A stability result of the Pósa lemma

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Abstract

For an integer α and a graph G, the *α-disintegration* of G is the graph obtained from G by recursively deleting vertices of degree at most α until that the resulting graph has no such vertex. Pósa proved that if a 2-connected graph contains a path on $m \geq k$ vertices with end-vertices in its $\lfloor (k-1)/2 \rfloor$-disintegration, then G contains a cycle of length at least k. We prove that if a 2-connected graph contains a path on $m \geq k$ vertices with end-vertices in its $\lfloor (k-3)/2 \rfloor$-disintegration, then G contains either a cycle of length at least k or a specific family of graphs. As an application, we strengthen the Erdős-Gallai stablity theorem of Füredi, Kostochka, Luo and Verstraëte.

1 Introduction

The *circumference* $c(G)$ of a graph G is the length of a longest cycle in G. For an integer α and a graph G, the *α-disintegration* of G, denoted by $H(G, \alpha)$, is the graph obtained from G by recursively deleting vertices of degree at most α until that the resulting graph has no such vertex. We also call $H(G, \alpha)$ the α-core of G, and moreover this core is unique for every α.[1] Pósa [12] proved the following well-known lemma which is widely used in graph theory.

Lemma 1.1 (Pósa [12])**.** *Let $\ell = \lfloor (k-1)/2 \rfloor$ and $k \geq 5$. Let G be a 2-connected graph and H be the ℓ-disintegration of G. If the longest H-path in G has $m \geq k$ vertices, then G contains a cycle of length at least k.*

The following theorem, which combines the ideas of Pósa's lemma [12] and Kopylov's work [8], is the main result of this paper. Denote by $K_{3,3}^{+}$ the graph obtained from taking a copy of $K_{3,3}$ and a new edge xy and joining each of x, y to the same two vertices in one part of $K_{3,3}$.

Theorem 1.2. *Let $\ell = \lfloor (k-1)/2 \rfloor$ and $k \geq 5$. Let G be a 2-connected graph with $c(G) < k$ and H be the $(\ell - 1)$-disintegration of G. Let m be the number of vertices in a largest H-path in G. If $m \geq k$, then G contains a subgraph $F \in \mathcal{F}(m, k, r)$ for some $r \leq \ell$ or a copy of $K_{3,3}^{+}$ when $m = k + 1 = 9$.*

Remark. We give the definition of the graph family $\mathcal{F}(m, k, r)$ in Section 2. The graph $K_{3,3}^{+}$ contains a copy of $F \in \mathcal{F}(8, 8, 1)$.

For integers $n \geq k \geq 2a$, let $H(n, k, a)$ be the n-vertex graph whose vertex set is partitioned into three sets A, B, C such that $|A| = a$, $|B| = n - k + a$, $|C| = k - 2a$ and whose edge set consists of all edges between A and B together with all edges in $A \cup C$ (see Figure 1, the subgraphs induced by A and C are complete graphs and the subgraph induced by B contains no edge). Note that any path/cycle in $H(n, k, a)$ cannot contain consecutive vertices in B. One may check that the longest path in $H(n, k, a)$ contains k vertices and the longest cycle in $H(n, k, a)$ contains $k - 1$ vertices.

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[1] One can see that $H(G, \alpha)$ is unique in G and has minimum degree at least $\alpha + 1$ (if non-empty).

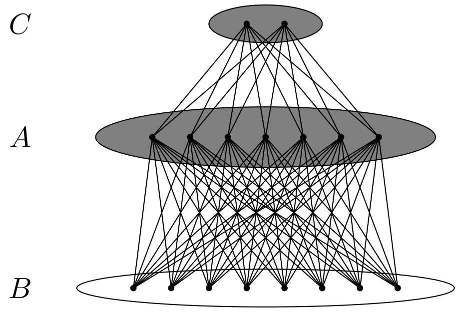


Figure 1. $H(17,16,7)$.

Let
$$h(n,k,a) := e(H(n,k,a)) = \binom{k-a}{2} + (n-k+a)a. \tag{1}$$

The celebrated Erdős-Gallai theorem [2] states that any n-vertex graph G with $c(G) < k$ has at most $(k-1)(n-1)/2$ edges. This was improved by Kopylov [8] by showing that any n-vertex 2-connected graph G with $c(G) < k$ has at most $\max\{h(n,k,2), h(n,k,\lfloor (k-1)/2 \rfloor)\}$ edges. Combined with the results in [5], Füredi, Kostochka, Luo and Verstraëte [6] proved a stability version of Kopylov's theorem, which says that for any 2-connected graph G with $c(G) < k$, if $e(G)$ is close to the above maximum number from Kopylov's theorem, then G must be a subgraph of some well-specified graphs.

Theorem 1.3 (Füredi, Kostochka, Luo and Verstraëte [5,6])**.** *Let G be an n-vertex 2-connected graph with $c(G) < k$. Let $\ell = \lfloor (k-1)/2 \rfloor$. Then*
$$e(G) \leq \max\{h(n,k,\ell-1), h(n,k,3)\}$$
unless

(a) $k = 2\ell+1$, $k \neq 7$, and $G \subseteq H(n,k,\ell)$;

(b) $k = 2\ell+2$ or $k = 7$, and $G - A$ is a star forest for some $A \subseteq V(G)$ of size at most ℓ;[2] *or*

(c) $G \subseteq H(n,k,2)$.

The proof of Theorem 1.3 is mainly based on contracting edges and the following fact. If a graph contains a cycle of length at least k and is obtained from another graph by contracting edges, then that other graph also contains a cycle of length at least k. Theorem 1.3 was further extended in [10].

The aim of this paper is to study a new approach and provide some potential tools in this line of research. In order to explain our main idea of this paper, we restate Kopylov's theorem as follows. If an n-vertex 2-connected graph G has more than $\max\{h(n,k,2)), h(n,k,\lfloor (k-1)/2 \rfloor)\}$ edges, then G contains a copy of graph $F \in \mathcal{C}_k$, where $\mathcal{C}_k$ is the set of cycles of length at least k. Roughly speaking, our proof shows that if an n-vertex 2-connected graph G has more than $\max\{h(n,k,3)), h(n,k,\lfloor (k-1)/2 \rfloor - 1)\}$ edges, then G contains a copy of graph $F \in (\mathcal{C}_k \cup \mathcal{F})$, where $\mathcal{F}$ is a set of special graphs (see Subsection 2.2). From this generalization of Kopylov's theorem, we can deduce that if an n-vertex 2-connected graph contains a copy of $F \in \mathcal{F}$ with $c(G) < k$, then G is a subgraph of some graphs in Theorem 1.3. As an application, we get the following theorem strengthening Theorem 1.3 for odd $k \geq 9$.

Theorem 1.4. *Let $k = 2\ell + 1 \geq 5$ be an odd integer and $n \geq k$. Let G be an n-vertex 2-connected graph with $c(G) < k$. Then $e(G) < \max\{h(n,k,3), h(n,k,\ell-1)\}$ unless*

(a) G is a subgraph of $H(n,k,2)$;

(b) G is a subgraph of $H(n,k,\ell)$;

[2] A *star forest* is a graph in which every component is a star.

(c) $G = H(n, k, 3)$*;*

(d) $G = H(n, k, \ell - 1)$*; or*

(e) $G - A$ *is a star forest for some* $A \subseteq V(G)$ *of size at most two for* $k = 7$*.*

Remark. Although Theorem 1.4 improves Theorem 1.3 only for odd $k \geq 9$ with the case $e(G) = \max\{h(n, k, 3), h(n, k, \ell - 1)\}$, it will be used to prove [13] a longstanding conjecture of Erdős, Simonovits and Sós [3] (determining the maximum number of edge colors in a complete graph such that there is no rainbow path of given length). We will prove Theorem 1.3 for even k in [11].

The organization of this paper is as follows. In Section 2, we give a formal definition of a family of graphs for the use of our characterization. In Section 3, we prove our main result which builds on an integration of Pósa's rotation lemma and Kopylov's proof in [8]. In Section 4, as an application, we strengthen Theorem 1.3 for odd $k \geq 9$.

2 Notation and a family of graphs

2.1 Notation

The general notation used in this paper is standard (see, e.g., [1]). For disjoint subsets $A, B \subseteq V(G)$, we denote $G(A, B)$ to be the induced bipartite subgraph of G with parts A, B. Let $E(A, B) = E(G(A, B))$ for short. When defining a graph, we will only specify these adjacent pairs of vertices. That is if a pair $\{a, b\}$ is not discussed as a possible edge, then it is assumed to be a non-edge.

Denote by $N_G(x)$ the set of neighbors of x in G and let $d_G(x)$ be the size of $N_G(x)$. For $U \subseteq V(G)$, let $N_U(x) = N_G(x) \cap U$ and $d_U(x) = |N_U(x)|$. Let $P = x_1 x_2 \cdots x_m$ be a path in G and call P and an (x_1, x_m)-path or an x_1-path (a path starting from x_1). For $x \in V(G)$, let $N_P(x) = N_G(x) \cap V(P)$ and $N_P[x] = N_P(x) \cup \{x\}$, with $d_P(x) = |N_P(x)|$. For $x_i, x_j \in V(P)$, we use $x_i P x_j$ to denote the subpath of P between x_i and x_j. For $x \in V(P)$, denote x^- and x^+ to be the immediate predecessor and immediate successor of x on P, respectively. For $S \subseteq V(P)$, let $S^+ = \{x^+ : x \in S\}$ and $S^- = \{x^- : x \in S\}$. We call $(x_i, x_j)_P$ a *crossing pair* of P if $i < j$, $x_i \in N_P(x_m)$ and $x_j \in N_P(x_1)$. If there is no ambiguity, we write this pair as (i, j) for short. We call a path a *crossing path* if it has a crossing pair. For a crossing pair (i, j), let $\ell(i, j) = j - i - 1$ and call $\ell(i, j)$ the *length* of the minimal crossing pair (i, j). A crossing pair (i, j) is *minimal* in P if $x_h \notin N_P(x_1) \cup N_P(x_m)$ for each $i < h < j$. For $S \subseteq V(G)$, we call $P = x_1 x_2 \cdots x_m$ an *S-path* if $x_1, x_m \in S$. For a graph G, let $\omega(G)$ be the order of a maximum clique in G.

