On judicious bipartitions of graphs

Jie Ma*
Xingxing Yu†

Abstract

For a positive integer m, let $f(m)$ be the maximum value t such that any graph with m edges has a bipartite subgraph of size at least t, and let $g(m)$ be the minimum value s such that for any graph G with m edges there exists a bipartition $V(G) = V_1 \cup V_2$ such that G has at most s edges with both incident vertices in V_i. Alon proved that the limsup of $f(m) - (m/2 + \sqrt{m/8})$ tends to infinity as m tends to infinity, establishing a conjecture of Erdős. Bollobás and Scott proposed the following judicious version of Erdős' conjecture: the limsup of $m/4 + \sqrt{m/32} - g(m)$ tends to infinity as m tends to infinity. In this paper, we confirm this conjecture. Moreover, we extend this conjecture to k-partitions for all even integers k. On the other hand, we generalize Alon's result to multi-partitions, which should be useful for generalizing the above Bollobás-Scott conjecture to k-partitions for odd integers k.

*School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui, 230026, China.
†School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332-0160, USA. Partially supported by NSF grants DMS-1265564 and AST-1247545 and NSA grant H98230-13-1-0255.

1 Introduction

Let G be a graph and $e \in E(G)$. We will use $V(e)$ to denote the set of the vertices of G incident with e. Let $S, T \subseteq V(G)$. We write $e_G(S) := |\{e \in E(G) : V(e) \subseteq S\}|$, $(S,T)_G := \{e \in E(G) : V(e) \cap S \neq \emptyset \neq V(e) \cap T\}$, and $e_G(S,T) := |(S,T)_G|$. If $S = \{v\}$, then we simply write $e_G(v,T)$ for $e_G(\{v\},T)$. For any integer $k \geq 2$ and any k-partition $V_1, V_2, ..., V_k$ of $V(G)$, let $e_G(V_1, V_2, ..., V_k) = \sum_{1 \leq i < j \leq k} e(V_i, V_j)$. When understood, the reference to G in the subscript will be dropped.

Again let G be a graph. We write $e(G) := |E(G)|$. For any integer $k \geq 2$, let $f_k(G)$ denote the maximum number of edges in a k-partite subgraph of G. For any integers $k \geq 2$ and $m \geq 1$, let $f_k(m) := \min\{f_k(G) : e(G) = m\}$. Let $f(G) := f_2(G)$ and $f(m) := f_2(m)$.

The problem for deciding $f(G)$, known as the *Max Cut Problem*, is NP-hard, and there has been extensive work on approximating $f(G)$, see [4, 14–16, 19]. On the other hand, it is easy to prove that any graph with m edges has a partition V_1, V_2 with $e(V_1, V_2) \geq m/2$. Edwards [11, 12] improved this lower bound by showing

$$f(m) \geq \frac{m}{2} + \frac{1}{4}\left(\sqrt{2m + 1/4} - \frac{1}{2}\right) = m/2 + \sqrt{m/8} + O(1).$$

This is best possible, as K_{2n+1} are extremal graphs. Erdős [13] conjectured that the limsup of

$$f(m) - (m/2 + \sqrt{m/8})$$

tends to infinity as m tends to infinity. Alon [1] confirmed this conjecture with the following

Theorem 1.1 (*Alon*) *There exist absolute constants $c > 0$ and $M > 0$ such that for every even integer $n > M$, if $m = n^2/2$ then*

$$f(m) \geq m/2 + \sqrt{m/8} + cm^{1/4}.$$

When m is sufficiently large, the function $f(m)$ was determined exactly in a recursive formula by Alon and Halperin [3] and Bollobás and Scott [7] independently.

Bollobás and Scott [5] considered problems in which one needs to find a partition of a given graph or hypergraph to optimize several quantities simultaneously. Such problems are called *judicious partitioning problems*, and we refer the reader to [7–9, 17] for more problems and references.

Let G be a graph and k a positive integer; we use $g_k(G)$ to denote the minimal number t such that there exists a k-partition $V(G) = V_1 \cup \ldots \cup V_k$ satisfying $e(V_i) \leq t$ for $i = 1, ..., k$. For any positive integers k and m, let $g_k(m) := \max\{g_k(G) : e(G) = m\}$. Let $g(G) := g_2(G)$ and $g(m) := g_2(m)$.

The problem of determining $g(G)$ is known as the *Bottleneck Graph Bipartition Problem* and is also NP-hard. The NP-hardness of determining $g(G)$ can be derived from the NP-hardness of Max Cut: taking a graph G consisting of two disjoint copies of graph H, then one can see that $g(G) = e(H) - f(H)$. On the other hand, Bollobás and Scott [6] showed that for any positive integer m

$$g(m) \leq \frac{m}{4} + \frac{1}{8}(\sqrt{2m + 1/4} - 1/2) = m/4 + \sqrt{m/32} + O(1).$$

This bound is sharp, as K_{2n+1} are extremal graphs. In fact, the result they proved is stronger, which states that this bound as well as the Edwards bound can be achieved simultaneously by some bipartite in every graph with m edges. Bollobás and Scott [7] (also see [17]) proposed the following judicious version of Erdős' conjecture.

Conjecture 1.2 (*Bollobás and Scott*) *The limsup of*

$$m/4 + \sqrt{m/32} - g(m)$$

tends to infinity as m tends to infinity.

The main goal of this paper is to prove the coming result which confirms Conjecture 1.2.

Theorem 1.3 *There exist absolute constants $d > 0$ and $N > 0$ such that for every even integer $n > N$, if $m = n^2/2$ then*

$$g(m) \leq m/4 + \sqrt{m/32} - dm^{1/4}.$$

We describe the main ideas of the proofs as follows. To establish an upper bound of $g(m)$, i.e., to find a bipartition such that each of its two parts contains a small number of edges, we start with a maximum cut, say (V_1, V_2). In light of Theorem 1.1, this already guarantees that $e(V_1) + e(V_2)$ is relatively small. We then modify the bipartition by moving some vertices from the part with more edges (say V_1) to the other part until V_1 has a small enough number of edges left, and keeping the number of edges in V_2 not growing too much. Therefore, at each step we wish to choose a vertex in V_1 to move such that the number of edges lost in V_1 and the number of new edges added in V_2 both are under control. To prove the existence of such a vertex (say a "good" one), we show that there cannot be too many "bad" vertices. Similar approaches have been employed in [2, 6, 10], while the challenges for us are to find suitable measures of "goodness" in all different cases, which are classified by the size of the maximum cut.

