (with equality iff $p = 1/2$),

$$4\varepsilon \le \frac{\varepsilon}{p} + \frac{\varepsilon}{1-p}$$

So we have $a' + b' + x' + y' + z' \le 1$.

If $a' + b' + x' + y' + z' = 1$ then $p = 1/2$ and $\mathbf{v}' \in \mathscr{D}$. Since $f_i(\mathbf{v}') = f_i(\mathbf{v})$, we have $\mathbf{v}' \in \mathscr{M}$; and hence (3) holds with $\mathbf{v}'$. We may thus assume that $a' + b' + x' + y' + z' < 1$. Let

$$\alpha = \frac{\varepsilon}{p} + \frac{\varepsilon}{1-p} - 4\varepsilon,$$

and let

$$\mathbf{v}'' := (a'', b'', x'', y'', z'', e'', p'') = (a' + \alpha, b', x', y', z', 0, p'')$$

with $p'' \in [0, 1]$.

Note that $e'' = 0$, $z'' = x'' + y''$, $a'' + b'' + x'' + y'' + z'' = 1$, and $\mathbf{v}''$ satisfies (1). So by (2), we may choose $p'' \in (0, 1)$ such that $f_1(\mathbf{v}'') = f_2(\mathbf{v}'')$, and hence $\mathbf{v}'' \in \mathscr{D}$. If $p'' \ge p'$ then $f_1(\mathbf{v}'') > f_1(\mathbf{v}') = f_1(\mathbf{v})$ (since $a'' > a'$ and f_1 increases with p). If $p'' < p'$ then $f_2(\mathbf{v}'') > f_2(\mathbf{v}') = f_2(\mathbf{v})$ (since f_2 decreases with p). In either case, we obtain a contradiction to the assumption that $\mathbf{v} \in \mathscr{M}$. Thus, (3) holds.

Let $\mathscr{M}' = \{(a, b, x, y, z, e, p) \in \mathscr{M} : a = b = e = 0 \text{ and } z = x + y\}$. We may assume that

(4) $\mathscr{M}' = \emptyset$.

For otherwise, let $\mathbf{v} = (0, 0, x, y, x + y, 0, p) \in \mathscr{M}'$. Then $f_1(\mathbf{v}) = px$, $f_2(\mathbf{v}) = (1-p)y$, and $x + y = 1/2$. Since $f_1(\mathbf{v}) = f_2(\mathbf{v})$, we have $px = (1-p)(1/2 - x)$. Hence, $p = 1 - 2x$, and $f_1(\mathbf{v}) = x(1 - 2x) = 1/8 - 2(1/4 - x)^2 \le 1/8 < 1/7$. So the assertion of the lemma holds; and thus we may assume (4).

By (3) and (4), we may assume without losing generality that there exists $\mathbf{v} = (0, b, x, y, x + y, 0, p) \in \mathscr{M}$ such that $b \ne 0$. Then $b + 2(x + y) = 1$, and hence $x = (1-b)/2 - y$. So

$$f_1(\mathbf{v}) = xp = (1-b)p/2 - yp, \text{ and } f_2(\mathbf{v}) = y(1-p) + b(1-p)^2.$$

Since $\mathbf{v} \in \mathscr{M}$, $f_1(\mathbf{v})$ is the maximum value of f_1 over $\mathscr{D}$ subject to $g := f_1 - f_2 = 0$, where f_1, f_2, g are considered as functions of b, y, p.

Case 1. $y \ne 0$.

Then $y \in (0, 1)$ and $b \in (0, 1)$; so $\mathbf{v}$ is a critical point of f_1 (as a function of b, y). Hence $\mathbf{v}$ must satisfy $\partial f_1/\partial b = \lambda \partial g/\partial b$ and $\partial f_1/\partial y = \lambda \partial g/\partial y$, where λ is a Lagrange multiplier. Thus

$$p = \lambda\left(p + 2(1-p)^2\right), \text{ and } p = \lambda\left(p + (1-p)\right) = \lambda.$$

Since $p \in (0,1)$, we have $\lambda \neq 0$. So from the above equations we deduce that $(1-p) = 2(1-p)^2$. Again since $p \neq 1$, we have $p = 1/2$. Let

$$\mathbf{v}' := (a', b', x', y', z', e', p') = (0, 0, x, y + b/2, z + b/2, 0, p).$$

Then $a' + b' + x' + y' + z' + e' = 1$, $z' = x' + y'$, and $f_1(\mathbf{v}') = f_1(\mathbf{v})$. Since $p = 1/2$,

$$f_2(\mathbf{v}') = (1-p)(y + b/2) = (1-p)y + (1-p)b/2 = (1-p)y + (1-p)^2 b = f_2(\mathbf{v}).$$

This implies $\mathbf{v}' \in \mathscr{M}'$, contradicting (4).

Case 2. $y = 0$.

Then $f_1(\mathbf{v}) = (1-b)p/2$ and $f_2(\mathbf{v}) = b(1-p)^2$. By (1) and (2) and since $f_1(\mathbf{v}) = f_2(\mathbf{v})$, we have $b \in (0,1)$ and $p \in (0,1)$. Since $f_1(\mathbf{v})$ is the maximum of f_1 over $\mathscr{D}$ subject to $g := f_1 - f_2 = 0$ (considered as functions of p and b), $\mathbf{v}$ satisfies $\partial f_1/\partial p = \lambda \partial g/\partial p$ and $\partial f_1/\partial b = \lambda \partial g/\partial b$ for some λ. Therefore,

$$(1-b)/2 = \lambda\left((1-b)/2 + 2b(1-p)\right), \text{ and } p/2 = \lambda\left(p/2 + (1-p)^2\right).$$

Since $p \in (0,1)$, we have $\lambda \neq 0$; so we derive from above that $b = (1-p)/(1+p)$. From $f_1(\mathbf{v}) = f_2(\mathbf{v})$, we deduce $b = \frac{p}{p+2(1-p)^2}$. Hence

$$\frac{p}{p + 2(1-p)^2} = \frac{1-p}{1+p}.$$

Simplifying this we get $p^3 - 2p^2 + 3p - 1 = 0$. Since the function $p^3 - 2p^2 + 3p - 1$ is always increasing and takes value 0.036125 when $p = 9/20$, so $p < 9/20$.

We now claim that $f_1 \leq 1/7$. For otherwise, we have $f_1 > 1/7$, i.e.,

$$\frac{(1-b)p}{2} = \frac{p^2}{1+p} > 1/7.$$

But this gives $p > \frac{1+\sqrt{29}}{14} > 9/20$, a contradiction. ∎

In the next lemma we show that under certain conditions two functions can be made equal and bounded from above. The proof is similar to that of Lemma 3.1.

Lemma 3.2 *Let $\mathscr{D}$ denote the set of all points (a, b, x, y, e, p) such that $a, b, x, y, e \in \mathbb{R}^+$, $p \in [0.18, 1]$, $a+b+2(x+y+e) = 1$, and $p^2a+px+p^3e = (1.18-p)^2b+(1.18-p)y+(1.18-p)^3e$. Suppose $\mathscr{D} \neq \emptyset$. Then for any $(a, b, x, y, e, p) \in \mathscr{D}$, $p^2a + px + p^3e \leq (1.18^2/8)(1 - 0.82e)$.*

Proof. For convenience, let

$$\begin{aligned} g_1(a, b, x, y, e, p) &:= p^2a + px + p^3e, \text{ and} \\ g_2(a, b, x, y, e, p) &:= (1.18-p)^2b + (1.18-p)y + (1.18-p)^3e. \end{aligned}$$

A point $\mathbf{v} := (a, b, x, y, e, p) \in \mathscr{D}$ is said to be *maximal* if $g_1(\mathbf{v})$ is the maximum of g_1 over $\mathscr{D}$. Let $\mathscr{M}$ denote the set of all maximal points. Since $\mathscr{D}$ is compact and $\mathscr{D} \neq \emptyset$, $\mathscr{M} \neq \emptyset$. Let $M := g(\mathbf{v})$ for $\mathbf{v} \in \mathscr{M}$. We claim that

(1) for any $\mathbf{v} = (a, b, x, y, e, p) \in \mathscr{D}$, we have $e = 0$ and $g_1(\mathbf{v}) \le M(1 - 0.82e)$.

It is clear that (1) holds when $e = 0$. So assume $e \neq 0$. Let

$$\mathbf{v}' := (a', b', x', y', e', p') = \left(\frac{a + pe}{1 - 0.82e}, \frac{b + (1.18 - p)e}{1 - 0.82e}, \frac{x}{1 - 0.82e}, \frac{y}{1 - 0.82e}, 0, p \right).$$

Then $a' + b' + 2(x' + y' + e') = 1$, and $g_1(\mathbf{v}') = g_1(\mathbf{v})/(1 - 0.82e) = g_2(\mathbf{v})/(1 - 0.82e) = g_2(\mathbf{v}')$; so $\mathbf{v}' \in \mathscr{D}$. Now $g_1(\mathbf{v}) = g_1(\mathbf{v}')(1 - 0.82e) \le M(1 - 0.82e)$, proving (1).