2.2 A family of graphs

Let $m \geq k \geq 5$ and $1 \leq r \leq \ell$ be integers. We now devote the rest of this subsection to the definition of a family of m-vertex graphs $\mathcal{F}(m, k, r)$ [3]. We divide $\mathcal{F}(m, k, r)$ into the following four classes, namely Types I, II, III and IV (see Figures 2, 3, 4 and 5).

Type I: Each graph $F \in \mathcal{F}(m, k, r)$ of Type I satisfies:

- $k = 2\ell + 1$, $r \leq \ell - 1$, and $c(F) < k$;
- F contains a Hamiltonian path $v_1 v_2 \ldots v_m$ such that $A = \{v_1, \ldots, v_r\}$; $B = \{v_{m-r+1}, \ldots, v_m\}$; and either
  - $m = k$, $r \leq \ell - 1$, and $C = \{v_{r+1}, v_{r+3}, \ldots, v_{m-r-2}, v_{m-r}\}$; or
  - $m \geq k$, $r = \ell - 1$, and $C = \{v_{r+1}, v_{m-r}\} = \{v_\ell, v_{m-\ell+1}\}$;

[3] For the parameter r, roughly speaking we may view it as something close to $\omega(F)$, though its own meaning will be clear in the proof of Theorem 1.2. Readers may treat the coming lengthy definition as a handout and skip to next sections.

- each vertex in A has degree exactly ℓ in $F[A \cup C]$ and each vertex in B has degree exactly ℓ in $F[B \cup C]$.

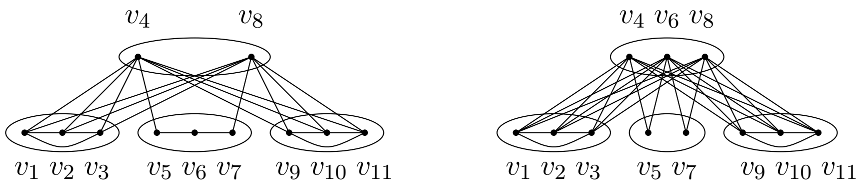


Figure 2. Graphs of **Type I**.

Type II: Each graph $F \in \mathcal{F}(m, k, r)$ of Type II satisfies:

- $k = 2\ell + 2$, $r \le \ell - 1$, and $c(F) < k$;
- F contains a Hamiltonian path $v_1 v_2 \ldots v_m$ such that $B = \{v_{m-r+1}, \ldots, v_m\}$; and either
  - $m = k$, $r \le \ell - 1$, $A = \{v_1, \ldots, v_r\}$ and $C = \{v_{r+1}, v_{r+3}, \ldots, v_{r+2i+1}, v_{r+2i+4}, \ldots, v_{m-r-2}, v_{m-r}\}$, where $0 \le i \le (m - 2r - 4)/2$ (Figure 3(a));
  - $m = k + 1$, $r = \ell - 2 \ge 2$, $A = \{v_1, \ldots, v_r\}$, and $C = \{v_{r+1}, v_{r+4}, v_{r+7}\}$ (Figure 3(b));
  - $m = k$, $r \le \ell - 1$, $A = \{v_1, \ldots, v_{r+1}\}$, and $C = \{v_{r+2}, v_{r+4}, \ldots, v_{m-r-2}, v_{m-r}\}$ (Figure 3(c));
  - $m \ge k$, $r = \ell - 1$, $A = \{v_1, \ldots, v_{r+1}\}$, and $C = \{v_{r+2}, v_{m-r}\} = \{v_{\ell+1}, v_{m-\ell+1}\}$; or
  - $m \ge k$, $r = \ell - 1$, $A = \{v_1, \ldots, v_r\}$, and $C = \{v_{r+1}, v_{m-r}\} = \{v_\ell, v_{m-\ell+1}\}$;
- each vertex in A has degree exactly ℓ in $F[A \cup C]$ such that there are two independent edges between $\{v_{r+2}, v_{m-r}\}$ and A when $|A| = r + 1$ [4] and each vertex in B has degree exactly ℓ in $F[B \cup C]$.

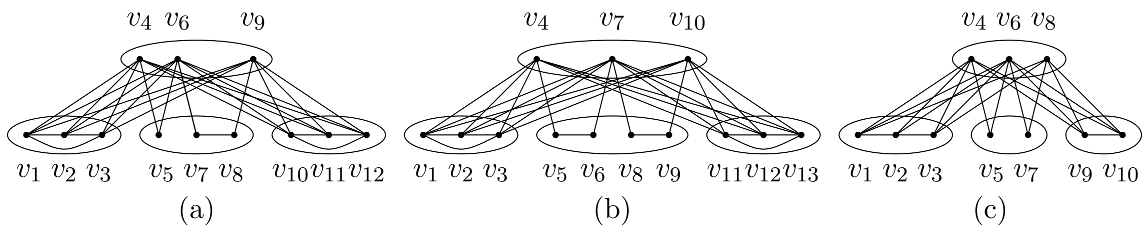


Figure 3. Graphs of **Type II**.

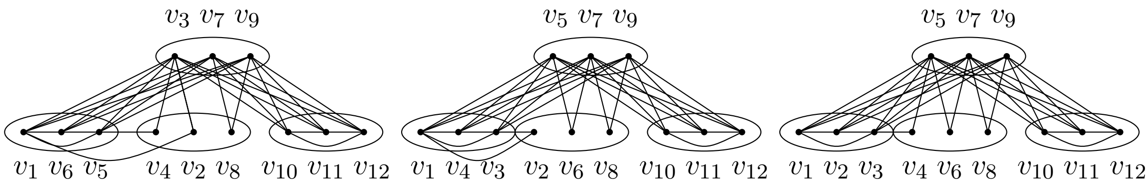
Type III: Each graph $F \in \mathcal{F}(m, k, r)$ of Type III satisfies:

- $k = 2\ell + 2$, $r \le \ell - 1$, and $c(F) < k$;
- F contains a Hamiltonian path $v_1 v_2 \ldots v_m$ such that when $B = \{v_{m-r+1}, \ldots, v_m\}$; and either
  - $m = k$, $r \le \ell - 1$, $C = \{v_3, v_5, \ldots, v_{1+2i}, v_{r+2+2i}, v_{r+4+2i}, \ldots, v_{m-r-2}, v_{m-r}\}$, and $A = \{v_1, v_{3+2i}, v_{4+2i}, \ldots, v_{r+1+2i}\}$ where $1 \le i \le \ell - r$ (Figure 4(a));
  - $m = k$, $r \le \ell - 1$, $A = \{v_1, v_3, \ldots, v_{r+1}\}$, and $C = \{v_{r+2}, v_{r+4}, \ldots, v_{m-r-2}, v_{m-r}\}$ (Figure 4(b));
  - $m = k$, $r \le \ell - 1$, $A = \{v_1, \ldots, v_r\}$, and $C = \{v_{r+2}, v_{r+4}, \ldots, v_{m-r-2}, v_{m-r}\}$ (Figure 4(c));
  - $m \ge k$, $r = \ell - 1$, $A = \{v_1, v_3, \ldots, v_{r+1}\}$, and $C = \{v_{r+2}, v_{m-r}\} = \{v_{\ell+1}, v_{m-\ell+1}\}$ (similar as Figure 4(b)); or

[4] This condition ensure the graphs in Type II are 2-connected and have some other good properties for the proofs in the forthcoming paper [11].

- $m \geq k$, $r = \ell - 1$, $A = \{v_1, \ldots, v_r\}$, and $C = \{v_{r+2}, v_{m-r}\} = \{v_{\ell+1}, v_{m-\ell+1}\}$(similar as Figure 4(c));

- each vertex in A has degree exactly ℓ in $F[A \cup C]$ and each vertex in B has degree exactly ℓ in $F[B \cup C]$.



(a) (b) (c)

Figure 4. Graphs of **Type III**.

Type IV: Each graph $F \in \mathcal{F}(m, k, r)$ of Type IV satisfies:

- $k = 2\ell + 2$, $r = \ell$, and $c(F) < k$;

- F contains a Hamiltonian path $v_1 v_2 \ldots v_m$ with $A = \{v_1, \ldots, v_r\}$ and $B = \{v_{m-r+1}, \ldots, v_m\}$;

- each vertex in A has degree exactly ℓ in $F[A \cup \{v_{r+1}, v_i\}]$ and each vertex in B has degree exactly ℓ in $F[B \cup \{v_{m-r}, v_i\}]$, where $r + 3 \leq i \leq m - r - 2$.

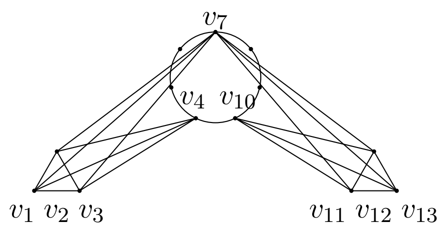


Figure 5. $F \in \mathcal{F}(13, 10, 4)$ of Type IV

3 A generalization of Pósa's lemma

The following well-known lemma is due to Pósa [12] and is extensively used in extremal graph theory.