Bollobás and Scott [8] (also see [7]) extended Edwards' bound to k-partitions of graphs and proved that for any graph G with m edges has a k-partition $V_1, \ldots, V_k$ such that

$$e(V_1, \ldots, V_k) \geq \frac{k-1}{k}m + \frac{k-1}{2k}(\sqrt{2m + 1/4} - 1/2) + O(k), \tag{1}$$

with equality when G is the complete graph of order $kn + 1$, where the $O(k)$ term is $(-k^2 + 4k - 4)/(8k)$. Bollobás and Scott [6] also showed that the vertex set of any graph G with m edges can be partitioned into $V_1, \ldots, V_k$ such that for $i \in \{1, 2, \ldots, k\}$,

$$e(V_i) \leq \frac{m}{k^2} + \frac{k-1}{2k^2}(\sqrt{2m + 1/4} - 1/2). \tag{2}$$

Recently, Xu and Yu [18] showed that a k-partition $V_1, \ldots, V_k$ can be found that satisfies both inequalities (1) and (2) simultaneously, generalizing an earlier bipartition result of Bollobás and Scott [6] as mentioned previously.

It is natural to ask whether Theorems 1.1 and 1.3 can be extended to k-partitions. We show that the generalization of Theorem 1.3 to k-partitions does hold for all even integers k.

Theorem 1.4 *Let $k > 0$ be an even integer, and let d and m be the same as in Theorem 1.3. Then,*

$$g_k(m) \le \frac{m}{k^2} + \frac{k-1}{2k^2}\sqrt{2m} - \frac{4d}{k^2} \cdot m^{1/4} + O(k).$$

We also prove the following generalization of Theorem 1.1, which we think should be useful for extending Theorem 1.3 to k-partitions for odd integers k.

Theorem 1.5 *For any integer $k \ge 2$, there exist positive constants $c(k) = O(1/\sqrt{k})$ and $N(k) = O(k^3)$ such that for every even integer $n > N(k)$, if $m = n^2/2$ then*

$$f_k(m) \ge \frac{k-1}{k}m + \frac{k-1}{2k}\sqrt{2m} + c(k)m^{1/4}.$$

We organize this paper as follows. In Section 2, we prove Theorem 1.3 by establishing a sequence of lemmas. In Section 3, we first present the proof of Theorem 1.4, and then prove Theorem 1.5 by refining Alon's original proof of Theorem 1.1. In Section 4, we offer some concluding remarks.

2 Bipartitions

In this section, we prove Theorem 1.3. Our proof is divided into several cases according to values of $f(G) - e(G)/2$. Since we will be using maximum bipartite subgraphs to find good judicious partitions, we need a result from [2] which establishes a connection between $f(G)$ and $g(G)$; a similar connection between $f_k(G)$ and $g_k(G)$ can be found in Bollobás and Scott [10].

Lemma 2.1 (*Alon, Bollobás, Krivelevich and Sudakov*) *Let G be a graph with m edges and $f(G) = m/2 + \delta$. If $\delta \ge m/30$ then there exists an absolute constant D such that, when $m \ge D$ there exists a bipartition $V(G) = V_1 \cup V_2$ satisfying*

$$\max\{e(V_1), e(V_2)\} \le \frac{m}{4} - \frac{m}{100}. \tag{3}$$

If $\delta \le m/30$, then there exists a partition $V(G) = V_1 \cup V_2$ such that

$$\max\{e(V_1), e(V_2)\} \le \frac{m}{4} - \frac{\delta}{2} + \frac{10\delta^2}{m} + 3\sqrt{m}. \tag{4}$$

The following easy consequence of Lemma 2.1 proves Theorem 1.3 for graphs G with $e(G) = m$ and $f(G) - m/2 \ge m/10^4$.

Lemma 2.2 *There exists an absolute constant $M_1 > 0$ such that the following holds. If G is a graph with m edges and $f(G) = m/2 + \delta$, $m \ge M_1$, and $\delta \ge m/10^4$, then there exists a bipartition $V(G) = V_1 \cup V_2$ such that for $i = 1, 2$,*

$$e(V_i) \le \frac{m}{4} - \frac{m}{4 \times 10^4}.$$

Proof. Let D be from Lemma 2.1. Let $M_1 \geq D$ be a sufficiently large constant (here and later we do not express the constants explicitly to keep the presentation clear). Then by Lemma 2.1, we may assume that $m/10^4 \leq \delta \leq m/30$, and that there exists a partition $V(G) = V_1 \cup V_2$ such that (4) holds. As the function $h(\delta) := -\delta/2 + 10\delta^2/m$ achieves its maximum at $\delta = m/40$, we see that $e(V_i) \leq \frac{m}{4} + h(m/40) + 3\sqrt{m} \leq m/4 - m/(4 \times 10^4)$ for each $i = 1, 2$. ∎

In view of Lemma 2.2, it suffices to prove Theorem 1.3 for graphs G with $f(G) - m/2 \leq m/10^4$. We will find special vertices that we could use to modify existing bipartitions. This will be done in the next lemma and Lemma 2.5, using the approach described in the introduction.

Lemma 2.3 *Let G be a graph with m edges, and assume $f(G) = m/2 + \delta$, where $\delta \leq m/10^4$. Suppose $V(G) = V_1 \cup V_2$ is a partition such that $e(v, V_1) \leq e(v, V_2)$ for every $v \in V_1$, and $e(V_1) \geq m/4 - \delta/2$. Then there exists $v \in V_1$ such that*

$$e(v, V_1) \leq \sqrt{m/2} + 3\sqrt{\delta} \text{ and } e(v, V_2) \leq (1 + 20\sqrt{\delta/m})e(v, V_1).$$

Proof. Let

$$T = \{v \in V_1 : e(v, V_1) > \sqrt{m/2} + 3\sqrt{\delta}\}$$

and

$$S = \{v \in V_1 : e(v, V_2) > \left(1 + 20\sqrt{\delta/m}\right) e(v, V_1)\}.$$

We will show that $\sum_{v \in V_1} e(v, V_1) > \sum_{v \in S \cup T} e(v, V_1)$, from which the existence of the desired vertex v follows. To this end, we bound $\sum_{v \in T} e(v, V_1)$ and $\sum_{v \in S} e(v, V_1)$.