Therefore, it suffices to prove that $M \le 1.18^2/8$. Let $\mathscr{M}' = \{(a, b, x, y, e, p) \in \mathscr{M} : x = y = e = 0\}$. We may assume

(2) $\mathscr{M}' = \emptyset$.

For, suppose there exists some $\mathbf{v} = (a, b, x, y, e, p) \in \mathscr{M}'$. Then $a + b = 1$,

$$g_1(\mathbf{v}) = p^2 a, \text{ and } g_2(\mathbf{v}) = (1.18 - p)^2 b.$$

Since $g_1(\mathbf{v}) = g_2(\mathbf{v})$, we have

$$b = \frac{p^2}{p^2 + (1.18 - p)^2}.$$

Note that for any $s, t \in \mathbb{R}^+$, we have $2\sqrt{st} \le s + t$ and $2st \le s^2 + t^2$; so $8s^2t^2 \le (s+t)^2(s^2+t^2)$, which implies

$$\frac{s^2t^2}{s^2 + t^2} \le \frac{1}{2} \left(\frac{s + t}{2} \right)^2.$$

Thus

$$M = g_2(\mathbf{v}) = \frac{p^2 \, (1.18 - p)^2}{p^2 + (1.18 - p)^2} \le \frac{1}{2} \left(\frac{1.18}{2} \right)^2 = \frac{1.18^2}{8},$$

and the assertion of the lemma holds. So we may assume (2).

By (1) and (2), there exists $\mathbf{v} = (a, b, x, y, e, p) \in \mathscr{M}$ such that $e = 0$, and $x \neq 0$ or $y \neq 0$. We now show that $\mathbf{v}$ may be chosen so that

(3) $y = 0$.

For, suppose $y \neq 0$. Since $a + b + 2(x + y + e) = 1$ and $e = 0$, $x = (1 - a - b - 2y)/2$. So

$$g_1(\mathbf{v}) = p^2 a + p\frac{1 - a - b - 2y}{2}, \text{ and}$$
$$g_2(\mathbf{v}) = (1.18 - p)^2 \, b + (1.18 - p) \, y.$$

Suppose $b \neq 0$. Then since we assume $y \neq 0$ and because $\mathbf{v} \in \mathscr{M}$, $\mathbf{v}$ is a critical point of g_1 subject to $g := g_1 - g_2 = 0$, where g_1, g_2, g are considered as functions of b and y. By applying the method of Lagrange multipliers, we have $\partial g_1/\partial b = \lambda \partial g/\partial b$ and $\partial g_1/\partial y = \lambda \partial g/\partial y$. Hence

$$-\frac{p}{2} = \lambda \left(-\frac{p}{2} - (1.18 - p)^2 \right), \text{ and } -p = \lambda \left(-p - (1.18 - p) \right).$$

Since $p \in [0.18, 1]$, $\lambda \neq 0$. Hence from the above expressions we deduce that $(1.18 - p)^2 = (1.18 - p)/2$. So $p = 0.68$, since $p \in [0.18, 1]$. Let

$$\mathbf{v}' := (a', b', x', y', e', p') = (a, b + 2y, x, 0, 0, p).$$

Then

$$\begin{aligned}
&a' + b' + 2(x' + y' + e') = a + b + 2(x + y) = 1,\\
&g_1(\mathbf{v}') = p^2 a + px = g_1(\mathbf{v}), \text{ and}\\
&g_2(\mathbf{v}') = (1.18 - p)^2 b' = (1.18 - p)^2 b + 2(1.18 - p)^2 y = (1.18 - p)^2 b + (1.18 - p)y = g_2(\mathbf{v}).
\end{aligned}$$

The last equality holds because $p = 0.68$. So $g_1(\mathbf{v}') = g_2(\mathbf{v}') = g_1(\mathbf{v})$. This means that $\mathbf{v}' \in \mathscr{M}$, with $e' = 0$ and $y' = 0$; and (3) holds by replacing $\mathbf{v}$ with $\mathbf{v}'$.

Now suppose $a = 0$ and $b = 0$. Then $g_1(\mathbf{v}) = p(1 - 2y)/2$ and $g_2(\mathbf{v}) = (1.18 - p)y$. So $g_1(\mathbf{v}) = g_2(\mathbf{v})$ implies $y = p/2.36$. Hence,

$$M = g_1(\mathbf{v}) = \frac{p}{2} - \frac{p^2}{2.36} = \frac{1.18}{8} - \frac{1}{2 \times 1.18}\left(p - \frac{1.18}{2}\right)^2 \leq \frac{1.18}{8} < \frac{1.18^2}{8},$$

and the assertion of the lemma holds.

So we may assume $a \neq 0$ and $b = 0$. Then

$$g_1(\mathbf{v}) = p^2 a + p(1 - a - 2y)/2, \text{ and } g_2(\mathbf{v}) = (1.18 - p)y.$$

Now $\mathbf{v}$ must be a critical point of g_1 subject to $g := g_1 - g_2 = 0$, where g_1, g_2, g are considered as functions of a and y. So there exists λ (Lagrange multiplier) such that $\partial g_1/\partial a = \lambda \partial g/\partial a$ and $\partial g_1/\partial y = \lambda \partial g/\partial y$. This gives

$$p^2 - \frac{p}{2} = \lambda\left(p^2 - \frac{p}{2}\right), \text{ and } -p = \lambda\left(-p - (1.18 - p)\right) = -1.18\lambda.$$

Since $p \in [0.18, 1]$, $\lambda \neq 1$ (from the second equation) and $p = 1/2$ (from the first equation). Hence, $g_1(\mathbf{v}) = (1 - 2y)/4$ and $g_2(\mathbf{v}) = 0.68y$. Since $g_1(\mathbf{v}) = g_2(\mathbf{v})$, we have $(1 - 2y)/4 = 0.68y$, and so $y = 1/4.72$. Hence $M = g_2(\mathbf{v}) = 0.68/4.72 < 1.18^2/8$. This completes the proof of (3).

By (2) and (3), $x \neq 0$ and $\mathbf{v} = (a, b, x, 0, 0, p)$. Hence $x = (1 - a - b)/2$,

$$g_1(\mathbf{v}) = p^2 a + p\frac{1 - a - b}{2}, \text{ and } g_2(\mathbf{v}) = (1.18 - p)^2 b.$$

Note that when $b = 0$, we have $M = g_2(\mathbf{v}) = 0 < 1.18^2/8$. Hence, we may assume

(4) $b \neq 0$.

We consider two cases: $a \neq 0$, and $a = 0$.

Case 1. $a \neq 0$.

Then $\mathbf{v}$ is a critical point of g_1 subject to $g := g_1 - g_2 = 0$, all considered as functions of a and b. So there exists λ such that $\partial g_1/\partial a = \lambda \partial g/\partial a$ and $\partial g_1/\partial b = \lambda \partial g/\partial b$, which give

$$p^2 - \frac{p}{2} = \lambda\left(p^2 - \frac{p}{2}\right), \text{ and } -\frac{p}{2} = \lambda\left(-\frac{p}{2} - (1.18 - p)^2\right).$$

Since $p \in [0.18, 1]$, we have $\lambda \neq 1$ from the second equation; so $p^2 - p/2 = 0$ (from the first equation), which implies $p = 1/2$. Define

$$\mathbf{v}' := (a', b', x', y', e', p') = (a + 2x, b, 0, 0, 0, p).$$

Then $a' + b' + 2(x' + y' + e') = a + b + 2x = 1$ and $g_2(\mathbf{v}) = g_2(\mathbf{v}')$. Also, because $p = 1/2$, $g_1(\mathbf{v}') = p^2 a' = p^2 a + 2p^2 x = p^2 a + px = g_1(\mathbf{v})$. Therefore, $\mathbf{v}' \in \mathscr{M}'$, contradicting (2).

Case 2. $a = 0$.

Then $g_1(\mathbf{v}) = p(1-b)/2$ and $g_2(\mathbf{v}) = (1.18 - p)^2 b$. Since $g_1(\mathbf{v}) = g_2(\mathbf{v})$, we have

$$b = \frac{p/2}{(1.18 - p)^2 + p/2}.$$

If $p = 0.18$ then $b = 0.18/2.18$; so $M = g_2(\mathbf{v}) = b < 1.18^2/8$. If $p = 1$ then $b = 1/1.0648$; so $M = g_2(v) = 0.18^2 b < 1.18^2/8$. Hence we may assume $p \in (0.18, 1)$.