Lemma 3.1 (Pósa [12])**.** *Let G be a 2-connected graph and $P = x_1 x_2 \cdots x_m$ be a path in G. Then G contains a cycle of length at least $\min\{m, d_P(x_1) + d_P(x_m)\}$ containing $N_P[x_1] \cup N_P[x_m]$. Moreover, if P is a non-crossing path with $N_P(x_1) \cap N_P(x_m) = \emptyset$, then G contains a cycle of length at least $\min\{m, d_P(x_1) + d_P(x_m) + 2\}$. If P is a non-crossing path with $N_P(x_1) \cap N_P(x_m) \neq \emptyset$, then G contains a cycle of length at least $\min\{m, d_P(x_1) + d_P(x_m) + 1\}$.*

Now we give the proof of our main result.

Proof of Theorem 1.2. Let G be a 2-connected graph with $c(G) < k$ and H be the $(\ell - 1)$-disintegration of G. Suppose to the contrary that G does not contain any subgraph in $\mathcal{F}(m, k, r)$ with $m \geq k$ and $r \leq \ell$. Let $\mathcal{P}$ be the family of all longest H-paths in G. We proceed by showing a sequence of claims in what follows.

Claim 1. *Every $P = x_1 x_2 \cdots x_m \in \mathcal{P}$ satisfies the following properties.*

(i) $N_H(x_1) \subseteq N_P(x_1)$ *and* $N_H(x_m) \subseteq N_P(x_m)$*,*

(ii) $d_P(x_1) \geq d_H(x_1) \geq \ell$ *and* $d_P(x_m) \geq d_H(x_m) \geq \ell$*, and*

(iii) $N_P^-(x_1) \cap N_P[x_m] = \emptyset$ *and* $N_P^+(x_m) \cap N_P[x_1] = \emptyset$*.*

Proof. (*i*). Suppose to the contrary that there exists a vertex $y \in (N_H(x_1) \setminus N_P(x_1))$. Then yx_1Px_m is an H-path longer than P, a contradiction. Therefore, we have $N_H(x_1) \subseteq N_P(x_1)$. Similarly, we have $N_H(x_m) \subseteq N_P(x_m)$.

(*ii*). Note that H is the $(\ell-1)$-disintegration of G. Each vertex of H has degree at least ℓ in H, that is $d_H(x_1) \geq \ell$ and $d_H(x_m) \geq \ell$. It follows from (*i*) that $d_P(x_1) \geq d_H(x_1) \geq \ell$ and $d_P(x_m) \geq d_H(x_m) \geq \ell$.

(*iii*). Suppose to the contrary that $N_P^-(x_1) \cap N_P[x_m] \neq \emptyset$. Let x_i be a vertex in $N_P^-(x_1) \cap N_P[x_m]$, i.e., x_1 is adjacent to x_{i+1} and x_m is adjacent to x_i. Thus, $x_1Px_ix_mPx_{i+1}x_1$ is a cycle of length $m \geq k$ in G, a contradiction to $c(G) < k$. Therefore, we have $N_P^-(x_1) \cap N_P[x_m] = \emptyset$. Similarly, we have $N_P^+(x_m) \cap N_P[x_1] = \emptyset$. □

Given a path P with a crossing pair (i, j), let

$$U_i = N_P[x_1] \cup (N_P^+(x_m) \setminus \{x_{i+1}\}) \text{ and } V_j = N_P[x_m] \cup (N_P^-(x_1) \setminus \{x_{j-1}\}).$$

Claim 2. *Let $P = x_1x_2 \cdots x_m$ be a crossing path in $\mathcal{P}$ and (i, j) be any minimal crossing pair of P. Then the following properties hold.*

(*i*) $d_P(x_1) + d_P(x_m) = |U_i| = |V_j| \geq 2\ell$,

(*ii*) $U_i \subseteq V(x_1Px_i) \cup V(x_jPx_m)$, $V_j \subseteq V(x_1Px_i) \cup V(x_jPx_m)$,

(*iii*) $m - k + 1 \leq \ell(i, j) \leq m - 2\ell$, *i.e.*, $2\ell \leq |V(x_1Px_i) \cup V(x_jPx_m)| \leq k - 1$, *and*

(*iv*) $|(V(x_1Px_i) \cup V(x_jPx_m)) \setminus U_i| = |(V(x_1Px_i) \cup V(x_jPx_m)) \setminus V_j| \leq 1$. *Moreover, if*

- k *is odd,*
- $d_P(x_1) + d_P(x_m) = 2\ell + 1$ *or*
- $\ell(i, j) = m - 2\ell$,

then $V(x_1Px_i) \cup V(x_jPx_m) = U_i = V_j$.

Proof. By Claim 1(*iii*) we have $N_P[x_1] \cap (N_P^+(x_m) \setminus \{x_{i+1}\}) = \emptyset$. Hence we have $|U_i| = d_P(x_1) + 1 + d_P(x_m) - 1 = d_P(x_1) + d_P(x_m)$. Similarly, we have $|V_j| = d_P(x_1) + d_P(x_m)$. It follows from Claim 1(*ii*) that $|U_i| = |V_j| \geq 2\ell$.

By the definition of a minimal crossing pair, we can easily obtain $U_i \subseteq V(x_1Px_i) \cup V(x_jPx_m)$ and $V_j \subseteq V(x_1Px_i) \cup V(x_jPx_m)$, proving (*ii*).

Since $c(G) < k$ and $x_1Px_ix_mPx_jx_1$ is a cycle of length $m - \ell(i, j)$, we have $m - \ell(i, j) < k$, i.e., $m - k + 1 \leq \ell(i, j)$. By (*i*) and (*ii*) we have $2\ell \leq |V_j| \leq |V(x_1Px_i) \cup V(x_jPx_m)|$. Therefore, we have $\ell(i, j) = m - |V(x_1Px_i) \cup V(x_jPx_m)| \leq m - 2\ell$, proving (*iii*).

Lastly, from (*i*) (*ii*) and (*iii*) we have $|(V(x_1Px_i) \cup V(x_jPx_m)) \setminus U_i| = |(V(x_1Px_i) \cup V(x_jPx_m)) \setminus V_j| \leq k-1-2\ell \leq 1$. If $k = 2\ell+1$ is odd, then $|(V(x_1Px_i) \cup V(x_jPx_m)) \setminus U_i| = |(V(x_1Px_i) \cup V(x_jPx_m)) \setminus V_j| \leq 2\ell + 1 - 1 - 2\ell = 0$, i.e., $V(x_1Px_i) \cup V(x_jPx_m) = U_i = V_j$. If $d_P(x_1) + d_P(x_m) = 2\ell + 1$, then by (*i*) we have $|U_i| = |V_j| = 2\ell + 1$, and hence $V(x_1Px_i) \cup V(x_jPx_m) = U_i = V_j$. If $\ell(i, j) = m - 2\ell$, then $|V(x_1Px_i) \cup V(x_jPx_m)| = 2\ell$, and hence by (*i*) and (*ii*) we have $V(x_1Px_i) \cup V(x_jPx_m) = U_i = V_j$. The proof of the claim is complete. □

Given a path P with a crossing pair (i, j), let

$$U_i^* = N_H[x_1] \cup (N_H^+(x_m) \setminus \{x_{i+1}\}) \text{ and } V_j^* = N_H[x_m] \cup (N_H^-(x_1) \setminus \{x_{j-1}\}).$$

Claim 2*.* *Let $P = x_1x_2 \cdots x_m$ be a crossing path in $\mathcal{P}$ and (i, j) be any minimal crossing pair of P. Then the following properties hold.*

(*i*) $d_H(x_1) + d_H(x_m) = |U_i^*| = |V_j^*| \geq 2\ell$,

(ii) $U_i^* \subseteq V(x_1Px_i) \cup V(x_jPx_m)$ *and* $V_j^* \subseteq V(x_1Px_i) \cup V(x_jPx_m)$,

(iii) $|(V(x_1Px_i) \cup V(x_jPx_m)) \setminus U_i^*| = |(V(x_1Px_i) \cup V(x_jPx_m)) \setminus V_j^*| \leq 1$ *and*

(iv) *if*

- k *is odd,*
- $d_H(x_1) + d_H(x_m) = 2\ell + 1$ *or*
- $\ell(i,j) = m - 2\ell$,

then $V(x_1Px_i) \cup V(x_jPx_m) = U_i^* = V_j^*$, $N_H(x_1) = N_P(x_1)$ *and* $N_H(x_m) = N_P(x_m)$.

Proof. By Claim 1(i), we have $U_i^* \subseteq U_i$ and $V_j^* \subseteq V_j$. Similar to the proof of Claim 2, we can easily prove (i), (ii) and (iii).

If $k = 2\ell+1$ is odd, then it follows from Claim 2 that $d_P(x_1)+d_P(x_m) = |V(x_1Px_i)\cup V(x_jPx_m)| = 2\ell$, and hence by (i) and (ii) we have $V(x_1Px_i)\cup V(x_jPx_m) = U_i^* = V_j^*$, implying $d_H(x_1)+d_H(x_m) = |V(x_1Px_i) \cup V(x_jPx_m)| \geq d_P(x_1) + d_P(x_m)$. Thus by Claim 1$(i)$ we have $N_H(x_1) = N_P(x_1)$ and $N_H(x_m) = N_P(x_m)$. The rest of the proof is similar and omitted. $\square$

The following Figure 6 shows the neighbors of x_1 and x_m in a crossing path P with a minimal crossing pair (i,j) (at most one blue or red edges are missing when k is even, $d_P(x_1) + d_P(x_m) = 2\ell$ and $\ell(i,j) = m - 2\ell - 1$).