Since $e(v, V_1) \leq e(v, V_2)$ for all $v \in V_1$,

$$e(V_1) = \frac{1}{2} \sum_{v \in V_1} e(v, V_1) \leq \frac{1}{2} \sum_{v \in V_1} e(v, V_2) = \frac{1}{2} e(V_1, V_2) \leq \frac{f(G)}{2} = \frac{m}{4} + \frac{\delta}{2}.$$

On the other hand,

$$2e(V_1) = \sum_{v \in V_1} e(v, V_1) \geq \sum_{v \in T} e(v, V_1) > \left(\sqrt{m/2} + 3\sqrt{\delta}\right) |T|,$$

implying that

$$|T| < \frac{2e(V_1)}{\sqrt{m/2} + 3\sqrt{\delta}} \leq \frac{m/2 + \delta}{\sqrt{m/2} + 3\sqrt{\delta}} < \sqrt{m/2} - \frac{3}{2}\sqrt{\delta},$$

where the final inequality holds because $\delta \leq m/10^4$. Hence

$$\sum_{v \in T} e(v, V_1) \leq e(V_1) + e(T) < e(V_1) + \frac{1}{2}|T|^2 < e(V_1) + \frac{1}{2}\left(\sqrt{m/2} - \frac{3}{2}\sqrt{\delta}\right)^2.$$

Since $\delta \leq m/10^4$, we have

$$\sum_{v \in T} e(v, V_1) < e(V_1) + \frac{m}{4} - \frac{3}{4}\sqrt{m\delta/2}. \tag{5}$$

Note that $e(V_1,V_2) = \sum_{v\in S} e(v,V_2) + \sum_{v\in V_1-S} e(v,V_2)$, so

$$e(V_1,V_2) \geq \left(1+20\sqrt{\delta/m}\right)\sum_{v\in S} e(v,V_1) + \sum_{v\in V_1-S} e(v,V_1) = 2e(V_1) + 20\sqrt{\delta/m}\sum_{v\in S} e(v,V_1).$$

Together with $e(V_1) \geq m/4 - \delta/2$ and $e(V_1,V_2) \leq f(G) = m/2+\delta$, we get

$$\sum_{v\in S} e(v,V_1) < \frac{1}{20}\sqrt{m/\delta}\cdot(e(V_1,V_2) - 2e(V_1)) \leq \frac{1}{10}\sqrt{m\delta}. \tag{6}$$

Combining (5) and (6), we have $\sum_{v\in S\cup T} e(v,V_1) < e(V_1) + \frac{m}{4} - \frac{3}{8}\sqrt{m\delta/2}$, implying that

$$\sum_{v\in V_1} e(v,V_1) = 2e(V_1) \geq e(V_1) + \frac{m}{4} - \frac{\delta}{2} > e(V_1) + \frac{m}{4} - \frac{3}{8}\sqrt{m\delta/2} > \sum_{v\in S\cup T} e(v,V_1),$$

where the first inequality follows from $e(V_1) \geq m/4 - \delta/2$ and the second inequality holds because $\delta \leq m/10^4$. So $V_1 - (S\cup T) \neq \emptyset$, completing the proof. ∎

The following result says that there exists an absolute constant $C > 0$ such that Theorem 1.3 holds for graphs G with m edges and $\sqrt{m/8} + Cm^{1/4} \leq f(G) - m/2 \leq m/10^4$.

Lemma 2.4 *Let c be the absolute constant in Theorem 1.1, let $d = c/4$, and let $C := 2(70+d)$. There exists an absolute constant $M_2 > 0$ such that the following holds. If G is a graph with m edges, $f(G) = m/2+\delta$, $m \geq M_2$, and $\sqrt{m/8} + Cm^{1/4} \leq \delta \leq m/10^4$, then there exists a partition $V(G) = V_1 \cup V_2$ such that for $i = 1,2$,*

$$e(V_i) \leq m/4 + \sqrt{m/32} - dm^{1/4}.$$

Proof. Let $M_2 > 0$ be a sufficiently large constant. Let (U_1,U_2) be a maximum cut of G, i.e.,

$$e(U_1,U_2) = f(G) = \frac{m}{2} + \delta.$$

Without loss of generality we may assume $e(U_1) \geq e(U_2)$. Note that $e(v,U_1) \leq e(v,U_2)$ for every $v \in U_1$; otherwise, $e(U_1 - \{v\}, U_2 \cup \{v\}) > e(U_1,U_2) = f(G)$, a contradiction. Hence

$$e(U_1) = \frac{1}{2}\sum_{v\in U_1} e(v,U_1) \leq \frac{1}{2}\sum_{v\in U_1} e(v,U_2) = \frac{1}{2}e(U_1,U_2) = \frac{m}{4} + \frac{\delta}{2}.$$

We now define a process to move vertices from U_1 to U_2, using Lemma 2.3, such that in the end we get the desired bipartition.

- Initially, we set $V_1^0 := U_1, V_2^0 := U_2$. Let $V_1^i \cup V_2^i$ denote the partition of $V(G)$ after the ith iteration.
- If $e(V_1^i) \leq m/4 - \delta/2 + (\sqrt{m/2} + 3\sqrt{\delta})/2$, set $V_1 := V_1^i$ and $V_2 := V_2^i$, and stop.
- If $e(V_1^i) > m/4 - \delta/2 + (\sqrt{m/2} + 3\sqrt{\delta})/2$ then by Lemma 2.3, there exists $u_i \in V_1^i$ such that

  $$e(u_i, V_1^i) \leq \sqrt{m/2} + 3\sqrt{\delta} \quad \text{and} \quad e(u_i, V_2^i) \leq \left(1+20\sqrt{\delta/m}\right)e(u_i,V_1^i).$$

  Set $V_1^{i+1} := V_1^i - \{u_i\}, V_2^{i+1} := V_2^i \cup \{u_i\}$, and repeat the above steps for V_1^{i+1}, V_2^{i+1}.