Since $b \neq 0$ (by (4)) and $p \in (0.18, 1)$, $\mathbf{v}$ is a critical point of g_1 subject to $g := g_1 - g_2 = 0$, all considered as functions of b and p. So there exists λ such that $\partial g_1/\partial b = \lambda \partial g/\partial b$ and $\partial g_1/\partial p = \lambda \partial g/\partial p$, which gives

$$-\frac{p}{2} = \lambda\left(-\frac{p}{2} - (1.18 - p)^2\right) \text{ and } \frac{1-b}{2} = \lambda\left(\frac{1-b}{2} + 2b\,(1.18 - p)\right).$$

Since $p \in (0.18, 1)$, we have $\lambda \neq 0$ (from the first equation). So

$$\frac{p}{2}\left(\frac{1-b}{2} + 2b\,(1.18 - p)\right) = \frac{1-b}{2}\left(\frac{p}{2} + (1.18 - p)^2\right).$$

By a simple calculation, we derive

$$b = \frac{1.18 - p}{1.18 + p}.$$

Therefore, we have $(1.18 - p)^3 = p^2$.

Note that $h(p) := (1.18 - p)^3 - p^2$ is a decreasing function over $(0.18, 1)$, and a simple calculation shows $h(0.53) = -0.006275 < 0$. So $p < 0.53$. Also note that $g_1(\mathbf{v}) = p^2/(1.18 + p)$ is an increasing function over $(0.18, 1)$. So

$$g_1(\mathbf{v}) = \frac{p^2}{1.18 + p} < \frac{(0.53)^2}{1.18 + 0.53} < 0.165 < \frac{1.18^2}{8}.$$

This completes the proof of the lemma. ∎

4 Proofs of Lemmas 2.1 and 2.2

Proof of Lemma 2.1. For any permutation ijk of $\{1, 2, 3\}$, let

$$\alpha_i := b_{jk} + x_j + x_k,\ \beta_i := a_j + a_k, \text{ and } \gamma_i := \alpha_i + \beta_i + c.$$

Then for $i = 1, 2, 3$,

$$f_i(p_i) = (1 - p_i)\alpha_i + (1 - p_i)^2 \beta_i + (1 - p_i)^3 c.$$

By symmetry, we may assume that

$$\gamma_1 \leq \gamma_2 \leq \gamma_3.$$

We may further assume that

(1) $\gamma_1 \geq 0.35$.

For, suppose $\gamma_1 < 0.35$. Let $p_1 = 0$; then $f_1 = \gamma_1 < 0.35$. We wish to apply Lemma 3.1 to show that there exist $p_2, p_3 \in (0,1)$ such that $p_2 + p_3 = 1$ and $f_2 = f_3 \leq 0.35$. Let

$$m = \alpha_2 + \alpha_3 + \beta_2 + \beta_3 + (\alpha_2 + \alpha_3) + c.$$

Let $x = \alpha_2/m$, $y = \alpha_3/m$, $a = \beta_2/m$, $b = \beta_3/m$, $z = (\alpha_2 + \alpha_3)/m$, and $e = c/m$. Then $a + b + x + y + z + e = 1$ and $z \geq x + y$. Thus by Lemma 3.1, there exist $p_2, p_3 \in (0,1)$ such that $p_2 + p_3 = 1$ and $f_2/m = f_3/m \leq 1/7$.

Note that

$$m = 2(b_{13} + x_1 + x_3 + b_{12} + x_1 + x_2) + (a_1 + a_2 + a_1 + a_3) + c \leq 2 + 2x_1.$$

Since $b_{ij} \geq \max\{2x_i, 2x_j\}$ for $1 \leq i \neq j \leq 3$, we have $5x_1 \leq x_1 + b_{12} + b_{13} \leq 1$. Hence $x_1 \leq 1/5$, and so $m \leq 12/5$. Therefore, $f_2 = f_3 \leq (12/5)/7 < 0.35$; so (i) holds and we may assume (1).

We now write $f_i(p_i)$ for f_i, considering it as a function of p_i over $[0,1]$ (while fixing the other parameters). Differentiating with respect to p_i, we have $f_i'(p_i) = -\alpha_i - 2(1-p_i)\beta_i - 3(1-p_i)^2 c \leq 0$ and $f_i''(p_i) = 2\beta_i + 6(1-p_i)c \geq 0$. Note from (1) that $f'(p_i) < 0$ with the possible exception when $p_i = 1$. So

(2) each $f_i(p_i)$ is both decreasing and convex over $[0,1]$.

Because of (2), we approximate $f_i(p_i)$ (for each i) with the line $h_i(p_i)$ through the the points $(0, f_i(0))$ and $(1, f_i(1))$ in the Euclidean plane. Hence $h_i(p_i) = (1-p_i)\gamma_i$. It is also convenient to consider the reflection of $f_3(p_3)$ with respect to the line $p_3 = 1/2$, namely $f_4(p_3) = f_3(1-p_3) = p_3\alpha_3 + p_3^2\beta_3 + p_3^3 c$. Let $h_4(p_3) = \gamma_3 p_3$, which is the reflection of $h_3(p_3)$ with respect to the line $p_3 = 1/2$.

By (2) and by definition, we have

(3) $f_4(p_3)$ is convex and increasing over $[0,1]$; and for $i = 1,2,3,4$, $f_i(p_i) \leq h_i(p_i)$ when $p_i \in [0,1]$.

For each $0 \leq \alpha \leq \gamma_1$ and for $i = 1,2,3,4$, let $p_i(\alpha)$ denote the unique root of $f_i(p_i) = \alpha$ in $[0,1]$, and $q_i(\alpha)$ the unique root of $h_i(q_i) = \alpha$ in $[0,1]$. Note that from (2) and (3), we have

(4) for $\alpha \in [0, \gamma_1]$ and for $i = 1,2,3$, $p_i(\alpha) \leq q_i(\alpha)$, $p_i(\alpha)$ and $q_i(\alpha)$ decreases with α; and $p_4(\alpha)$ and $q_4(\alpha)$ increases with α.

Let (a,b) be the point where f_2 and f_4 intersect, that is, $f_2(a) = f_4(a) = b$; so $p_2(b) = p_4(b) = a$. Let (a', b') be the point where h_2 and h_4 intersect, i.e., $h_2(a') = h_4(a') = b'$. By (2) and (3), we have $b \leq b'$. By solving $h_2(a') = h_4(a') = b'$, we have

$$a' = \frac{\gamma_2}{\gamma_2 + \gamma_3}, \text{ and } b' = \frac{\gamma_2\gamma_3}{\gamma_2 + \gamma_3}.$$

Since $h_3(1-a') = h_4(a') = b'$ and by definition, we have $q_3(b') = 1 - q_2(b')$; and so $q_2(b') + q_3(b') = 1$.

We may assume

(5) $b' = \frac{\gamma_2\gamma_3}{\gamma_2+\gamma_3} \geq \gamma_1$.

For, suppose $b' < \gamma_1$. Then $b < \gamma_1$; so $p_i(b)$ is defined for $i = 1, 2, 3, 4$. Since f_3 and f_4 are reflections through the line $p_3 = 1/2$, $p_3(b) + p_4(b) = 1$. Since $p_2(b) = p_4(b) = a$ and $p_1(b) > 0$, we have $p_1(b)+p_2(b)+p_3(b) = p_1(b)+1 > 1$. Also, $p_1(\gamma_1) = 0$, and $p_2(\gamma_1)+p_3(\gamma_1) \leq q_2(\gamma_1)+q_3(\gamma_1) < q_2(b')+q_3(b') = 1$; so $p_1(\gamma_1)+p_2(\gamma_1)+p_3(\gamma_1) < 1$. Since $p_1(\alpha)+p_2(\alpha)+p_3(\alpha)$ is a decreasing function of α, there exists $\alpha \in (b, \gamma_1)$ (and hence by (4), $p_i(\alpha) \in (0, 1)$ for $i = 1, 2, 3$) such that $p_1(\alpha) + p_2(\alpha) + p_3(\alpha) = 1$; so (ii) holds with $f_i(p_i) = \alpha$ for $i = 1, 2, 3$.

We claim that

(6) $\gamma_1 \leq 1/2$, $0.4 \leq \gamma_2 \leq 1$, $0.7 \leq \gamma_3 \leq 1$, $\gamma_2 + \gamma_3 \geq 1.4$, and $c - \sum_{1\leq i<j\leq 3} b_{ij} \geq -0.25$.