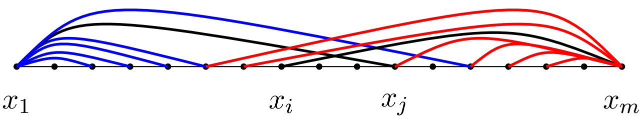


Figure 6. The neighbors of x_1 and x_m in the crossing path P

Next we consider the neighbors of end-vertices of a path with a crossing pair. The following claim strengthens Claims 1 and 2 and will be used many times throughout the proof.

Let $N_P^{+1}(x_m) = N_P^+(x_m)$ and $N_P^{+i}(x_m) = (N_P^{+(i-1)}(x_m))^+$ for $i \geq 2$.

Claim 3. *Let* $P = x_1x_2\cdots x_{m'}$ *be a crossing path in* G *with* $d_P(x_1) \geq \ell$, $d_P(x_{m'}) \geq \ell$ *and* $m' \geq k$. *Let* $X = V(P) \setminus (\bigcup_{i=1}^{m'-k+1} N_P^{+i}(x_{m'}) \cup \{x_1\})$. *Then the following holds.*

(i) *If* k *is even, then* x_1 *is adjacent to all vertices but at most one of* X.

(ii) *If* k *is odd or* $d_P(x_1) = |X|$, *then* x_1 *is adjacent to each vertex of* X.

Proof. Clearly, since $c(G) < k$, we have $N_P(x_1) \subseteq V(P) \setminus (\bigcup_{i=1}^{m'-k+1} N_P^{+i}(x_{m'}) \cup \{x_1\}) = X$. Since P has a minimal crossing pair, say (i,j), by Claim 2(iii) we have $\ell(i,j) \geq m' - k + 1$. Thus x_1 is not adjacent to $x_{i+1}, \ldots, x_{i+m'-k+1}$ (see Figure 7, x_1 can not be adjacent to the red empty vertices and adjacent to all but at most one vertices of the black vertices). Hence we have

$$|X| \leq m' - (d_P(x_{m'}) - 1) - (m' - k + 1) - 1 = k - 1 - d_P(x_{m'}).$$

If $k = 2\ell + 2$ is even, then since $d_P(x_{m'}) \geq \ell$, we have $|X| \leq \ell + 1$. Hence, using $d_P(x_1) \geq \ell$, x_1 must be adjacent to all vertices but at most one in X. The proof for the rest of Claim 3 is similar and omitted. $\square$

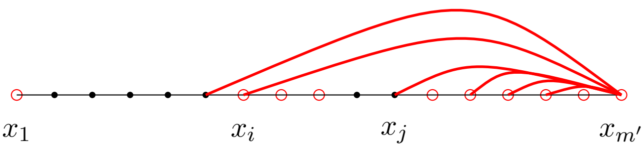


Figure 7. The possible neighbors of x_1 in P

Remark. In Claim 3, the length of P may be less than m and the end-vertices of P may not belong to H.

For $P = x_1x_2\cdots x_m \in \mathcal{P}$, let $s_P = \min\{h : x_{h+1} \in N_P(x_m)\}$ and $t_P = \max\{h : x_{h-1} \in N_P(x_1)\}$.[5]

Claim 4. *Let $P = x_1x_2\cdots x_m$ be a crossing path in $\mathcal{P}$ with a minimal crossing pair (i, j). If $x_s \in V(H)$ and $x_{s+1} \in N_P(x_1)$, then*

- *$N_P[x_s] = N_P[x_1]$, or k is even and x_s is adjacent to all but at most one vertex in $N_P[x_1]$;*
- *x_1 cannot be adjacent to two consecutive vertices of x_jPx_{t-1} and $N_P[x_s] \subseteq N_P[x_1]$.*

Similar result holds when $x_t \in V(H)$ and $x_{t-1} \in N_P(x_m)$.

Proof. By symmetry between x_1 and x_m, we will prove the first statement. We consider the path $R = x_sPx_1x_{s+1}Px_m$. It follows from $x_s, x_m \in V(H)$ that $R \in \mathcal{P}$. We have $N_H[x_s] \subseteq V(R)$ by the maximality of m and $N_H[x_m] \subseteq V(x_{s+1}Rx_m)$ by the definition of s. Hence, R has a crossing pair, as otherwise we have $|V(x_1Px_s)| \geq |N_R[x_s]| \geq \ell + 1$ and hence $x_1Px_ix_mPx_jx_1$ is a cycle of length at least $|V(x_1Px_{s+1})| + |N_R^+(x_m) \setminus \{x_{i+1}\}| + |\{x_q, x_{q+1}\}| = \ell + 1 + \ell - 1 + 2 \geq k$, a contradiction. By Claim 3, we have $N_P[x_s] \subseteq N_P[x_1]$, whence $N_P[x_s] = N_P[x_1]$, or k is even and x_s is adjacent to all but at most one vertex in $N_P[x_1]$.

Now, suppose to the contrary that x_1 is adjacent to x_q and x_{q+1} for some $j \leq q \leq t - 2$. Note that $x_q, x_{q+1} \in N_P[x_1] \subseteq V(P) \setminus \bigcup_{i=1}^{\theta} N_P^{+i}(x_m)$, where $\theta = m - k + 1$. Thus by Claim 3, x_s must be adjacent to one of x_q, x_{q+1}. If x_s is adjacent to x_q, then $x_sx_qPx_{s+1}x_mPx_{q+1}x_1Px_s$ is a cycle of length m; If x_s is adjacent to x_{q+1}, then $x_sx_{q+1}Px_mx_{s+1}Px_qx_1Px_s$ is a cycle of length m, both are contradictions. This completes the proof of the claim. $\square$

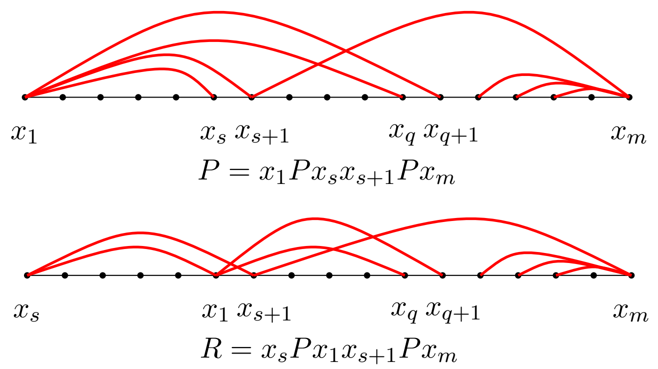


Figure 8. The structure of the crossing path P and R

Now according to the parity of k, we divide the remaining proof into two subsections. First, we consider the odd case, whose proof is comparably easier, yet reveals essential ideas of our arguments.

3.1 k is odd.

In this subsection, we have $k = 2\ell + 1$. From Claim 2*, we have $N_H(x_1) = N_P(x_1)$ and $N_H(x_m) = N_P(x_m)$.

Claim 5. *There exists a crossing path in $\mathcal{P}$.*

Proof. Suppose to the contrary that all paths in $\mathcal{P}$ are non-crossing. Then this is a $P = x_1x_2\cdots x_m \in \mathcal{P}$. By Lemma 3.1, G contains a cycle of length at least $\min\{m, 2\ell + 1\} \geq k$, a contradiction. $\square$

[5]When there is no ambiguity, we often omit the subscript index in s_P and t_P (such as in the coming claim).

By Claim 5, there is a crossing path $P \in \mathcal{P}$. Within P, let (i_1, j_1) and (i_2, j_2) be two minimal crossing pairs of P such that i_1 is as small as possible and j_2 is as large as possible.[6]

Claim 6. *P has a unique minimal crossing pair (i,j) with $\ell(i,j) = m-2\ell$ when $m \geq k+1$. Moreover, if $m = k$, then each minimal crossing pair (i',j') in P satisfies that $\ell(i',j') = 1$.*

Proof. Let $m \geq k+1$. Suppose to the contrary that there exist two minimal crossing pairs in P, say $i_1 < j_1 \leq i_2 < j_2$. By Claim 2(iii), we have $\ell(i_1, j_1) \geq m-k+1 \geq 2$ and $\ell(i_2, j_2) \geq m-k+1 \geq 2$. Then we have contradicted Claim 2(ii) since $x_{i_2+1} \in V(x_{j_1}Px_m)$ but $x_{i_2+1} \notin N_P(x_m)$ and $x_{i_2+2} \notin N_P(x_1)$. Let $m = k$. Then by Claim 2(iii) again, each minimal crossing pair (i',j') satisfies $1 = k-k+1 \leq \ell(i',j') \leq k - 2\ell = 1$. The proof of Claim 6 is complete. $\square$

Claim 7. *$i_1 = s+1$ and $j_2 = t-1$.*

Proof. We may assume that $j_2 < t-1$. By the definition of s,t, we have $x_{s+1} \in N_P(x_m)$ and $x_{t-1} \in N_P(x_1)$. Since k is odd, by Claim 2$^*(iv)$ we have $U^*_{i_1} = V^*_{j_1} = V(x_1Px_{i_1}) \cup V(x_{j_1}Px_m)$, and hence $x_s, x_{s+1} \in N_H(x_1)$. Thus it follows from Claim 4 that x_1 is not adjacent to x_{t-2} which implies that $j_2 < t-2$. By $U^*_{i_1} = V^*_{j_1} = V(x_1Px_{i_1}) \cup V(x_{j_1}Px_m)$ again, x_m is adjacent to x_{t-3}. Therefore, $(t-3, t-1)$ is a minimal crossing pair in P, contradicting the choice of j_2. Thus we have $j_2 = t-1$. Similarly, we have $i_1 = s+1$. $\square$