Note that, after i iterations (for each i), we always have $e(v, V_1^i) \le e(v, V_2^i)$ for every $v \in V_1^i$; so Lemma 2.3 may be applied to V_1^i, V_2^i. Let $V_1 = V_1^k, V_2 = V_2^k$ denote the final partition, after k iterations. Then

$$e(V_1) \le \frac{m}{4} - \frac{\delta}{2} + \sqrt{m/8} + \frac{3}{2}\sqrt{\delta}.$$

Moreover, since V_1 is obtained from V_1^{k-1} by moving u_{k-1} to V_2^{k-1}, we have

$$e(V_1) > \frac{m}{4} - \frac{\delta}{2} - \frac{1}{2}(\sqrt{m/2} + 3\sqrt{\delta}).$$

Since $e(V_1) = e(U_1) - \sum_{i=0}^{k-1} e(u_i, V_1^i)$, we have

$$\sum_{i=0}^{k-1} e(u_i, V_1^i) = e(U_1) - e(V_1) < e(U_1) - \frac{m}{4} + \frac{\delta}{2} + \frac{1}{2}(\sqrt{m/2} + 3\sqrt{\delta}).$$

Hence, since $e(U_2) = m - e(U_1, U_2) - e(U_1) = m/2 - \delta - e(U_1)$ and $e(u_i, V_2^i) < (1 + 20\sqrt{\delta/m})e(u_i, V_1^i)$, we have

$$\begin{aligned}
e(V_2) &= e(U_2) + \sum_{i=0}^{k-1} e(u_i, V_2^i) \\
&\le \frac{m}{2} - \delta - e(U_1) + (1 + 20\sqrt{\delta/m}) \sum_{i=0}^{k-1} e(u_i, V_1^i) \\
&\le \frac{m}{2} - \delta - e(U_1) + (1 + 20\sqrt{\delta/m}) \left(e(U_1) - \frac{m}{4} + \frac{\delta}{2} + \frac{1}{2}\sqrt{m/2} + \frac{3}{2}\sqrt{\delta} \right) \\
&= \frac{m}{4} - \frac{\delta}{2} + \frac{1}{2}\sqrt{m/2} + \frac{3}{2}\sqrt{\delta} + 20\sqrt{\delta/m} \left(e(U_1) - \frac{m}{4} + \frac{\delta}{2} + \frac{1}{2}\sqrt{m/2} + \frac{3}{2}\sqrt{\delta} \right) \\
&\le \frac{m}{4} - \frac{\delta}{2} + \frac{1}{2}\sqrt{m/2} + \frac{3}{2}\sqrt{\delta} + 20\sqrt{\delta/m} \left(\delta + \frac{1}{2}\sqrt{m/2} + \frac{3}{2}\sqrt{\delta} \right) \quad (\text{since } e(U_1) \le m/4 + \delta/2) \\
&= \frac{m}{4} - \frac{\delta}{2} + \sqrt{m/8} + (3/2 + 5\sqrt{2})\sqrt{\delta} + 20\frac{\delta^{3/2}}{\sqrt{m}} + 30\frac{\delta}{\sqrt{m}}.
\end{aligned}$$

Therefore,

$$\max\{e(V_1), e(V_2)\} \le \frac{m}{4} + \sqrt{m/8} + h(\delta),$$

where

$$h(\delta) := -\frac{\delta}{2} + (3/2 + 5\sqrt{2})\sqrt{\delta} + 20\frac{\delta^{3/2}}{\sqrt{m}} + 30\frac{\delta}{\sqrt{m}}.$$

Since $m \ge M_2$ is sufficiently large, one can verify that $h(\delta)$ is a decreasing function in the domain $\sqrt{m/8} + Cm^{1/4} \le \delta \le m/10^4$. Therefore,

$$\begin{aligned}
&\max\{e(V_1), e(V_2)\} \\
\le& \frac{m}{4} + \sqrt{m/8} + h(\sqrt{m/8} + Cm^{1/4}) \\
<& \frac{m}{4} + \sqrt{m/32} - \frac{C}{2}m^{1/4} + 70m^{1/4} \quad (\text{since } \sqrt{m/8} + Cm^{1/4} < \sqrt{m}) \\
=& \frac{m}{4} + \sqrt{m/32} - dm^{1/4} \quad (\text{since } C = 2(70 + d)),
\end{aligned}$$

completing the proof of this lemma. ∎

The next result is similar to Lemma 2.3, which will be used to prove Theorem 1.3 for graphs G with m edges and $f(G) - m/2 \le \sqrt{m/8} + Cm^{1/4}$.

Lemma 2.5 *Let c be the constant in Theorem 1.1 and d, C be the constants in Lemma 2.4. There exists an absolute constant $M_3 > 0$ such that the following holds. If*

- *G is a graph with $m \ge M_3$ edges,*
- *$f(G) = m/2 + \delta$,*
- *$\sqrt{m/8} + cm^{1/4} \le \delta \le \sqrt{m/8} + Cm^{1/4}$,*
- *$V(G) = V_1 \cup V_2$ is a bipartition such that $e(v, V_1) \le e(v, V_2)$ for every $v \in V_1$, and $e(V_1) \ge m/4 + \sqrt{m/32} - dm^{1/4}$*

then there exists $v \in V_1$, such that

$$e(v, V_1) \le \sqrt{m/2} + \frac{c}{6}\sqrt{\delta} \quad \text{and} \quad e(v, V_2) \le \left(1 + \frac{c}{4}\sqrt{\delta/m}\right) e(v, V_1).$$

Proof. Let $M_3 > 0$ be a sufficiently large constant (compared to c, d and C). And let

$$T = \left\{v \in V_1 : e(v, V_1) > \sqrt{m/2} + \frac{c}{6}\sqrt{\delta}\right\}$$

and

$$S = \left\{v \in V_1 : e(v, V_2) > \left(1 + \frac{c}{4}\sqrt{\delta/m}\right) e(v, V_1)\right\}.$$

Since $e(v, V_1) \le e(v, V_2)$ for all $v \in V_1$, we have

$$e(V_1) \le \frac{1}{2} e(V_1, V_2) \le \frac{f(G)}{2} = \frac{m}{4} + \frac{\delta}{2}.$$

On the other hand,

$$2e(V_1) \ge \sum_{v \in T} e(v, V_1) > \left(\sqrt{m/2} + \frac{c}{6}\sqrt{\delta}\right) |T|.$$

Thus

$$|T| < \frac{2e(V_1)}{\sqrt{m/2} + \frac{c}{6}\sqrt{\delta}} \le \frac{m/2 + \delta}{\sqrt{m/2} + \frac{c}{6}\sqrt{\delta}} < \sqrt{m/2} - \frac{c}{12}\sqrt{\delta},$$

where the last inequality holds because $\sqrt{m/8} + cm^{1/4} \le \delta \le \sqrt{m/8} + Cm^{1/4}$ and $m \ge M_3$ is large enough. From $\sum_{v \in T} e(v, V_1) \le e(V_1) + e(T)$, we obtain that

$$\sum_{v \in T} e(v, V_1) \le e(V_1) + \frac{1}{2}\left(\sqrt{m/2} - \frac{c}{12}\sqrt{\delta}\right)^2 \le e(V_1) + \frac{m}{4} - \frac{c}{24}\sqrt{m\delta/2}. \tag{7}$$