By (5), $\frac{\gamma_2\gamma_3}{\gamma_2+\gamma_3} \geq \gamma_1$. So by Cauchy-Schwarz,

$$\gamma_2 + \gamma_3 \geq \frac{4}{\frac{1}{\gamma_2} + \frac{1}{\gamma_3}} \geq 4\gamma_1.$$

Hence by (1), $\gamma_2 + \gamma_3 \geq 1.4$. Then $\gamma_2 \geq 0.4$ and, since $\gamma_3 \geq \gamma_2$, $\gamma_3 \geq (\gamma_2 + \gamma_3)/2 \geq 0.7$. Since

$$\gamma_1 + \gamma_2 + \gamma_3 = 2 + c - \sum_{1\leq i<j\leq 3} b_{ij},$$

we have $5\gamma_1 \leq \gamma_1 + \gamma_2 + \gamma_3 = 2 + c - \sum_{i<j} b_{ij}$, and so $\gamma_1 \leq 2/5 + (c - \sum_{i<j} b_{ij})/5$. Therefore, since $\gamma_2 + \gamma_3 \leq 2$,

$$2 + c - \sum_{i<j} b_{ij} = \gamma_1 + \gamma_2 + \gamma_3 \leq 2 + \frac{2}{5} + \frac{c - \sum_{i<j} b_{ij}}{5}.$$

So $c - \sum_{i<j} b_{ij} \leq 1/2$, which in turn implies $5\gamma_1 \leq 2 + c - \sum_{i<j} b_{ij} \leq 5/2$. Thus, $\gamma_1 \leq \frac{1}{2}$. By (1), $1.75 \leq 5\gamma_1 \leq 2 + c - \sum_{i<j} b_{ij}$, which implies $c - \sum_{i<j} b_{ij} \geq -0.25$.

We also claim that

(7) $x_i \leq 1.25/9$, for $i = 1, 2, 3$.

Since $b_{ij} \geq 2x_i$ and $b_{ij} \geq 2x_j$, $c + 5x_i \leq 1$. By (6), $c - \sum b_{ij} \geq -0.25$; so $c - 4x_i \geq -0.25$. Hence $1 - 5x_i \geq 4x_i - 0.25$, which gives (7).

We now prove that

(8) $f_1(0.18) \leq 0.35$.

This is true if $\gamma_1 \leq 0.35/0.82$ as $f_1(0.18) \leq 0.82\gamma_1$. So we may assume that $\gamma_1 > 0.35/0.82$. From the proof of (6) we see that $c \geq \sum_{i<j} b_{ij} + 5\gamma_1 - 2$. Then, since $b_{12} \geq 2x_2, b_{13} \geq 2x_3$ and $\alpha_1 = b_{23} + x_2 + x_3$, we have $c \geq \alpha_1 + 5\gamma_1 - 2$. Also, $\gamma_1 \geq \alpha_1 + c$. So $\gamma_1 - \alpha_1 \geq \alpha_1 + 5\gamma_1 - 2$. Therefore, $2\gamma_1 + \alpha_1 \leq 1$. Hence, since $\gamma_1 > 0.35/0.82$, we have $\alpha_1 \leq 1 - 0.7/0.82$ and $c \geq 5\gamma_1 - 2 \geq 5 \times (0.35/0.82) - 2 = 0.11/0.82$. This implies that $0.82\alpha_1 + 0.82^3 c < 0.7(\alpha_1 + c)$. Hence, since $0.82^2 < 0.7$, $f_1(0.18) < 0.7\gamma_1 \leq 0.35$ (as $\gamma_1 \leq 1/2$ by (6)). So we have (8).

Now let $p_1 = 0.18$; then by (8), $f_1(p_1) \leq 0.35$. We wish to apply Lemma 3.2 to prove the existence of p_2 and p_3 such that $p_2 + p_3 = 1 - p_1 = 0.82$, $f_2(p_2) \leq 0.35$ and $f_3(p_3) \leq 0.35$. Let $1 - p_2 = p$ and $1 - p_3 = 1.18 - p$. Let

$$m = \beta_2 + \beta_3 + 2(\alpha_2 + \alpha_3 + c),$$

and let $a = \beta_2/m$, $b = \beta_3/m$, $x = \alpha_2/m$, $y = \alpha_3/m$, $e = c/m$, $g_1(p) = f_2(p)/m$, and $g_2(p) = f_3(p)/m$. Then $a + b + 2(x + y + e) = 1$,

$$g_1(p) = p^2 a + px + p^3 e, \text{ and } \quad g_2(p) = (1.18 - p)^2 b + (1.18 - p)y + (1.18 - p)^3 e.$$

Note that

$$m = 2a_1 + a_2 + a_3 + 2(b_{12} + b_{13} + 2x_1 + x_2 + x_3 + c) = 2 + 2x_1 - (a_2 + a_3 + 2b_{23}) \leq 2 + 2x_1,$$

and

$$\begin{aligned} m &= 2 + 2x_1 - (a_2 + a_3 + 2b_{23}) \\ &= 2 + 2x_1 - \gamma_1 + x_2 + x_3 + c - b_{23} \\ &\leq 2 + 2x_1 - \gamma_1 + c \quad (\text{since } b_{23} \geq \max\{2x_1, 2x_3\}) \\ &\leq 2 + 2(1.25/9) - 0.35 + c \quad (\text{by (1) and (7)}). \end{aligned}$$

We claim that

(9) $\gamma_2/m > 0.18$ and $\gamma_3/m > 0.18$.

By (7), $m \leq 2 + 2(1.25/9)$; so by (6), $\gamma_3/m \geq 0.7/(2 + 2.5/9) > 0.18$. If $\gamma_2 \geq 0.5$, then $\gamma_2/m \geq 0.5/(2 + 2.5/9) > 0.18$. So we may assume that $\gamma_2 < 0.5$. Then by (6), $\gamma_3 > 0.9$. Hence, $2x_1 \leq b_{13} \leq b_{13} + b_{23} + x_3 + a_3 = 1 - \gamma_3 < 0.1$. So $m \leq 2 + 2x_1 < 2.1$ and, by (6), $\gamma_2/m \geq 0.4/2.1 > 0.18$. Thus, we have (9).

In order to apply Lemma 3.2, we need to show that there exists $p \in [0.18, 1]$ such that $g_1(p) = g_2(p)$. To see this, consider g_1, g_2 as functions of p. By (9), we note that

$$\begin{aligned} &g_1(0.18) \leq 0.18(a + x + e) \leq 0.18, \text{ and} \\ &g_2(0.18) = b + y + e = \gamma_3/m > 0.18. \end{aligned}$$

So $g_1(0.18) < g_2(0.18)$. Similarly, we can show $g_1(1) > 0.18 \geq g_2(1)$. By (2), $g_1(p)$ is an increasing function, and $g_2(p)$ is a decreasing function. So there exists $p \in (0.18, 1)$ such that $g_1(p) = g_2(p)$.

We can now apply Lemma 3.2. As a consequence, $g_1(p) = g_2(p) \leq (1.18^2/8)(1 - 0.82e)$, so $f_2(p) = f_3(p) \leq (1.18^2/8)(m - 0.82c)$. If $c \leq 0.35$ then, since $m \leq 2 + 2(1.25/9) - 0.35 + c$,

$$f_2(p) = f_3(p) \leq \frac{1.18^2}{8}(2 + 2.5/9 - 0.35 + 0.18 \times 0.35) < 0.347 < 0.35.$$

So we may assume $c > 0.35$. Then, since $m \leq 2 + 2x_1 \leq 2 + 2.5/9$ by (7),

$$f_2(p) = f_3(p) \leq \frac{1.18^2}{8}(2 + 2.5/9 - 0.82 \times 0.35) < 0.35.$$

Note that $p_2 = 1 - p$ and $p_3 = p - 0.18$. Since $p \in (0.18, 1)$, we have $p_2, p_3 \in (0,1)$. Clearly, $p_1 + p_2 + p_3 = 1$. So (i) holds, which completes the proof of the lemma. ∎

In order to prove Lemma 2.2, we first deal with the special case when $b_{ij} = x_i + x_j$ for $1 \le i < j \le 3$.

Lemma 4.1 *Let $b_i, y_i \in \mathbb{R}^+$ for $i = 1,2,3$ such that $\sum_{i=1}^{3}(3y_i + b_i) = 2$. Suppose there exist $q_i \in (0,1)$, $i = 1,2,3$, such that $q_1+q_2+q_3 = 2$ and $2y_1q_1 + b_1q_1^2 = 2y_2q_2 + b_2q_2^2 = 2y_3q_3 + b_3q_3^2$. Then for $i = 1,2,3$, $2y_iq_i + b_iq_i^2 \le 0.35$.*

Proof. For convenience, let $f_i := 2y_iq_i + b_iq_i^2$, $i = 1,2,3$. Let $\mathscr{D}$ denote the set of all points $(b_1, b_2, b_3, y_1, y_2, y_3, q_1, q_2, q_3)$ such that $b_i, y_i \in \mathbb{R}^+$ and $q_i \in [0,1]$ for $i = 1,2,3$,

$$\begin{aligned}
&\sum_{i=1}^{3}(3y_i + b_i) = 2,\\
&q_1 + q_2 + q_3 = 2, \text{ and}\\
&f_1 = f_2 = f_3.
\end{aligned}$$

So $\mathscr{D}$ is a compact subset of $[0,2]^3 \times [0,2/3]^3 \times [0,1]^3$. Note that $\mathscr{D} \neq \emptyset$ by assumption of the lemma. Let

$$\mathbf{v} := (b_1, b_2, b_3, y_1, y_2, y_3, q_1, q_2, q_3) \in \mathscr{D}$$

such that $f_1(\mathbf{v})$ is the maximum of f_1 over $\mathscr{D}$. It suffices to show that $f_1(\mathbf{v}) \le 0.35$.