Now we are ready to finish the proof of Theorem 1.2 when k is odd. For each minimal crossing pair, by Claim 2(iii) we have

$$|V(x_1Px_i) \cup V(x_jPx_m)| = 2\ell. \tag{2}$$

By Claim 2$^*(iv)$, we have $V(x_1Px_i) \cup V(x_jPx_m) = U^*_i = V^*_j$, $N_H(x_1) = N_P(x_1)$ and $N_H(x_m) = N_P(x_m)$, and hence we have (note that $x_s \notin N_H(x_m)$ and $x_t \notin N_H(x_1)$)

$$V(x_1Px_{s+1}) \subseteq N_H[x_1] \text{ and } V(x_{t-1}Px_m) \subseteq N_H[x_m]. \tag{3}$$

Moreover, we have

$$d_H(x_1) = d_P(x_1) = \ell \text{ and } d_H(x_m) = d_P(x_m) = \ell. \tag{4}$$

By Claim 7, we derive that $i_1 = s+1$ and $j_2 = t-1$. If $(i_1, j_1) = (i_2, j_2)$, then let $C = \{x_{s+1}, x_{t-1}\}$, otherwise let $C = \{x_{s+1}, x_{s+3}, \cdots, x_{t-3}, x_{t-1}\}$. Let

$$A = V(x_1Px_s), \quad B = V(x_tPx_m), \quad D = V(P) \setminus (A \cup B \cup C).$$

If $m \geq k+1$, then by Claim 6, P has a unique minimal crossing pair (i,j), and hence by Claim 7 we have $(i,j) = (s+1, t-1)$. Hence, by (3) and the definitions of s and t, we have $N_H[x_1] = A \cup \{x_{s+1}, x_{t-1}\}$, $N_H[x_m] = B \cup \{x_{s+1}, x_{t-1}\}$ and $|A| = |B| = \ell - 1$. From (3), $R_\gamma = x_\gamma P x_1 x_{\gamma+1} P x_m$ is an H-path on m vertices for $2 \leq \gamma \leq s$ and hence $d_{R_\gamma}(x_\gamma) \geq \ell$ and $d_{R_\gamma}(x_m) \geq \ell$. If R_γ is not a crossing path, then by Lemma 3.1 we have $c(G) \geq \min\{m, 2\ell+1\} = k$, a contradiction. Thus R_γ is a crossing path, and by Claim 3(ii) x_γ is adjacent to each vertex of $(A \cup \{x_{s+1}, x_{t-1}\}) \setminus \{x_\gamma\}$. Thus we have $N_H[x_\gamma] = N_H[x_1]$. Similarly, we have $N_H[x_\lambda] = N_H[x_m]$ for $t \leq \lambda \leq m$. Hence $G[A]$ and $G[B]$ are complete graphs. Therefore, it is easy to check that $G[V(P)]$ gives a copy in $\mathcal{F}(m,k,s)$ of Type I, a contradiction (see Figure 9).

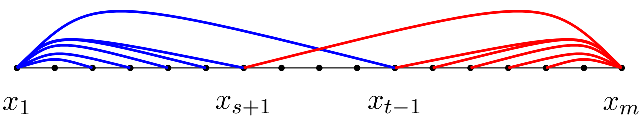


Figure 9. The structure of the crossing path P with $m \geq k+1$

[6] Note that it is possible that $(i_1, j_1) = (i_2, j_2)$.

Let $m = k$. From (3), we have $x_s, x_{s+1} \in N_H(x_1)$ and $x_t, x_{t-1} \in N_H(x_m)$. By Claim 4, x_1 is not adjacent to consecutive vertices of $V(x_{j_1}Px_{t-1})$ and x_m is not adjacent to consecutive vertices of $V(x_{s+1}Px_{i_2})$. Since x_m is adjacent to x_{s+1}, we have that x_m is not adjacent to x_{s+2}, whence x_1 is not adjacent to x_{s+2} by Claim 1(iii). By Claim 6, x_1 is adjacent to x_{s+3}. Consider the minimal crossing pair $(s+1, s+3)$. By Claim 2$^*(iv)$, we have $V(x_1Px_{s+1}) \cup V(x_{s+3}Px_m) = U^*_{s+1} = V^*_{s+3}$, and hence x_m is adjacent to x_{s+3}. Repeating the above arguments, we have x_1 and x_m are adjacent to $x_{s+5}, x_{s+7}, \ldots, x_{t-1}$, and hence we have $N_H[x_1] = A \cup C$ and $N_H[x_m] = B \cup C$. Consider the paths $R_\gamma = x_\gamma P x_1 x_{\gamma+1} P x_m$ for $2 \le \gamma \le s$. Since $c(G) < k$, $R_\gamma \in \mathcal{P}$ is a crossing path by Lemma 3.1. By Claim 3(ii), the neighbors of x_γ in H are determined by the neighbors of x_m in R_γ, that is $N_H[x_\gamma] = N_H[x_1]$. Similarly, $N_H[x_\lambda] = N_H[x_m]$ for $t \le \lambda \le m$. Thus $G[A]$ and $G[B]$ are complete graphs. Now it is straightforward to check that $G[V(P)]$ gives a copy in $\mathcal{F}(m,k,s)$ of Type I, a contradiction (see Figure 10). This completes the proof of Theorem 1.2 for odd k.

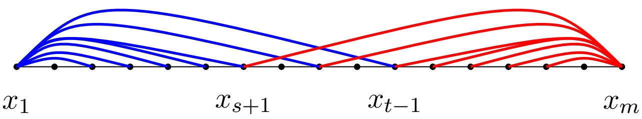


Figure 10. The structure of the crossing path P when $m = k$

3.2 k is even.

In this subsection, we have $k = 2\ell + 2$.

Claim 8. *There exists a crossing path in $\mathcal{P}$.*

Proof. Suppose to the contrary that all paths in $\mathcal{P}$ are non-crossing. Let $P = x_1x_2\cdots x_m \in \mathcal{P}$. Let α be the maximum integer such that x_α is adjacent to x_1 and β be the minimum integer such that x_β is adjacent to x_m. Note that $\alpha \le \beta$.

If $\alpha < \beta$, then by Lemma 3.1, G contains a cycle of length at least $\min\{m, 2\ell+2\} \ge k$, a contradiction. Therefore, we have $\alpha = \beta$. Since G is 2-connected, there exists a path Q in G with $V(Q) \cap V(P) = \{x_u, x_v\}$ for $1 \le u < \alpha < v \le m$. Let $p = \min\{h : h > u, x_h \in N_P(x_1)\}$ and $q = \max\{h : h < v, x_h \in N_P(x_m)\}$. Then $C_0 = x_1Px_uQx_vPx_mx_qPx_px_1$ is a cycle containing $N_P[x_1] \cup N_P[x_m]$. By Claim 1, C_0 has length at least $k-1$. Note that $c(G) < k$. This forces that C_0 has length $k-1$. It follows that $d_H(x_1) = d_H(x_m) = \ell$, $N_H(x_1) = V(x_2Px_u) \cup V(x_pPx_\alpha)$, $N_H(x_m) = V(x_\alpha Px_q) \cup V(x_vPx_{m-1})$, $V(C_0) = N_H[x_1] \cup N_H[x_m]$ and $Q = x_ux_v$.

For any $2 \le \gamma \le u-1$, we consider the path $R_\gamma = x_\gamma Px_1x_{\gamma+1}Px_m$. Since $x_\gamma \in N_H(x_1) \subseteq V(H)$, $R_\gamma \in \mathcal{P}$ is an H-path. Also, by our assumption, R_γ is non-crossing. It follows that $N_H[x_\gamma] \subseteq V(x_1Px_\alpha)$. Suppose that x_γ has a neighbor y in $V(x_{u+1}Px_{p-1})$. Then $x_\gamma Px_1x_{\gamma+1}Px_uQx_vPx_mx_qPyx_\gamma$ is a cycle of length at least $k+1$, a contradiction. Therefore, we have

$$N_H[x_\gamma] = N_H[x_1] \text{ for any } 2 \le \gamma \le u-1. \tag{5}$$

By symmetry, we have

$$N_H[x_\gamma] = N_H[x_m] \text{ for any } v+1 \le \gamma \le m-1. \tag{6}$$

Suppose that $p < \alpha$ or $q > \alpha$. By symmetry, we may assume that $p < \alpha$. Then we have $x_{\alpha-1} \in N_P(x_1)$. Now, we consider the path $L = x_uPx_{\alpha-1}x_1Px_{u-1}x_\alpha Px_m$ (since $N_H[x_{u-1}] = N_H[x_1]$, x_{u-1} is adjacent to x_α). Clearly, $L \in \mathcal{P}$. Note that $x_v \in N_L(x_u)$, $x_\alpha \in N_L(x_m)$ and x_α precedes x_v in L. It follows that L is a crossing path in $\mathcal{P}$, a contradiction.

The last paragraph implies that $p = \alpha$ and $q = \alpha$. Suppose that $u = \alpha - 1$ or $v = \alpha + 1$. By symmetry, we may assume $u = \alpha - 1$. Then $x_\alpha x_u \in E(P)$. Now we consider the path $M = x_uPx_1x_\alpha Px_m$. Clearly, $M \in \mathcal{P}$. Note that $x_v \in N_M(x_u)$, $x_\alpha \in N_M(x_m)$ and x_α precedes x_v in M. It follows that M is a crossing path in $\mathcal{P}$, a contradiction.