Since $e(V_1, V_2) = \sum_{v \in S} e(v, V_2) + \sum_{v \in V_1 - S} e(v, V_2)$,

$$e(V_1, V_2) \ge \left(1 + \frac{c}{4}\sqrt{\delta/m}\right) \sum_{v \in S} e(v, V_1) + \sum_{v \in V_1 - S} e(v, V_1) = 2e(V_1) + \frac{c}{4}\sqrt{\delta/m} \cdot \sum_{v \in S} e(v, V_1).$$

Therefore

$$\sum_{v\in S} e(v,V_1) \le \frac{4}{c}\sqrt{m/\delta}\cdot (e(V_1,V_2)-2e(V_1)),$$

which, together with $e(V_1,V_2)\le m/2+\delta$, $e(V_1)\ge m/4+\sqrt{m/32}-dm^{1/4}$ and $\delta\le\sqrt{m/8}+Cm^{1/4}$, imply that

$$\sum_{v\in S} e(v,V_1) \le \frac{4}{c}(C+2d)\sqrt{m/\delta}\cdot m^{1/4}. \tag{8}$$

Again, because $e(V_1)\ge m/4+\sqrt{m/32}-dm^{1/4}$, we have

$$\sum_{v\in V_1} e(v,V_1) = 2e(V_1) \ge e(V_1)+m/4+\sqrt{m/32}-dm^{1/4}. \tag{9}$$

When $m\ge M_3$, in view of $\delta=\Theta(\sqrt{m})$, we see

$$\sqrt{m/32}-dm^{1/4} > \frac{4}{c}(C+2d)\sqrt{m/\delta}\cdot m^{1/4} - \frac{c}{24}\sqrt{m\delta/2},$$

which, combining with (7), (8) and (9), imply that

$$\sum_{v\in V_1} e(v,V_1) > \sum_{v\in S\cup T} e(v,V_1).$$

So we get $V_1-(S\cup T)\ne\emptyset$ as desired. ∎

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Let c, M be the absolute constants in Theorem 1.1, and let M_1, M_2, M_3 be the absolute constants in Lemmas 2.2, 2.4 and 2.5, respectively. Let $d:=c/4$, $M':=\max\{M_1,M_2,M_3\}$, and $N\ge\max\{M,(2M')^{1/2}\}$ be a sufficiently large constant.

Let $m=n^2/2$, where $n\ge N$ is even; so $n\ge M$ and $m\ge M_i$ for $i=1,2,3$, and hence we may apply Theorem 1.1 and Lemmas 2.2 and 2.4 and 2.5.

Let G be a graph with m edges and $f(G)=m/2+\delta$. By Theorem 1.1, Lemma 2.2 and 2.4, we may assume that

$$\sqrt{m/8}+cm^{1/4}\le\delta\le\sqrt{m/8}+Cm^{1/4}. \tag{10}$$

Let $V(G)=U_1\cup U_2$, with $e(U_1,U_2)=f(G)=\frac{m}{2}+\delta$ and $e(U_1)\ge e(U_2)$. Then, as before, we see that $e(v,U_1)\le e(v,U_2)$ for every $v\in U_1$.

We now describe a process similar to that in the proof of Lemma 2.4, to obtain the desired bipartition.

- Set $V_1^0:=U_1, V_2^0:=U_2$, and let $V(G)=V_1^i\cup V_2^i$ be the partition obtained after i iterations.
- If $e(V_1^i)\le m/4+\sqrt{m/32}-dm^{1/4}$, then set $V_1:=V_1^i$ and $V_2:=V_2^i$, and stop.

- If $e(V_1^i) > m/4 + \sqrt{m/32} - dm^{1/4}$ then by Lemma 2.5, there exists a vertex $u_i \in V_1^i$ such that

$$e(u_i, V_1^i) \le \sqrt{m/2} + \frac{c}{6}\sqrt{\delta} \quad \text{and} \quad e(u_i, V_2^i) \le \left(1 + \frac{c}{4}\sqrt{\delta/m}\right) e(u_i, V_1^i).$$

  Set $V_1^{i+1} := V_1^i - \{u_i\}, V_2^{i+1} := V_2^i \cup \{u_i\}$, and repeat the above steps for V_1^{i+1}, V_2^{i+1}.

Note that, in each iteration of the above procedure, we always have $e(v, V_1^i) \le e(v, V_2^i)$ for every $v \in V_1^i$; thus Lemma 2.5 may be applied for V_1^i, V_2^i. Let $V_1 = V_1^k, V_2 = V_2^k$ be the final partition, obtained after k steps. Then

$$e(V_1) \le m/4 + \sqrt{m/32} - dm^{1/4},$$

and

$$e(V_1) > m/4 + \sqrt{m/32} - dm^{1/4} - \left(\sqrt{m/2} + \frac{c}{6}\sqrt{\delta}\right).$$

It suffices to show the upper bound of $e(V_2)$. Since $e(v, U_1) \le e(v, U_2)$ for every $v \in U_1$,

$$e(U_1) \le \frac{1}{2} e(U_1, U_2) = \frac{m}{4} + \frac{\delta}{2}.$$

Since $e(V_1) = e(U_1) - \sum_{i=0}^{k-1} e(u_i, V_1^i)$, we have

$$\sum_{i=0}^{k-1} e(u_i, V_1^i) = e(U_1) - e(V_1) < e(U_1) - m/4 - \sqrt{m/32} + dm^{1/4} + \sqrt{m/2} + \frac{c}{6}\sqrt{\delta}.$$