We may assume that $q_i \neq 0$ for $i = 1,2,3$; as otherwise we have $f_i(\mathbf{v}) = 0 < 0.35$ for $i = 1,2,3$. Thus, since $f_1 = f_2 = f_3$, we see that if $f_i = 0$ for some $i \in \{1,2,3\}$ then $b_i = y_i = 0$ for $i = 1,2,3$, contradicting the condition that $\sum_{i=1}^{3}(3y_i + b_i) = 2$. Hence, we have

(1) for each $i \in \{1,2,3\}$, $q_i > 0$, and $b_i > 0$ or $y_i > 0$.

We may assume that

(2) there exists some $i \in \{1,2,3\}$ such that $b_i > 0$.

For, suppose $b_i = 0$ for $i = 1,2,3$. Then $f_i = 2y_iq_i$ and $y_i > 0$ (by (1)) for $i = 1,2,3$, and $y_1 + y_2 + y_3 = 2/3$. Hence, by Cauchy-Schwarz,

$$\frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3} \ge \frac{9}{y_1 + y_2 + y_3} = \frac{27}{2}.$$

Setting $f_1 = f_2 = f_3 = \alpha$, we have $q_i = \alpha/2y_i$ for $i = 1,2,3$. Therefore, since $q_1 + q_2 + q_3 = 2$,

$$\alpha = \frac{4}{\frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3}} \le \frac{8}{27} < 0.35.$$

We may also assume that

(3) there exists some $j \in \{1,2,3\}$ such that $y_j > 0$.

For, otherwise, $y_1 = y_2 = y_3 = 0$. Then $f_i = b_i q_i^2$ and $b_i > 0$ (by (1)) for $i = 1, 2, 3$, and $b_1 + b_2 + b_3 = 2$. Setting $f_1 = f_2 = f_2 = \alpha$, we have $q_i = \sqrt{\alpha/b_i}$. Since $q_1 + q_2 + q_3 = 2$, we have (by Cauchy-Schwarz),

$$\alpha = \frac{4}{\left(\frac{1}{\sqrt{b_1}} + \frac{1}{\sqrt{b_2}} + \frac{1}{\sqrt{b_3}}\right)^2} \leq \frac{4}{81}\left(\sqrt{b_1} + \sqrt{b_2} + \sqrt{b_3}\right)^2 \leq \frac{4}{9}\frac{b_1 + b_2 + b_3}{3} = \frac{8}{27} < 0.35.$$

We may further assume that

(4) there exists some $i \in \{1, 2, 3\}$ such that $b_i y_i \neq 0$.

Otherwise, we have two cases (by symmetry): $y_1 = y_2 = b_3 = 0$, or $b_1 = b_2 = y_3 = 0$

First, assume $y_1 = y_2 = b_3 = 0$. Then, $b_1 > 0$, $b_2 > 0$, $y_3 > 0$, $b_1 + b_2 + 3y_3 = 2$,

$$f_1 = b_1 q_1^2, \; f_2 = b_2 q_2^2, \text{ and } f_3 = 2y_3 q_3.$$

Setting $\alpha = f_1 = f_2 = f_3$ and using $q_1 + q_2 + q_3 = 2$, we have

$$\frac{\sqrt{\alpha}}{\sqrt{b_1}} + \frac{\sqrt{\alpha}}{\sqrt{b_2}} + \frac{\alpha}{2y_3} = 2.$$

So

$$\sqrt{\alpha} = \frac{4}{\sqrt{(1/\sqrt{b_1} + 1/\sqrt{b_2})^2 + 4/y_3} + (1/\sqrt{b_1} + 1/\sqrt{b_2})}.$$

Note that

$$\left(\frac{1}{\sqrt{b_1}} + \frac{1}{\sqrt{b_2}}\right)^2 \geq \frac{4}{\sqrt{b_1 b_2}} \geq \frac{8}{b_1 + b_2} = \frac{8}{2 - 3y_3},$$

so

$$\sqrt{\alpha} \leq \frac{4}{\sqrt{\frac{8}{2-3y_3} + \frac{4}{y_3}} + \sqrt{\frac{8}{2-3y_3}}}.$$

Let $f(y_3) := \sqrt{8/(2-3y_3) + 4/y_3} + \sqrt{8/(2-3y_3)}$. Note that $y_3 \in (0, 2/3)$, and

$$f(y_3) \geq \begin{cases} \sqrt{4+20} + \sqrt{4}, & \text{if } y_3 \in (0, 1/5]; \\ \sqrt{8/(7/5) + 16} + \sqrt{8/(7/5)}, & \text{if } y_3 \in (1/5, 1/4]; \\ \sqrt{8/(5/4) + 12} + \sqrt{8/(5/4)}, & \text{if } y_3 \in (1/4, 1/3]; \\ \sqrt{8+8} + \sqrt{8}, & \text{if } y_3 \in (1/3, 1/2]; \\ \sqrt{16+6} + \sqrt{16}, & \text{if } y_3 \in (1/2, 2/3). \end{cases}$$

Therefore, $f(y_3) \geq 6.819$, and hence $\alpha \leq (4/6.819)^2 < 0.35$

Now assume $b_1 = b_2 = y_3 = 0$. Then $y_1 > 0$, $y_2 > 0$, $b_3 > 0$, $3(y_1 + y_2) + b_3 = 2$,

$$f_1 = 2y_1 q_1, \; f_2 = 2y_2 q_2, \text{ and } f_3 = b_3 q_3^2.$$

Again, setting $\alpha = f_1 = f_2 = f_3$ and using $q_1 + q_2 + q_3 = 2$, we have

$$\frac{\alpha}{2y_1} + \frac{\alpha}{2y_2} + \frac{\sqrt{\alpha}}{\sqrt{b_3}} = 2,$$

So

$$\sqrt{\alpha} = \frac{4}{\sqrt{1/b_3 + 4(1/y_1 + 1/y_2)} + 1/\sqrt{b_3}}.$$

Note that $1/y_1 + 1/y_2 \geq 4/(y_1 + y_2) = 12/(2 - b_3)$. Hence

$$\sqrt{\alpha} \leq \frac{4}{\sqrt{1/b_3 + 48/(2 - b_3)} + 1/\sqrt{b_3}}.$$

Let $g(b_3) := \sqrt{1/b_3 + 48/(2 - b_3)} + 1/\sqrt{b_3}$. Note that $b_3 \in (0, 2)$, and

$$g(b_3) \geq \begin{cases} \sqrt{3 + 48/(2-0)} + \sqrt{3}, & \text{if } b_3 \in (0, 1/3]; \\ \sqrt{2 + 48/(2-1/3)} + \sqrt{2}, & \text{if } b_3 \in (1/3, 1/2]; \\ \sqrt{3/2 + 48/(2-1/2)} + \sqrt{3/2}, & \text{if } b_3 \in (1/2, 2/3]; \\ \sqrt{2/3 + 48/(2-2/3)} + \sqrt{2/3}, & \text{if } b_3 \in (2/3, 3/2]; \\ \sqrt{1/2 + 48/(2-3/2)} + \sqrt{1/2}, & \text{if } b_3 \in (3/2, 2). \end{cases}$$

Therefore, $g(b_3) \geq 6.87$, and hence $\alpha \leq (4/6.87)^2 < 0.35$.

By (4) and by symmetry, we may assume that

(5) $b_3 y_3 \neq 0$.

We may further assume that

(6) $b_1 y_1 = 0$ and $b_2 y_2 = 0$.