Thus, we may suppose that $u < \alpha - 1$ and $v > \alpha + 1$. Let

$$A = V(x_1Px_u),\ B = V(x_vPx_m) \text{ and } C = V(P) \setminus (A \cup B).$$

By (5) and (6), $G[A]$ and $G[B]$ are complete graphs. Hence, it is easy to check that $G[V(P)]$ gives a copy in $\mathcal{F}(m,k,\ell)$ of Type IV (with $k = 2\ell+2$, $w = x_\alpha$ and $\{w_1, w_2\} = \{x_u, x_v\}$), a contradiction. $\square$

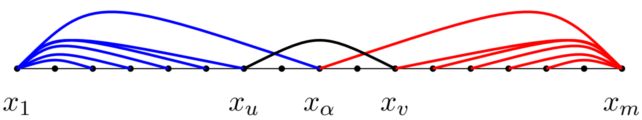


Figure 11. The structure of the non-crossing path P

We choose a longest H-path $P = x_1Px_m \in \mathcal{P}$ satisfying the following.

(a) $d_H(x_1) + d_H(x_m)$ is as large as possible;

(b) subject to (a), $\ell(i,j)$ is as large as possible, where (i,j) is a minimal crossing pair (i,j) in P;

(c) and subject to (a) and (b), P has as many minimal crossing pairs as possible.

Within P, let (i_1, j_1) and (i_2, j_2) be two minimal crossing pairs of P such that i_1 is as small as possible and j_2 is as large as possible. [7]

Claim 9. *The followings hold for the crossing pairs of P.*

- *There is a unique minimal crossing pair in P when $m \geq k+2$,*
- *there are at most two minimal crossing pairs in P when $m = k+1$ and*
- *each minimal crossing pair $(i', j') \neq (i,j)$ in P satisfies $\ell(i', j') = 1$ when $m = k$.*

Proof. Let $m \geq k+2$. Suppose to the contrary that there exist two minimal crossing pairs in P, that is $i_1 < j_1 \leq i_2 < j_2$. By Claim 2(iii), we have $\ell(i_1, j_1) \geq m-k+1 \geq 3$ and $\ell(i_2, j_2) \geq m-k+1 \geq 3$. Note that $V(x_{i_2+1}Px_{j_2-2}) \cap ((N_P^-(x_1) \setminus \{x_{j_1-1}\}) \cup N_P[x_m]) = \emptyset$, $|V(x_{i_2+1}Px_{j_2-2})| = \ell(i_2, j_2) - 1 \geq 2$ and $|(N_P^-(x_1) \setminus \{x_{j_1-1}\}) \cup N_P[x_m]| = |V_{j_1}| \geq 2\ell$ by Claim 2(i) and Claim 2(iii) (recall the definitions of U_i and V_j). It follows that $x_1Px_{i_1}x_mPx_{j_1}x_1$ is a cycle of length at least $2\ell+2 = k$, a contradiction.

Let $m = k+1$. Suppose to the contrary that there exist three minimal crossing pairs $(\alpha_1, \beta_1), (\alpha_2, \beta_2)$ and (α_3, β_3) in P. Without loss of generality, we may assume that $\alpha_1 < \beta_1 \leq \alpha_2 < \beta_2 \leq \alpha_3 < \beta_3$. Note that

$$(V(x_{\alpha_2+1}Px_{\beta_2-2}) \cup V(x_{\alpha_3+1}Px_{\beta_3-2})) \cap ((N_P^-(x_1) \setminus \{x_{\alpha_1-1}\}) \cup N_P[x_m]) = \emptyset,$$

$|V(x_{\alpha_2+1}Px_{\beta_2-2}| = |V(x_{\alpha_3+1}Px_{\beta_3-2})| = 1$ and $|(N_P^-(x_1) \setminus \{x_{j_1-1}\}) \cup N_P[x_m]| = |V_j| \geq 2\ell$ by Claim 2(i). Then $x_1Px_{\alpha_1}x_mPx_{\beta_1}x_1$ is a cycle of length at least $2\ell+1+1 = k$, a contradiction.

Finally, let $m = k$. By Claim 2(iii), we have $\ell(i,j) = 1$ or $\ell(i,j) = 2$. We may assume that $\ell(i,j) = 2 = m - 2\ell$, since otherwise the result follows by the choice of (i,j). Hence Claim 2(iv) implies $V(x_1Px_i) \cup V(x_jPx_m) = U_i = V_j$. Suppose to the contrary that there exists a minimal crossing pair (i', j') other than (i,j) in P with $\ell(i', j') = 2$. It is clear that $V(x_{i'+1}Px_{j'-2}) \cap ((N_P^-(x_1) \setminus \{x_{j-1}\}) \cup N_P[x_m]) = \emptyset$ and $|V(x_{i'+1}Px_{j'-2})| = 1$, contradicting $V(x_1Px_i) \cup V(x_jPx_m) = U_i = V_j$. $\square$

There are two possibilities for the size of m: $m \geq k+1$ or $m = k$. We now split the rest of the proof into two cases based on these two possibilities.

[7] It is possible that $(i_1, j_1) = (i_2, j_2)$.

3.2.1 $m \geq k+1$.

Since $m \geq k+1$, by Claim 9, there are at most two minimal crossing pairs in P. Suppose that there are two minimal crossing pairs $(i_1, j_1), (i_2, j_2)$ in P. By Claim 9 again, we have $m = k+1$. By Claim 2(iii), we have $2 = m-k+1 \leq \ell(i_1, j_1), \ell(i_2, j_2) \leq m - 2\ell = 2\ell+2+1-2\ell = 3$. Consider the crossing pair (i_1, j_1). From Claim 2(iv) we have $|(V(x_1Px_{i_1}) \cup V(x_{j_1}Px_m)) \setminus U_{i_1}| \leq 1$ implying (recall the definition of U_{i_1}) $|V(x_{i_2}Px_{j_2})| \leq 4$, i.e., $\ell(i_2, j_2) \leq 2$. Similarly, we have $\ell(i_1, j_1) \leq 2$. Thus we obtain $\ell(i_1, j_1) = \ell(i_2, j_2) = 2$. Consider the crossing pair (i_1, j_1). We have $x_{i_2+2} \notin U_{i_1}$ (recall the definition of U_{i_1}). Hence, by Claim 2(ii) and (iv), we have

$$U_{i_1} = (V(x_1Px_{i_1}) \cup V(x_{j_1}Px_m)) \setminus \{x_{i_2+2}\} \tag{7}$$

and hence Claim 2(i) and (iii) imply that $|U_{i_1}| = 2\ell$. Moreover, by Claim 2*(iv), we have $N_P(x_1) = N_H(x_1)$, $N_P(x_m) = N_H(x_m)$ and $d_P(x_1) = d_P(x_m) = d_H(x_1) = d_H(x_m) = \ell$. Hence from the definition of s and t, we have $x_s, x_{s+1} \in N_H(x_1)$ and $x_{t-1}, x_t \in N_H(x_m)$.

Assume that $j_2 < t-1$. Since $x_s, x_{s+1} \in N_H(x_1)$ and x_1 is adjacent to x_{t-1} by the definition of t, it follows from Claim 4 that x_1 is not adjacent to x_{t-2} and hence $j_2 \leq t-3$. Thus, by (7), x_m is adjacent to x_{t-3}. Therefore, $(t-3, t-1)$ is minimal crossing pair in P with $j_2 \leq t-3$, contradicting that there are two minimal crossing pairs. Hence, we have $j_2 = t-1$. Similarly, we have $i_1 = s+1$. Note that x_1 is not adjacent to x_{i_2+2} and x_m is not adjacent to x_{i_2+1}. Consider the crossing pair $(s+1, j_1) = (i_1, j_1)$. We have $x_{i_2+2} \notin U^*_{i_1}$. By Claim 2*$(iii)$, we have

$$(V(x_1Px_{s+1}) \cup V(x_{j_1}Px_m)) \setminus U^*_{i_1} = \{x_{i_2+2}\}. \tag{8}$$

By Claim 4, x_1 is not adjacent to x_{j_1+1} and hence by (8) x_m is adjacent to x_{j_1}. By Claim 4 again, x_m is not adjacent to x_{j_1+1} implying $j_1 = i_2$. Let

$$A = V(x_1Px_{\ell-2}),\ B = V(x_{\ell+6}Px_{k+1}),\ \text{and}\ C = \{x_{\ell-1}, x_{\ell+2}, x_{\ell+5}\}.$$

Combing the above arguments, we have $N_H[x_1] = A \cup C$, $N_H[x_m] = B \cup C$ and $|A| \geq 2$. Consider the paths $x_\gamma Px_1x_{\gamma+1}Px_m$ for $2 \leq \gamma \leq \ell-2$ and $x_\lambda Px_{k+1}x_{\lambda-1}Px_1$ for $\ell+6 \leq \lambda \leq k$. Since $|V(x_1Px_{\ell-1})| = \ell-1$, $|V(x_mPx_{\ell+5})| = \ell-1$, $x_\gamma \in V(H)$ and $x_\lambda \in V(H)$, x_γ is adjacent to some vertex of $V(x_\ell Px_{k+1})$ and x_λ is adjacent to some vertex of $V(x_1Px_{\ell+4})$. Thus those paths are crossing H-paths. By Claim 3, we can determine the neighbors of x_γ and x_λ in H, that is $N_H[x_1] = N_H[x_\gamma]$ and $N_H[x_{k+1}] = N_H[x_\lambda]$. Hence $G[A]$ and $G[B]$ are complete graphs, $G[V(P)]$ gives a copy of $F(k+1, k, \ell-2)$ of Type II with $|A| \geq 2$, a contradiction (see Figure 12.).