Then, since $e(V_2) = e(U_2) + \sum_{i=0}^{k-1} e(u_i, V_2^i)$, $e(u_i, V_2^i) \le \left(1 + \frac{c}{4}\sqrt{\frac{\delta}{m}}\right) e(u_i, V_1^i)$ and $e(U_1) \le m/4 + \delta/2$, we have

$$\begin{aligned}
e(V_2) &\le \frac{m}{2} - \delta - e(U_1) + \left(1 + \frac{c}{4}\sqrt{\frac{\delta}{m}}\right) \sum_{i=0}^{k-1} e(u_i, V_1^i) \\
&< \frac{m}{2} - \delta - e(U_1) + \left(1 + \frac{c}{4}\sqrt{\delta/m}\right) \left(e(U_1) - \frac{m}{4} - \sqrt{m/32} + dm^{1/4} + \sqrt{m/2} + \frac{c}{6}\sqrt{\delta}\right) \\
&= \frac{m}{4} - \delta + 3\sqrt{m/32} + dm^{1/4} + \frac{c}{6}\sqrt{\delta} + \frac{c}{4}\sqrt{\delta/m} \cdot \left(e(U_1) - \frac{m}{4} + 3\sqrt{m/32} + dm^{1/4} + \frac{c}{6}\sqrt{\delta}\right) \\
&\le \frac{m}{4} - \delta + 3\sqrt{m/32} + dm^{1/4} + \frac{c}{6}\sqrt{\delta} + \frac{c}{4}\sqrt{\delta/m} \cdot \left(\frac{\delta}{2} + 3\sqrt{m/32} + dm^{1/4} + \frac{c}{6}\sqrt{\delta}\right).
\end{aligned}$$

By (10), it holds that $dm^{1/4} + (c/6)\sqrt{\delta} < \sqrt{m/32}$. Hence,

$$e(V_2) \le \frac{m}{4} + 3\sqrt{m/32} + dm^{1/4} + h(\delta),$$

where

$$h(\delta) := -\delta + \left(\frac{c}{6} + \frac{c}{\sqrt{32}}\right)\sqrt{\delta} + \frac{c}{8\sqrt{m}}\delta^{3/2}$$

is a decreasing function in the domain of (10), provided that $m \geq N^2/2$ is sufficiently large. Therefore, we have

$$\begin{aligned}
e(V_2) &\leq \frac{m}{4} + 3\sqrt{m/32} + dm^{1/4} + h(\sqrt{m/8} + cm^{1/4}) \\
&\leq \frac{m}{4} + 3\sqrt{m/32} + dm^{1/4} - \sqrt{m/8} - \frac{c}{4} \cdot m^{1/4} \quad (\text{using } \sqrt{m/8} + cm^{1/4} < \sqrt{m}) \\
&= \frac{m}{4} + \sqrt{m/32} - dm^{1/4} \quad (\text{since } d = c/4).
\end{aligned}$$

This completes the proof of Theorem 1.3. ∎

3 k-Partitions

In this section, we first show Theorem 1.4, extending Theorem 1.3 to all multiple partitions with even parts.

Proof of Theorem 1.4. Write $k := 2t$ for some integer $t > 0$ and let d, m be the same as in Theorem 1.3. Consider an arbitrary graph G with m edges. By Theorem 1.3, there exists a bipartition $V(G) = V^1 \cup V^2$ such that for $i = 1, 2$,

$$e_i := e(V^i) \leq m/4 + \sqrt{m/32} - dm^{1/4}.$$

Using the inequality $\sqrt{1+x} \leq 1 + \frac{x}{2}$ for $x \geq -1$, we have

$$\sqrt{e_i} \leq \sqrt{m/4} \cdot \Big(1 + (1/2) \cdot (\sqrt{m/32} - dm^{1/4})/(m/4)\Big) = \sqrt{m}/2 + O(1).$$

By the equation (2), each $G[V^i]$ has a t-partition $V_1^i, \ldots, V_t^i$ such that for $j \in \{1, 2, \ldots, t\}$,

$$\begin{aligned}
e(V_j^i) &\leq \frac{e_i}{t^2} + \frac{t-1}{2t^2} \cdot \sqrt{2e_i} + O(t) \leq \frac{m}{4t^2} + \left(\frac{1}{8t^2} + \frac{t-1}{4t^2}\right) \cdot \sqrt{2m} - \frac{d}{t^2} \cdot m^{1/4} + O(t) \\
&= \frac{m}{k^2} + \frac{k-1}{2k^2}\sqrt{2m} - \frac{4d}{k^2} \cdot m^{1/4} + O(k).
\end{aligned}$$

These two t-partitions together form a desired k-partition of G. ∎

Next we generalize the proof of Alon in [1] to prove Theorem 1.5. We need the following lemma which appears in several articles, for example, as Lemma 2.1 in [1].

Lemma 3.1 *Let $G = (V, E)$ be an s-colorable graph with m edges. Then for any positive integer $k \leq s$,*

$$f_k(G) \geq \frac{t(s,k)}{\binom{s}{2}} m, \quad \textit{where} \quad t(s,k) = \sum_{1 \leq i < j \leq k} \left\lfloor \frac{s+i-1}{k} \right\rfloor \left\lfloor \frac{s+j-1}{k} \right\rfloor.$$

In the coming proof of Theorem 1.5, we will be using Lemma 3.1 for $t(ks, k)$. Note that

$$t(ks, k) = s^2 \binom{k}{2} \text{ and } t(ks,k)/\binom{ks}{2} = \frac{k-1}{k} + \frac{k-1}{k} \frac{1}{ks-1}.$$

Proof of Theorem 1.5. Fix $k \geq 2$, and let $\epsilon = \frac{1}{16\sqrt{k}}$, $c(k) = \frac{2^{1/4}}{8}\epsilon$, and $N(k) = 32^2k^3$. (We do not attempt to optimize these constants.) Let n be an even integer such that $n \geq N(k)$. Consider an arbitrary graph G with $m = n^2/2$ edges. Let s denote the unique integer satisfying $n - \epsilon\sqrt{n} + 1 < ks \leq n - \epsilon\sqrt{n} + k + 1$.

Claim 1. We may assume that $\chi(G) \geq ks + 1$.

For, suppose G is ks-colorable. Then

$$\begin{aligned}
f_k(G) &\geq \frac{t(ks,k)}{\binom{ks}{2}} m \quad \text{(by Lemma 3.1)} \\
&= \frac{k-1}{k}m + \frac{k-1}{k}\frac{m}{ks-1} \\
&\geq \frac{k-1}{k}m + \frac{k-1}{2k}\frac{n^2}{n-\epsilon\sqrt{n}+k} \quad \text{(since } m = n^2/2 \text{ and } ks \leq n - \epsilon\sqrt{n} + k + 1) \\
&\geq \frac{k-1}{k}m + \frac{k-1}{2k}\left(n + \frac{\epsilon}{2}\sqrt{n}\right) \quad \text{(since } n \geq N(k) = 32^2k^3 \text{ and } \epsilon = 1/(16\sqrt{k})) \\
&\geq \frac{k-1}{k}m + \frac{k-1}{2k}\sqrt{2m} + \frac{1}{8}\epsilon(2m)^{1/4} \quad \text{(since } m = n^2/2 \text{ and } (k-1)/(4k) \geq 1/8).
\end{aligned}$$

This proves Claim 1.