For, otherwise, by symmetry, assume $b_2 y_2 > 0$. Then $\mathbf{v}$ is a solution to the following optimization problem:

Maximize f_1
subject to

$$\begin{aligned} h_1 &:= f_1 - f_2 = 0, \\ h_2 &:= f_1 - f_3 = 0, \\ h_3 &:= 3(y_1 + y_2 + y_3) + (b_1 + b_2 + b_3) - 2 = 0, \\ h_4 &:= q_1 + q_2 + q_3 - 2 = 0. \end{aligned}$$

Applying the method of Lagrange multipliers, we have, for each $u \in \{y_i, b_i : i = 2, 3\}$,

$$\partial f_1/\partial u = \lambda_1 \partial h_1/\partial u + \lambda_2 \partial h_2/\partial u + \lambda_3 \partial h_3/\partial u + \lambda_4 \partial h_4/\partial u.$$

Thus,

$$\begin{aligned} &\text{for } u = y_2, \text{ we have } 0 = \lambda_1(-2q_2) + 3\lambda_3, \\ &\text{for } u = y_3, \text{ we have } 0 = \lambda_2(-2q_3) + 3\lambda_3, \\ &\text{for } u = b_2, \text{ we have } 0 = \lambda_1(-q_2^2) + \lambda_3, \\ &\text{for } u = b_3, \text{ we have } 0 = \lambda_2(-q_3^2) + \lambda_3. \end{aligned}$$

Clearly, if $\lambda_i = 0$ for some $i \in \{1, 2, 3\}$ then $\lambda_i = 0$ for all $i = 1, 2, 3$ (since $q_i > 0$ by (1)). In fact, $\lambda_i \neq 0$ for all $i = 1, 2, 3$. To see this we notice that either $b_1 > 0$ or $y_1 > 0$, so $\mathbf{v}$ also satisfies $\partial f_1/\partial u = \lambda_1 \partial h_1/\partial u + \lambda_2 \partial h_2/\partial u + \lambda_3 \partial h_3/\partial u + \lambda_4 \partial h_4/\partial u$ for $u = b_1$ or $u = y_1$. For $u = b_1$, we have $q_1^2 = \lambda_1 q_1^2 + \lambda_2 q_1^2 + \lambda_3$, and for $u = y_1$ we have $2q_1 = \lambda_1 2q_1 + \lambda_2 2q_1 + 3\lambda_3$. In either case, we see that $\lambda_i \neq 0$ (since $q_1 > 0$).

Now using the partial derivatives with respect to b_2 and y_2, we get $q_2 = 2/3$; and using the partial derivatives with respect to b_3 and y_3 we obtain $q_3 = 2/3$. So $q_1 = 2/3$ since $q_1 + q_2 + q_3 = 2$. Then for $i = 1, 2, 3$,

$$f_i = \frac{4}{3} y_i + \frac{4}{9} b_i = \frac{4}{9}(3y_i + b_i).$$

Since $f_1 = f_2 = f_3$ and $\sum_{i=1}^{3}(3y_i + b_i) = 2$, we get $3y_i + b_i = 2/3$ for $i = 1, 2, 3$, and hence $f_i = 8/27 < 0.35$. This proves (6)

By (5) and (6), we have three cases to consider: $b_1 = b_2 = 0$; $y_1 = y_2 = 0$; $y_1 = b_2 = 0$ or $b_1 = y_2 = 0$. Let h_1, h_2, h_3, h_4 be defined as in the proof of (6).

Case 1. $b_1 = b_2 = 0$.

Then $y_1 > 0$, $y_2 > 0$, $f_1 = 2y_1 q_1$, $f_2 = 2y_2 q_2$, $f_3 = 2y_3 q_3 + b_3 q_3^2$. Moreover, $\mathbf{v}$ is a critical point of f_1 subject to $h_1 = h_2 = h_3 = h_4 = 0$, all considered as functions of y_1, y_2, y_3, b_3. Hence for $u \in \{y_1, y_2, y_3, b_3\}$, $\mathbf{v}$ satisfies

$$\partial f_1/\partial u = \lambda_1 \partial h_1/\partial u + \lambda_2 \partial h_2/\partial u + \lambda_3 \partial h_3/\partial u + \lambda_4 \partial h_4/\partial u.$$

So

$$\begin{aligned}
&\text{for } u = y_1, \text{ we have } 2q_1 = \lambda_1(2q_1) + \lambda_2(2q_1) + 3\lambda_3,\\
&\text{for } u = y_2, \text{ we have } 0 = \lambda_1(-2q_2) + 3\lambda_3,\\
&\text{for } u = y_3, \text{ we have } 0 = \lambda_2(-2q_3) + 3\lambda_3,\\
&\text{for } u = b_3, \text{ we have } 0 = \lambda_2(-q_3^2) + \lambda_3.
\end{aligned}$$

Clearly, $\lambda_i \neq 0$ for $i = 1, 2, 3$. So from the partial derivatives with respect to b_3 and y_3, we have $q_3 = 2/3$, and hence $q_1 + q_2 = 4/3$. Set $\alpha := 2y_1 q_1 = 2y_2 q_2 = 4(3y_3 + b_3)/9$. In particular, $\alpha = 4(3y_3 + b_3)/9 = 4(2 - 3(y_1 + y_2))/9$, and so $y_1 + y_2 = 2/3 - 3\alpha/4$. Using $q_1 + q_2 = 4/3$ and Cauchy-Schwarz, we get

$$\frac{4}{3} = \frac{\alpha}{2y_1} + \frac{\alpha}{2y_2} \geq \frac{2\alpha}{y_1 + y_2} = \frac{2\alpha}{2/3 - 3\alpha/4}.$$

This implies $\alpha \leq 8/27 < 0.35$.

Case 2. $y_1 = y_2 = 0$.

Then $b_1 > 0$, $b_2 > 0$, $f_1 = b_1 q_1^2$, $f_2 = b_2 q_2^2$ and $f_3 = 2y_3 q_3 + b_3 q_3^2$. Now $\mathbf{v}$ is a critical point of f_1 subject to $h_1 = h_2 = h_3 = h_4 = 0$, all considered as functions of b_1, b_2, b_3, y_3. Hence for $u \in \{b_1, b_2, b_3, y_3\}$, $\mathbf{v}$ satisfies

$$\partial f_1/\partial u = \lambda_1 \partial h_1/\partial u + \lambda_2 \partial h_2/\partial u + \lambda_3 \partial h_3/\partial u + \lambda_4 \partial h_4/\partial u.$$

Thus,

$$\begin{aligned}
&\text{for } u = b_1, \text{ we have } q_1^2 = \lambda_1(q_1^2) + \lambda_2(q_1^2) + \lambda_3,\\
&\text{for } u = b_2, \text{ we have } 0 = \lambda_1(-q_2^2) + \lambda_3,\\
&\text{for } u = b_3, \text{ we have } 0 = \lambda_2(-q_3^2) + \lambda_3\\
&\text{for } u = y_3, \text{ we have } 0 = \lambda_2(-2q_3) + 3\lambda_3.
\end{aligned}$$

Clearly, $\lambda_i \neq 0$ for $i = 1, 2, 3$. So from the partial derivatives with respect to b_3 and y_3, we have $q_3 = 2/3$, and hence $q_1 + q_2 = 4/3$. Setting $\alpha := y_1 q_1^2 = y_2 q_2^2 = 4(3y_3 + b_3)/9$, we have $q_i = \sqrt{\alpha}/\sqrt{b_i}$ for $i = 1, 2$, $3y_3 + b_3 = 9\alpha/4$, and $b_1 + b_2 = 2 - 9\alpha/4$. So

$$\frac{4}{3} = \frac{\sqrt{\alpha}}{\sqrt{b_1}} + \frac{\sqrt{\alpha}}{\sqrt{b_2}} \geq \frac{2\sqrt{\alpha}}{\sqrt{\sqrt{b_1}\sqrt{b_2}}} \geq \frac{2\sqrt{2\alpha}}{\sqrt{b_1 + b_2}} = \frac{2\sqrt{2\alpha}}{\sqrt{2 - 9\alpha/4}}.$$

This gives $\alpha \leq 8/27 < 0.35$.

Case 3. $y_1 = b_2 = 0$, or $y_2 = b_1 = 0$.

By symmetry, we may assume that $y_1 = b_2 = 0$. Then $b_1 > 0$, $y_2 > 0$, $b_1 + 3y_2 + (3y_3 + b_3) = 2$, $f_1 = b_1 q_1^2$, $f_2 = 2y_2 q_2$, and $f_3 = 2y_3 q_3 + b_3 q_3^2$.

So $\mathbf{v}$ is a critical point of f_1 subject to $h_1 = h_2 = h_3 = h_4 = 0$, all considered as functions of b_1, y_2, b_3, y_3. Hence $\mathbf{v}$ satisfies $\partial f_1/\partial u = \lambda_1 \partial h_1/\partial u + \lambda_2 \partial h_2/\partial u + \lambda_3 \partial h_3/\partial u + \lambda_4 \partial h_4/\partial u$ for $u \in \{b_1, y_2, b_3, y_3\}$. Thus,

$$\begin{aligned}
&\text{for } u = b_1, \text{ we have } q_1^2 = \lambda_1(q_1^2) + \lambda_2(q_1^2) + \lambda_3,\\
&\text{for } u = y_2, \text{ we have } 0 = \lambda_1(-2q_2) + 3\lambda_3,\\
&\text{for } u = b_3, \text{ we have } 0 = \lambda_2(-q_3^2) + \lambda_3\\
&\text{for } u = y_3, \text{ we have } 0 = \lambda_2(-2q_3) + 3\lambda_3.
\end{aligned}$$

Clearly, $\lambda_i \neq 0$ for $i = 1, 2, 3$. So from the partial derivatives with respect to b_3 and y_3, we have $q_3 = 2/3$, and hence $q_1 + q_2 = 4/3$.