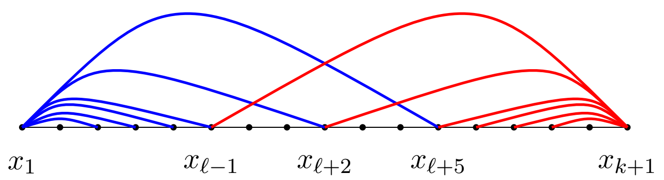


Figure 12. The structure of the crossing path P with two crossing pairs when $m = k+1$

Thus, we may assume that there is a unique minimal crossing pair (i, j) in P.

Claim 10. *Consider the crossing pair (i, j). We have $i = s+1$ and $j = t-1$.*

Proof. If $d_H(x_1) + d_H(x_m) \geq 2\ell+1$, then by Claim 2$(iii)$ and Claim 2*(i), (ii), we have $d_H(x_1) + d_H(x_m) = 2\ell+1$. Hence by Claim 2*$(iv)$, we have $V(x_1Px_i) \cup V(x_jPx_m) = U^*_i = V^*_j$, $N_P(x_1) = N_H(x_1)$ and $N_P(x_m) = N_H(x_m)$. If $m - \ell(i, j) = 2\ell$, by Claim 2*(iv), we also have $V(x_1Px_i) \cup V(x_jPx_m) = U^*_i = V^*_j$, $N_P(x_1) = N_H(x_1)$ and $N_P(x_m) = N_H(x_m)$. Therefore, we have $x_s, x_{s+1} \in N_H(x_1)$, and hence by Claim 4, x_1 is not adjacent to consecutive vertices of $V(x_jPx_{t-1})$. Since x_1 is adjacent to x_j, x_1 is not adjacent to x_{j+1}. Then by $V(x_1Px_i) \cup V(x_jPx_m) = U^*_i = V^*_j$, x_m is adjacent to x_j. Since there is a unique minimal crossing pair, x_1 is not adjacent to any vertex of $V(x_{j+1}Px_m)$, implying $j = t-1$. By symmetry we have $i = s+1$.

Now, suppose that $m - \ell(i,j) = 2\ell+1$ and $d_H(x_1) + d_H(x_m) = 2\ell$, i.e., $d_H(x_1) = d_H(x_m) = \ell$. Then by Claim $2^*(ii)$, there exists a unique vertex $x_p \in V(x_1Px_i) \cup V(x_jPx_m)$ such that

$$\{x_p\} \cup U_i^* = V(x_1Px_i) \cup V(x_jPx_m). \tag{9}$$

By symmetry between x_1 and x_m, we may assume that $1 \le p \le i$. Then we have

$$V(x_jPx_m) \subseteq U_i^*, \tag{10}$$

implying that $x_t, x_{t-1} \in N_H(x_m)$ ($m \neq t$ by $m \geq k+1$). Suppose to the contrary that $i > s+1$. Since $x_t, x_{t-1} \in N_H(x_m)$, $x_i \in N_H(x_m)$, by Claim 4, we have $x_{i-1} \notin N_P(x_m)$. Since there is only one minimal crossing pair in P, with Claim 1(iii), we have $V(x_{s+1}Px_i) \cap N_P(x_1) = \emptyset$. Thus by $x_{i-1} \notin N_P(x_m)$ and $x_i \notin N_P(x_1)$, x_i does not belongs to U_i^*, i.e., $p = i$. Since x_1 is not adjacent to x_{i-1} (by $i-1 \geq s+1$), by (9) we have $x_{i-2} \in N_H(x_m)$ and hence x_{i-3} is not adjacent to x_m by $x_t, x_{t-1} \in N_H(x_m)$ and Claim 4. Thus by (9) and $p = i$, x_1 is adjacent x_{i-2}. Since there is a unique minimal crossing pair, we have $s = i-3$. Hence, we have $V(x_1Px_s) \subseteq U_i^*$, implying that $x_s, x_{s+1} \in N_H(x_1)$ by $m \geq k+1$ (see Figure 13.). By Claim 4, x_1 is not adjacent to x_{j+1}. From (10), x_m is adjacent to x_j, and since there is a unique minimal crossing pair, from (10) we deduce that $V(x_jPx_{m-1}) \subseteq N_H(x_m)$. We consider the path $R_1 = x_tPx_mx_{t-1}Px_1$. Since $x_t \in V(H)$, we have $d_{R_1}(x_t) \geq d_H(x_t) \geq \ell$. Note that $|V(x_tPx_m)| \leq d_H(x_m) - 1 = \ell - 1$. Thus x_t is adjacent to some vertices of $V(x_1Px_{t-2})$. Since x_1 is adjacent to x_{t-1}, $R_1 \in \mathcal{P}$ is a crossing path. Hence by Claim 3, x_t must be adjacent to one of x_{i-2}, x_{i-1} (x_{i-2}, x_{i-1} are possible neighbors of x_t in R_1). Then $x_\gamma Px_1x_{t-1}Px_ix_mPx_tx_\gamma$ is a cycle of length at least $m-1 \geq k$, where $\gamma \in \{i-2, i-1\}$, a contradiction. This contradiction shows that $i = s+1$.

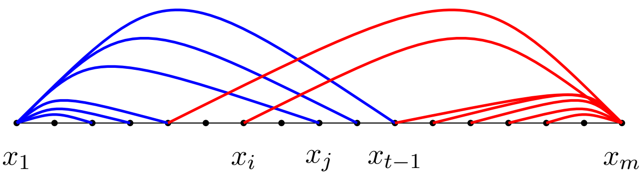


Figure 13. $P = x_1Px_sx_{s+1}Px_m$ with $d_H(x_1) = d_H(x_m) = \ell$ and $m - \ell(i,j) = 2\ell$

Next, we will show that $j = t-1$. First, we suppose that $x_{j+1} \in N_P(x_1)$ (see Figure 14.). Then we have

$$|V(x_1Px_i)| \leq \ell - 1. \tag{11}$$

By Claim 4, we have $x_s \notin V(H)$ or $x_{s+1} \notin N_P(x_1)$. If $x_s \notin V(H)$, then $x_s \notin N_H(x_1)$. Since x_{s-1} is not adjacent to x_m by the definition of s, then we have $x_s = x_p$, i.e., $p = i-1$. If $x_{s+1} \notin N_P(x_1)$, then $x_{s+1} \notin N_H(x_1)$. Sine x_s is not adjacent to x_m by the definition of s, then we have $x_{s+1} = x_p$, i.e., $p = i$. Hence, we can consider the following two cases.

Case 10.1. $p = i = s+1$. Then x_1 is not adjacent to x_i. It follows from (9) that $V(x_1Px_{i-1}) \subseteq N_H[x_1]$. If $i \leq 3$, then $x_1x_jPx_ix_mPx_{j+1}x_1$ is a cycle of length $m-1 \geq k$, a contradiction. Therefore, we have $i \geq 4$. Then we consider the path $R_3 = x_{i-2}Px_1x_{i-1}Px_m$. Since $x_{i-2} \in V(H)$ (by (9)), by (11) x_{i-2} is adjacent to at least one vertex of $V(x_{i+1}R_3x_m)$, and hence $R_3 \in \mathcal{P}$ is a crossing path. Hence by Claim 3, x_{i-2} must be adjacent to at least one vertex in $\{x_j, x_{j+1}\}$ (as in the proof of Claim 4). If x_{i-2} is adjacent to x_j, then $x_1Px_{i-2}x_jPx_ix_mPx_{j+1}x_1$ is a cycle of length $m-1 \geq k$, a contradiction. Similarly, if x_{i-2} is adjacent to x_{j+1}, then $x_1Px_{i-2}x_{j+1}Px_mx_iPx_jx_1$ is a cycle of length $m-1 \geq k$, a contradiction.

Case 10.2. $p = i-1 = s$. Then $x_i \in N_H(x_1)$ and $x_{i-1} \notin V(H)$. Clearly, by (9) we have $x_{i-1} \notin N_H(x_1)$ and $x_{i-2} \in N_H(x_1)$. Suppose that x_{i-2} has a neighbor $y \in V(H)$ not in P. Then we consider the H-path $R_4 = yx_{i-2}Px_1x_iPx_m$ on m vertices. Since R_4 is a longest H-path, we have $d_{R_4}(y) \geq d_H(y) \geq \ell$. From (11) we have $|V(yx_{i-2}Px_1)| = |V(x_1Px_i)| - 1 \leq \ell - 2$. Thus y is adjacent to at least one vertex of $V(x_{i+1}R_4x_m)$, and hence $R_4 \in \mathcal{P}$ is a crossing path. Therefore, by Claim 3, y must be adjacent to one of $\{x_j, x_{j+1}\}$ (as in the proof of Claim 4). If y is adjacent to x_j (or x_{j+1}), then $x_1Px_{i-2}yx_jPx_ix_mPx_{j+1}x_1$ (or $x_1Px_{i-2}yx_{j+1}Px_mx_iPx_jx_1$) is a cycle of length at least k, a contradiction. Therefore, we have $N_H(x_{i-2}) \subseteq V(P)$. Then we consider the path $R_5 = x_{i-2}Px_1x_iPx_m$.[8] Clearly, we have $d_{R_5}(x_{i-2}) \geq d_H(x_{i-2}) \geq \ell$ ($x_{i-1} \notin V(H)$) and $d_{R_5}(x_m) \geq \ell$ ($x_m \in V(H)$ is not adjacent to x_{i-1} by Claim 4 and $x_{t-1}, x_t \in N_H(x_m)$). By (11), we have $|V(x_{i-2}Px_1)| = |V(x_1Px_i)| - 2 \leq \ell - 3$. Then x_{i-2} is adjacent to at least one vertex of $V(x_{i+1}R_5x_m)$, and hence R_5 has a crossing pair. Therefore, by Claim 3, x_{i-2} must be adjacent to at least one of $\{x_j, x_{j+1}\}$. If x_{i-2} is adjacent to x_j (or x_{j+1}), then $x_1Px_{i-2}x_jPx_ix_mPx_{j+1}x_1$ (or $x_1Px_{i-2}x_{j+1}Px_mx_iPx_jx_1$) is a cycle of length $m - 1 \geq k$, a contradiction.