Let $H \subseteq G$ such that $\chi(H) = \chi(G)$ and H is vertex-critical. Then

$$\delta(H) \geq \chi(G) - 1 \geq ks \geq n - \epsilon\sqrt{n}.$$

Since $e(H) \leq n^2/2$, $|V(H)| \leq 2e(H)/\delta(H) \leq n + 2\epsilon\sqrt{n}$. Then there are at least $n - 4\epsilon\sqrt{n}$ color classes of size 1 in any proper coloring of H using $\chi(G)$ colors. So there exists a complete subgraph R of G with $|V(R)| := n - r \geq n - 4\epsilon\sqrt{n}$.

Note that $r \leq 4\epsilon\sqrt{n}$ and

$$e(R) = \binom{n-r}{2} = \frac{n^2}{2} - (2r+1)\frac{n}{2} + \frac{r(r+1)}{2}.$$

Hence,

$$\sqrt{e(R)} = \frac{n - r + 1/2}{\sqrt{2}} + O(1) \geq \sqrt{m} - \epsilon\sqrt{8n} + O(1),$$

where the $O(1)$ term is $1/(2\sqrt{2}) - 1/8$. Let $W := V(G) - V(R)$. Then

$$e(W) + e(R, W) = m - e(R) = r(n-r) + \frac{n}{2} + \frac{r^2 - r}{2}.$$

Claim 2. We may assume that $e(W) \leq n/(8k)$.

Otherwise, assume $e(W) \geq n/(8k)$. By the equation (1), there exist k-partitions $W = \bigcup_{i=1}^k W_i$ and $V(R) = \bigcup_{i=1}^k R_i$ such that

$$\begin{aligned}
e(W_1, ..., W_k) &\geq \frac{k-1}{k}e(W) + \frac{k-1}{2k}\sqrt{2e(W)} + O(k) \\
&\geq \frac{k-1}{k}e(W) + \frac{k-1}{4k}\sqrt{n/k} + O(k),
\end{aligned}$$

and
$$
\begin{aligned}
e(R_1, ..., R_k) &\geq \frac{k-1}{k} e(R) + \frac{k-1}{2k}\sqrt{2e(R)} + O(k)\\
&\geq \frac{k-1}{k} e(R) + \frac{k-1}{2k}\sqrt{2m} - \frac{k-1}{2k}4\epsilon\sqrt{n} + O(k).
\end{aligned}
$$

For any permutation $\pi \in [k]$, we define $e(\pi) := \sum_{i=1}^{k} e(W_i, R_{\pi(i)})$; then

$$
\sum_{\pi \in [k]} e(\pi) = (k-1)!\ e(W, R).
$$

Thus there exists a permutation $\pi \in [k]$ such that $e(\pi) \leq (k-1)!\ e(W,R)/k! = e(W,R)/k$. So

$$
\begin{aligned}
&e(W_1 \cup R_{\pi(1)}, ..., W_k \cup R_{\pi(k)})\\
=&e(W,R) - e(\pi) + e(W_1, ..., W_k) + e(R_1, ..., R_k)\\
\geq&\frac{k-1}{k} e(W,R) + \frac{k-1}{k} e(W) + \frac{k-1}{4k}\sqrt{n/k} + \frac{k-1}{k} e(R) + \frac{k-1}{2k}\sqrt{2m} - \frac{k-1}{2k} 4\epsilon\sqrt{n} + O(k)\\
=&\frac{k-1}{k} m + \frac{k-1}{2k}\sqrt{2m} + \frac{k-1}{k}(2m)^{1/4}\left(\frac{1}{4\sqrt{k}} - \frac{2}{16\sqrt{k}}\right) + O(k)\\
\geq&\frac{k-1}{k} m + \frac{k-1}{2k}\sqrt{2m} + \frac{1}{8\sqrt{k}}(2m)^{1/4} + O(k).
\end{aligned}
$$

Note that the $O(k)$ term here is $2(-k^2+4k-4)/(8k) + 1/(2\sqrt{2}) - 1/8$. Since $m > n^2/2 > (32^2k^3)^2/2$, we see that

$$
e(W_1 \cup R_{\pi(1)}, ..., W_k \cup R_{\pi(k)}) \geq \frac{k-1}{k} m + \frac{k-1}{2k}\sqrt{2m} + c(k)m^{1/4}.
$$

So the assertion of the theorem holds with the partition $W_1 \cup R_{\pi(1)}, ..., W_k \cup R_{\pi(k)}$, completing the proof of Claim 2.

Let $v_1, v_2, ..., v_{n-r}$ be the vertices of R and let $d_i := |N(v_i) \cap W|$ for $1 \leq i \leq n-r$, where $d_1 \leq d_2 \leq ... \leq d_{n-r}$. Since R is complete, a balanced k-partition of $V(R)$ (i.e., the sizes of parts differ by at most 1) gives $f_k(R)$. So let $V(R) = \bigcup_{i=1}^{k} R_i$ be a k-partition such that $\lfloor \frac{n-r}{k} \rfloor = |R_1| \leq |R_2| \leq ... \leq |R_k| = \lceil \frac{n-r}{k} \rceil$, and let $R_1 = \{v_1, ..., v_{\lfloor \frac{n-r}{k} \rfloor}\}$. Then

$$
e(R_1, ..., R_k) \geq \frac{k-1}{k} e(R) + \frac{k-1}{2k}\sqrt{2e(R)} + O(k),
$$

where the $O(k)$ term is $(-k^2+4k-4)/(8k)$. Let

$$
D := \frac{\sum_{1 \leq i \leq n-r} d_i}{n-r} = \frac{e(W,R)}{n-r}, \text{ and } D_0 := D - \lfloor D \rfloor.
$$

Then, since $e(W) + e(R,W) = r(n-r) + n/2 + (r^2-r)/2$, we have

$$
D_0 = \frac{n/2 + (r^2-r)/2 - e(W)}{n-r}.
$$

Since $r \leq 4\epsilon\sqrt{n}, n \geq N(k)$ and $e(W) \leq n/(8k)$, we conclude that

$$
\min\{D_0, 1-D_0\} \geq \frac{n}{5(n-r)}.
$$

Hence, D differs from an integer by at least $\frac{n}{5(n-r)}$.