Set $\alpha = f_1(\mathbf{v}) = f_2(\mathbf{v}) = f_3(\mathbf{v})$. Then

$$2 = b_1 + 3y_2 + (3y_3 + b_3) = \left(\frac{1}{q_1^2} + \frac{3}{2q_2} + \frac{9}{4}\right)\alpha = \left(\frac{1}{q_1^2} + \frac{3}{2(4/3 - q_1)} + \frac{9}{4}\right)\alpha.$$

Let $h(q_1) := 1/q_1^2 + 3/(2(4/3 - q_1))$. Note that $q_1 \in (0, 4/3)$ and

$$h(q_1) \geq \begin{cases}
4 + 3/(2(4/3 - 0)), & \text{if } q_1 \in (0, 1/2];\\
9/4 + 3/(2(4/3 - 1/2)), & \text{if } q_1 \in (1/2, 2/3];\\
25/16 + 3/(2(4/3 - 2/3))), & \text{if } q_1 \in (2/3, 4/5];\\
1 + 3/(2(4/3 - 4/5)), & \text{if } q_1 \in (4/5, 1];\\
9/16 + 3/(2(4/3 - 1)), & \text{if } q_1 \in (1, 4/3).
\end{cases}$$

So $h(q_1) \geq 3.8125$, and hence $\alpha = 2/(h(q_1) + 9/4) \leq 2/(3.8125 + 9/4) < 0.35$. ∎

Proof of Lemma 2.2. For any permutation ijk of $\{1, 2, 3\}$, and let $y_k = x_i + x_j$ and $b_k = a_i + a_j$. Then

$$f_k = (1 - p_k)(b_{ij} + y_k) + (1 - p_k)^2 b_k.$$

Set $\alpha = f_1(p_1) = f_2(p_2) = f_3(p_3)$. Note that we may assume $\alpha > 0$ (otherwise we are done); and hence $b_{ij} + y_k + b_k > 0$ for $k = 1, 2, 3$. Since $p_k \in (0, 1)$, $1 - p_k \in (0, 1)$; and hence by solving $f_k(p_k) = \alpha$ we get

$$1 - p_k = \frac{2\alpha}{\sqrt{(b_{ij} + y_k)^2 + 4b_k\alpha} + (b_{ij} + y_k)}.$$

We wish to show that $\alpha \leq 0.35$; so we consider the following optimization problem.

Maximize α

Subject to

$$g_1 := \sum_{k=1}^{3} \frac{2\alpha}{\sqrt{(b_{ij} + y_k)^2 + 4b_k\alpha} + (b_{ij} + y_k)} - 2 = 0,$$

$$g_2 := b_{12} + b_{13} + b_{23} + \frac{1}{2}(y_1 + y_2 + y_3 + b_1 + b_2 + b_3) - 1 = 0,$$

$$b_{ij} \geq y_k \geq 0, \text{ for } \{i, j, k\} = \{1, 2, 3\}.$$

Here, g_1, g_2 are considered as functions of α, b_{ij}, b_k, y_k. By the assumption of the lemma, the feasible region of this optimization problem is nonempty.

Claim 1. α is maximized only when $b_{ij} = y_k$ or $y_k = 0$, for all $\{i, j, k\} = \{1, 2, 3\}$.
For, suppose $b_{ij} > y_k > 0$ for some permutation ijk of $\{1, 2, 3\}$. By applying the method of Lagrange multipliers, we have $\partial\alpha/\partial u = \lambda_1 \partial g_1/\partial u + \lambda_2 \partial g_2/\partial u$, where $u \in \{\alpha, b_{ij}, y_k\}$. So

$$\text{for } u = b_{ij}, \quad 0 = \lambda_1 \frac{-2\alpha\left(b_{ij} + y_k + \sqrt{(b_{ij} + y_k)^2 + 4b_k\alpha}\right)}{\sqrt{(b_{ij} + y_k)^2 + 4b_k\alpha}\left(\sqrt{(b_{ij} + y_k)^2 + 4b_k\alpha} + (b_{ij} + y_k)\right)^2} + \lambda_2,$$

$$\text{for } u = y_k, \quad 0 = \lambda_1 \frac{-2\alpha\left(b_{ij} + y_k + \sqrt{(b_{ij} + y_k)^2 + 4b_k\alpha}\right)}{\sqrt{(b_{ij} + y_k)^2 + 4b_k\alpha}\left(\sqrt{(b_{ij} + y_k)^2 + 4b_k\alpha} + (b_{ij} + y_k)\right)^2} + \frac{\lambda_2}{2},$$

$$\text{for } u = \alpha, \quad 1 = \lambda_1 \frac{\partial g_1}{\partial \alpha} + \lambda_2 \frac{\partial g_2}{\partial \alpha}.$$

The first two equations give $\lambda_1 = \lambda_2 = 0$, which contradicts the third equation.

Therefore, the maximum of α is achieved when $b_{ij} = y_k$ for some permutation ijk of $\{1, 2, 3\}$, or when $y_k = 0$ for some $k \in \{1, 2, 3\}$; so Claim 1 follows.

Claim 2. We may assume that α is maximized when $b_{ij} > y_k$ for some $\{i, j, k\} = \{1, 2, 3\}$.

For, otherwise, the maximum of α is achieved when $b_{ij} = y_k$ for all permutations ijk of $\{1, 2, 3\}$. Set $q_k = 1 - p_k$ for $k = 1, 2, 3$; and so $f_k = 2y_kq_k + b_kq_k^2$ and $3(y_1 + y_2 + y_3) + b_1 + b_2 + b_3 = 2$. We can now apply Lemma 4.1 and conclude that $f_k \leq 0.35$ for $k = 1, 2, 3$. So Claim 2 holds.

From Claim 1 and Claim 2, we deduce

Claim 3. α is maximized when there exists a permutation ijk of $\{1, 2, 3\}$ such that $b_{ij} > 0$ and $y_k = 0$ (so $x_i = x_j = 0$).

We consider three cases.

Case 1. α is maximized when $x_k = b_{ik} = b_{jk} = 0$ and $b_k = 0$.

Then $b_{ij} + a_k = 1$, $f_k = (1-p_k)b_{ij}$, $f_i = (1-p_i)^2 a_k$, and $f_j = (1-p_j)^2 a_k$.

Since $f_i = f_j$, we have $p_i = p_j$. In particular, $p_i \in (0, 1/2)$ as $p_i + p_j + p_k = 1$. Since $b_{ij} = 1 - a_k$ and $f_k = f_i$, we have $2p_i(1-a_k) = (1-p_i)^2 a_k$. Therefore, $a_k = 2p_i/(1+p_i^2)$, and so,

$$\alpha = \frac{2p_i(1-p_i)^2}{1+p_i^2} = \frac{4}{1+p_i^2} + 2p_i - 4.$$

Differentiating with respect to p_i, we have $\alpha'(p_i) = 2 - 8p_i/(1+p_i^2)^2$ and $\alpha''(p_i) < 0$. Thus $\alpha(p_i)$ has maximum when $\alpha'(p_i) = 0$, i.e., when $(1+p_i^2)^2 = 4p_i$. We now estimate $\alpha(p_i)$ subject to $(1+p_i^2)^2 = 4p_i$. Considering the function $g(x) := (1+x^2)^2 - 4x$ for $x \in (0, 1/2)$, we see that $g'(x) = 4(1+x^2)x - 4 < 0$, $g(0.3) < 0$, and $g(0.29) > 0$; so $g(x) = 0$ implies that $x \in (0.29, 0.3)$. Hence, $(1+p_i^2)^2 = 4p_i$ implies $p_i \in (0.29, 0.3)$. On the other hand, $(1+p_i^2)^2 = 4p_i$ implies $\alpha(p_i) = 2/\sqrt{p_i} + 2p_i - 4$. Since the function $h(t) := 2/\sqrt{t} + 2t - 4$ is decreasing over $[0.29, 0.3]$ (because $h' = 2 - t^{-3/2} < 0$ for $t \in [0.29, 0.3]$), we have $\alpha \le \alpha(p_i) = h(p_i) \le h(0.29) = 2/\sqrt{0.29} + 2(0.29) - 4 < 0.35$.