Claim 3. $e(W, R_2 \cup ... \cup R_k) \geq \frac{k-1}{k} e(W, R) + \frac{n}{5k}$.

To see this, let us consider two cases. If $d_{\lfloor \frac{n-r}{k} \rfloor} \geq D$ then $d_{\lfloor \frac{n-r}{k} \rfloor} \geq D + \frac{n}{5(n-r)}$ by integrality; so Claim 3 follows as

$$e(W, R_2 \cup ... \cup R_k) = \sum_{i > \lfloor \frac{n-r}{k} \rfloor} d_i \geq \frac{k-1}{k}(n-r)\left(D + \frac{n}{5(n-r)}\right) = \frac{k-1}{k} e(W, R) + \frac{k-1}{5k} n.$$

Otherwise, $d_{\lfloor \frac{n-r}{k} \rfloor} \leq D$; then $d_{\lfloor \frac{n-r}{k} \rfloor} \leq D - \frac{n}{5(n-r)}$ by integrality. Hence

$$e(W, R_1) = \sum_{i \leq \lfloor \frac{n-r}{k} \rfloor} d_i \leq \frac{n-r}{k}\left(D - \frac{n}{5(n-r)}\right) \leq \frac{1}{k} e(W, R) - \frac{n}{5k};$$

so $e(W, R_2 \cup ... \cup R_k) \geq \frac{k-1}{k} e(W, R) + \frac{n}{5k}$, completing the proof of Claim 3.

Now, consider the k-partition $W \cup R_1, R_2, ..., R_k$ of $V(G)$. Since $e(R_1, \ldots, R_k) \geq \frac{k-1}{k} e(R) + \frac{k-1}{2k}\sqrt{2e(R)} + O(k)$ and by Claim 3, We have

$$\begin{aligned}
&e(W \cup R_1, R_2, \ldots, R_k) \\
=&e(R_1, R_2, \ldots, R_k) + e(W, R_2 \cup \ldots \cup R_k) \\
\geq&\frac{k-1}{k} e(R) + \frac{k-1}{2k}\sqrt{2e(R)} + \frac{k-1}{k} e(W, R) + \frac{n}{5k} + O(k) \\
\geq&\frac{k-1}{k} m - \frac{k-1}{k} e(W) + \frac{k-1}{2k}\sqrt{2m} - \frac{k-1}{2k} 4\epsilon\sqrt{n} + \frac{n}{5k} + O(k) \quad (\text{as } \sqrt{e(R)} \geq \sqrt{m} - \epsilon\sqrt{8m} + O(1)) \\
\geq&\frac{k-1}{k} m + \frac{k-1}{2k}\sqrt{2m} - \frac{k-1}{k}\frac{\sqrt{2m}}{8k} - \frac{k-1}{2k} 4\epsilon(2m)^{1/4} + \frac{\sqrt{2m}}{5k} + O(k) \quad (\text{by Claim 2}) \\
\geq&\frac{k-1}{k} m + \frac{k-1}{2k}\sqrt{2m} + c(k) m^{1/4},
\end{aligned}$$

where the last inequality holds because $n \geq N(k) \geq 32^2 k^3$ and the $O(k)$ term is $(-k^2 + 4k - 4)/(8k)$. ∎

4 Concluding remarks

We point out that Theorem 1.3 is best possible up to the constant d. To see this, let $m = \frac{n^2+1}{2}$. The lower bound on $f(m)$ remains unchanged since the proof in [1] works for values of m which differ from $n^2/2$ by a constant; so our proof also gives the same upper bound on $g(m)$. Let G be the vertex-disjoint union of K_n and K_k, where n is odd, k is even, and $n = k(k-1) - 1$. Let $m := e(G) = \binom{n}{2} + \binom{k}{2}$. An easy calculation shows that

$$g(G) = \binom{\frac{n+1}{2}}{2} + \binom{\frac{k}{2}}{2} = \frac{m}{4} + \sqrt{m/32} - \frac{(2m)^{1/4}}{4} + O(1).$$

This shows that for $m = \frac{n^2+1}{2}$, $g(m) \geq m/4 + \sqrt{m/32} - (2m)^{1/4}/8 + O(1)$.

In our proof of Theorem 1.3, we choose $d = c/4$, where c is the constant from Theorem 1.1. The calculation in the end of the proof of Theorem 1.3 in fact only requires that $d - c + O(1) \leq -d$, where the $O(1)$ term may be made arbitrarily small when m is sufficient large. So one can show that for any $\epsilon > 0$, there exists an integer $N = N(\epsilon)$ such that when $n \geq N$,

$$g(m) \leq \frac{m}{4} + \sqrt{m/32} - \left(\frac{c}{2} - \epsilon\right) m^{1/4}.$$

We do not know if one can get rid of the ϵ.

Our proofs of Lemmas 2.3 and 2.4 may be modified to show that if G is graph with m edges, $f(G) = m/2 + \delta$, and $\delta \leq \alpha m$, then for sufficiently large m, there is a partition $V(G) = V_1 \cup V_2$ such that

$$\max\{e(V_1), e(V_2)\} \leq \frac{m}{4} - \frac{\delta}{2} + \sqrt{m/8} + \beta\sqrt{\delta} + \gamma\frac{\delta^{3/2}}{\sqrt{m}},$$

where α is an absolute constant and β, γ are constants depend on α only. When $\delta = O(m^t)$, where $t < \frac{2}{3}$, this bound is better than Lemma 2.1, since $\sqrt{m/8}$ dominates $\sqrt{\delta}$, $\delta^{3/2}/\sqrt{m}$ (while $3\sqrt{m}$ dominates $10\delta^2/m$ in Lemma 2.1).

Another conclusion we may draw from the proofs of Lemmas 2.3 and 2.4 is that: If $f(G) \geq m/2 + \sqrt{m/8} + \alpha$, where $\alpha = \Theta(m^t)$ and $\frac{1}{4} < t < \frac{1}{2}$, then $g(G) \leq m/4 + \sqrt{m/32} + O(m^{1/4}) - \alpha/2$.

We conclude our discussion with the following natural question: Is it also true that for every integer k, the limsup of

$$\frac{m}{k^2} + \frac{k-1}{2k^2}(\sqrt{2m + 1/4} - 1/2) - g_k(m)$$

tends to infinity when m tends to infinity? Theorem 1.4 has answered this question when k is even. For the case that k is odd, Theorem 1.5 and results of Bollobás and Scott in [10] seem to be relevant.

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