Case 2. α is maximized when $x_k = b_{ik} = b_{jk} = 0$ and $b_k > 0$.

Then $b_{ij} + (b_i + b_j + b_k)/2 = 1$, $f_i = (1-p_i)^2 b_i$, $f_j = (1-p_j)^2 b_j$, and $f_k = (1-p_k)b_{ij} + (1-p_k)^2 b_k$. From $\partial\alpha/\partial b_k = \lambda_1 \partial g_1/\partial b_k + \lambda_2 \partial g_2/\partial b_k$, we obtain

$$0 = \lambda_1 \frac{-4\alpha^2}{\sqrt{(b_{ij}+y_k)^2 + 4b_k\alpha}\left(\sqrt{(b_{ij}+y_k)^2 + 4b_k\alpha} + (b_{ij}+y_k)\right)^2} + \frac{\lambda_2}{2}.$$

Using this and the partial derivatives with respect to $u \in \{\alpha, b_{ij}\}$ (as in the proof of Claim 1), we deduce that $\lambda_1 \ne 0$ and $\lambda_2 \ne 0$, and

$$4\alpha = b_{ij} + \sqrt{b_{ij}^2 + 4b_k\alpha}.$$

Therefore, α is maximized when $4\alpha = b_{ij} + \sqrt{b_{ij}^2 + 4b_k\alpha}$, that is $4\alpha = b_k + 2b_{ij}$ which implies $p_k = 1/2$ (since $f_k(p_k)$ is decreasing and $f_k(p_k) = \alpha$ has a unique solution).

Write $b'_k := b_k + 2b_{ij}$; then $f_k = (1-p_k)^2 b'_k$ (because $p_k = 1/2$). Note that $(b'_k + b_i + b_j)/2 = b_{ij} + (b_k + b_i + b_j)/2 = 1$. Since $\alpha = f_1 = f_2 = f_3$ and $(1-p_i) + (1-p_j) + (1-p_k) = 2$, we have

$$\frac{\sqrt{\alpha}}{\sqrt{b'_k}} + \frac{\sqrt{\alpha}}{\sqrt{b_i}} + \frac{\sqrt{\alpha}}{\sqrt{b_j}} = 2.$$

Applying Cauchy-Schwarz, we have

$$\alpha = \left(\frac{2}{\frac{1}{\sqrt{b'_k}} + \frac{1}{\sqrt{b_i}} + \frac{1}{\sqrt{b_j}}}\right)^2 \le 4\left(\frac{\sqrt{b'_k} + \sqrt{b_i} + \sqrt{b_j}}{9}\right)^2 \le \frac{4}{9}\frac{b'_k + b_i + b_j}{3} = \frac{8}{27} < 0.35.$$

Case 3. α is maximized when (i) $x_k > 0$, or (ii) $x_k = 0$ and $b_{ik} > 0$ or $b_{jk} > 0$.

We claim that there exist $a'_m, x'_m, b'_{mn} \in \mathbb{R}^+$, for any $1 \le m \ne n \le 3$, such that $b'_{mn} = b'_{nm}$,

$$\begin{aligned}
&b'_{mn} \ge \max\{2x'_m, 2x'_n\},\\
&b'_{12} + b'_{23} + b'_{31} + x'_1 + x'_2 + x'_3 + a'_1 + a'_2 + a'_3 = 1,\\
&b'_{mn} + x'_m + x'_n \ge b_{mn} + x_m + x_n,\\
&a'_m + a'_n = a_m + a_n, \text{ and}\\
&b'_{st} + x'_s + x'_t > b_{st} + x_s + x_t \text{ for some } 1 \le s \ne t \le 3.
\end{aligned}$$

There are two cases to consider. First, suppose $x_k > 0$. Then there exists $\delta > 0$ such that $x'_k = x_k - \delta > 0$ and $b'_{ij} = b_{ij} - 2\delta \ge 2\delta$. Let $b'_{ik} = b_{ik} + \delta, b'_{jk} = b_{jk}$ and $x'_i = x'_j = \delta$. In particular, $x_k > \delta$; and so $b_{ik} \ge 2x_k \ge 2\delta$ and $b_{jk} \ge 2x_k \ge 2\delta$. It is easy to verify that the claim holds by setting $a'_i = a_i$, $a'_j = a_j$ and $a'_k = a_k$. Now assume that $x_k = 0$, and $b_{ik} > 0$ or $b_{jk} > 0$. We may assume $b_{ik} > 0$; the case $b_{jk} > 0$ is symmetric. Then there exists $\delta > 0$ such that $b'_{ik} = b_{ik} - \delta/2 \ge \delta$ and $b'_{ij} = b_{ij} - \delta/2 \ge \delta$. Let $b'_{jk} = b_{jk} + \delta/2$ and $x'_i = \delta/2$. It is easy to verify that the claim holds by setting $x'_j = x_j = 0$, $x'_k = x_k = 0$, $a'_i = a_i$, $a'_j = a_j$ and $a'_k = a_k$.

For every permutation mnl of $\{1, 2, 3\}$, let

$$f'_l := (1 - p_l)(b'_{mn} + x'_m + x'_n) + (1 - p_l)^2(a'_m + a'_n).$$

For convenience of comparison, recall that

$$\alpha := f_l = (1 - p_l)(b_{mn} + x_m + x_n) + (1 - p_l)^2(a_m + a_n).$$

By Lemma 2.1, there exist $p'_i \in [0, 1]$ with $p'_1 + p'_2 + p'_3 = 1$ such that $f'_l(p'_l) \le 0.35$ for $l = 1, 2, 3$, or $f'_1(p'_1) = f'_2(p'_2) = f'_3(p'_3)$ and $p'_i \in (0, 1)$. Since $p_i \in [0, 1]$ and $p_1 + p_2 + p_3 = 1$, there exists some l such that $1 - p_l \le 1 - p'_l$.

If $f'_i(p'_i) \le 0.35$ for $i = 1, 2, 3$ then, since $b'_{mn}+x'_m+x'_n \ge b_{mn}+x_m+x_n$ and $a'_m+a'_n = a_m+a_n$ for all $\{m, n, l\} = \{1, 2, 3\}$, we have $f_l(p_l) \le f'_l(p'_l) \le 0.35$. Hence $\alpha \le 0.35$.

We may thus assume $f'_1(p'_1) = f'_2(p'_2) = f'_3(p'_3)$. Suppose $1 - p_l < 1 - p'_l$. Then, since $b'_{mn}+x'_m+x'_n \ge b_{mn}+x_m+x_n$ and $a'_m+a'_n = a_m+a_n$, and because $b_{mn}+x_m+x_n+a_m+a_n > 0$ (see the beginning of the proof), we have $f_l(p_l) < f'_l(p'_l)$, contradicting the maximality of α. So $1 - p_l = 1 - p'_l$. Then $(1 - p'_m) + (1 - p'_n) = (1 - p_m) + (1 - p_n)$. So we may assume that $1 - p_n \le 1 - p'_n$. By the same argument above for $1 - p'_l = 1 - p_l$, we derive the contradiction $f_n(p_n) < f'_n(p'_n)$ if $1-p_n < 1-p'_n$; and so we must have $1-p'_n = 1-p_n$. Hence we have $p'_i = p_i$ for $i = 1, 2, 3$. Recall that there exist $1 \le s \ne t \le 3$ such that $b'_{st} + x'_s + x'_t > b_{st} + x_s + x_t$. Let $r \in \{1, 2, 3\} \setminus \{s, t\}$. Then $f_r(p_r) < f'_r(p'_r)$, again a contradiction to the maximality of α. ∎

5 Conclusion

We have shown that if G is a 3-uniform hypergraph with m edges then $V(G)$ admits a partition V_1, V_2, V_3 so that each V_i meets at least $0.65m - o(m)$ edges. Towards this end, we mention a conjecture of Bollobás and Scott in [5]: for integers $r, k \ge 2$, every r-uniform hypergraph with m edges has a vertex-partition into k sets, each of which meets at least $(1+o(1))(1-(1-1/r)^k)m$ edges. In particular, for $r = k = 3$, the bound in this conjecture is $19/27m + o(m)$, where $19/27 \approx 0.7037$. Although our method can be modified to make further improvement on the current bound of 0.65, it is unlikely to yield a bound close to 19/27 because of the bound in Lemma 2.1.

We also mention a related problem for graphs. It is conjectured in [5] that every graph with m edges admits a k-partition, $k \geq 3$, such that $d(V_i) \geq 2m/(2k-1)$. The complete graph on $2k-1$ vertices shows that the lower bound is best possible. This conjecture is shown to be true in [9] for sufficiently large m. In fact, it is shown [9] that $d(V_i) \geq m/(k-1) + o(m)$. It may be possible to demand $d(V_i) \geq (2k-1)m/k^2 + o(m)$.

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