Combining Cases 10.1 and 10.2, x_1 is not adjacent to x_{j+1}. By (10), x_m is adjacent to x_j. Since there is only one minimal crossing pair, x_1 is not adjacent to any vertex of $V(x_{j+1}Px_m)$, that is, $j = t - 1$. This completes the proof of the claim. $\square$

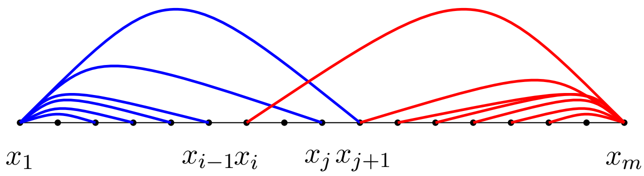


Figure 14. x_1 is adjacent to both of x_j and x_{j+1}

Therefore, there is only one minimal crossing pair (i, j) in P with $i = s+1$ and $j = t-1$ by Claim 10. It follows from Claim 2(iii) that $m - \ell(i,j) = 2\ell$ or $m - \ell(i,j) = 2\ell + 1$. Suppose that $m - \ell(i,j) = 2\ell$. Then applying Claim 2*(iv), we have $U_i^* = V_j^* = V(x_1Px_i) \cup V(x_jPx_m)$, $N_H(x_1) = N_P(x_1)$ and $N_H(x_m) = N_P(x_m)$. Moreover, we have $i = \ell$ and $j = m - \ell + 1$. Consider the paths $x_\gamma Px_1x_{\gamma+1}Px_m$ for $2 \leq \gamma \leq \ell - 1$. Since $|V(x_1Px_i)| = \ell$ and $x_\gamma \in V(H)$, x_γ is adjacent to some vertex of $V(x_{i+1}Px_m)$. Hence $x_\gamma Px_1x_{\gamma+1}Px_m$ is a crossing H-path. Then by Claim 3, x_γ is adjacent to all but at most one vertex of $V(x_1Px_i) \cup \{x_j, x_{j-1}\}$. If x_2 is adjacent to x_{j-1}, then $x_1x_{i-1}Px_2x_{j-1}Px_ix_mPx_jx_1$ is a cycle of length $m \geq k + 1$, a contradiction. Thus, we have $N_H[x_2] = N_H[x_1]$. If x_3 is adjacent to x_{j-1}, then $x_1x_2x_{i-1}Px_3x_{j-1}Px_ix_mPx_jx_1$ is a cycle of length $m \geq k + 1$, a contradiction. Thus, we have $N_H[x_3] = N_H[x_1]$. Progressively, we can show that $N_H[x_\gamma] = N_H[x_1]$. By symmetry of x_1 and x_m, we have $N_H[x_m] = N_H[x_\lambda]$ for $m - \ell + 2 \leq \lambda \leq m - 1$. Let

$$A = V(x_1Px_{\ell-1}), B = V(x_{m-\ell+2}Px_m) \text{ and } C = \{x_\ell, x_{m-\ell+1}\}.$$

Then $G[A]$ and $G[B]$ are complete graphes on $\ell - 1$ vertices. Hence, it is easy to check that $G[V(P)]$ gives a copy in $\mathcal{F}(m, k, \ell - 1)$ with Type II, a contradiction.

Therefore, we may assume that

$$m - \ell(i,j) = 2\ell + 1, \text{ i.e., } \ell(i,j) = m - 2\ell - 1. \tag{12}$$

Suppose that $d_H(x_1) + d_H(x_m) = 2\ell + 1$. Without loss of generality, let $d_H(x_1) = \ell + 1$ and $d_H(x_m) = \ell$. Then applying Claim 2*(iv), we have $U_i^* = V_j^* = V(x_1Px_i) \cup V(x_jPx_m)$, $N_H(x_1) = N_P(x_1)$ and $N_H(x_m) = N_P(x_m)$. Consider the paths $x_\gamma Px_1x_{\gamma+1}Px_m$ for $2 \leq \gamma \leq i - 1$ and $x_\lambda Px_mx_{\lambda-1}Px_1$ for $j + 1 \leq \lambda \leq m - 1$ (it is easy to check that those paths are crossing paths as before). By Claim 3, it is not hard to show that $N_H[x_\gamma] \subseteq N_H[x_1]$ for $2 \leq \gamma \leq \ell$ and $N_H[x_m] = N_H[x_\lambda]$ for $m - \ell + 2 \leq \lambda \leq m - 1$. Let

$$A = V(x_1Px_{i-1}), B = V(x_{j+1}Px_m) \text{ and } C = \{x_i, x_j\}.$$

[8] Note that R_5 has $m - 1$ vertices. Thus R_5 does not belong to $\mathcal{P}$.

Since we can keep x_1x_j, $x_{i-1}x_i$ and delete other edges between A and C to ensure $d_{F[A\cup C]}(z)=\ell$ for each $z\in A$, [9] $G[V(P)]$ gives a copy in $\mathcal{F}(m,k,\ell-1)$ with Type II, a contradiction.

Now we may assume that $d_H(x_1)=d_H(x_m)=\ell$. By Claim 2*(iii), without loss of generality, there exists a vertex x_p such that

$$\{x_p\}\cup U_i^*=V(x_1Px_i)\cap V(x_jPx_m) \text{ with } 1\le p\le i. \tag{13}$$

Claim 10 implies that

$$i=\ell+1 \text{ and } j=m-\ell+1. \tag{14}$$

Also, note that $N_H[x_m]=\{x_{\ell+1}\}\cup V(x_{m-\ell+1}Px_m)$ and $N_H[x_1]=\{x_{m-\ell+1}\}\cup(V(x_1Px_{\ell+1})\setminus\{x_p\})$. Then we consider the path $Q_\lambda=x_1Px_{\lambda-1}x_mPx_\lambda$, where $m-\ell+2\le\lambda\le m-1$. Since $d_{Q_\lambda}(x_\lambda)\ge d_H(x_\lambda)\ge\ell$, x_λ is adjacent to some vertices of $V(x_1Px_{j-1})=V(x_1Q_\lambda x_{j-1})$. Hence, $Q_\lambda\in\mathcal{P}$ is a crossing path. As in the previous proofs, x_λ is adjacent to all but at most one vertex of $N_H[x_m]\cup\{x_{p-1}\}$ by Claim 3.

Claim 11. *For each $m-\ell+2\le\lambda\le m-1$, we have $N_H[x_\lambda]=N_H[x_m]$.*

Proof. Suppose to the contrary that x_λ is adjacent to x_{p-1}. First we assume that $p<i$. Then $x_{p-1}Px_1x_{p+1}Px_{\lambda-1}x_mPx_\lambda x_{p-1}$ is a cycle of length $m-1\ge k$, a contradiction.

Therefore, we have $p=i$ (see Figure 15.). Then we consider the path $L_\lambda=x_1Px_\lambda x_mPx_{\lambda+1}$. Note that $x_{\lambda+1}\in V(H)$, $|V(x_jPx_m)|\le\ell$ and x_1 is adjacent to x_j. Clearly, $L_\lambda\in\mathcal{P}$ is a crossing path. By Claim 3, $x_{\lambda+1}$ must be adjacent to at least one of $\{x_{p-1},x_p\}$. By the maximality of $\ell(p,j)$ in L_γ, $x_{\lambda+1}$ is adjacent to x_p, as otherwise we have $\ell(p-1,j)>\ell(p,j)$ where $\ell(p-1,j)$ is in P, a contradiction. Then $x_{p-1}Px_1x_jPx_px_{\lambda+1}Px_mx_{j+1}Px_\lambda x_{p-1}$ is a cycle of length m, a contradiction. Therefore, x_λ is not adjacent to $x_{i-1}=x_{p-1}$. This completes the proof of the claim. $\square$

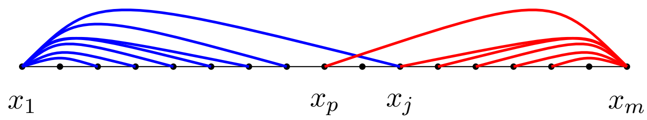


Figure 15. The structure of the crossing path P with one crossing pair

From Claim 11,

$$G[V(x_{m-\ell+2}Px_m)] \text{ is a complete graph.} \tag{15}$$

Suppose that $x_p\notin V(H)$. Let $2\le\gamma\le\ell$ and $\gamma\ne p,p-1$. Then we consider the m-vertex H-path $M_\gamma=x_\gamma Px_1x_{\gamma+1}Px_m$. Note that $x_\gamma\in V(H)$ which implies $x_p\in N_P(x_\gamma)$, $|V(x_1Px_i)|=\ell$ and x_m is adjacent to x_i. Clearly, $M_\gamma\in\mathcal{P}$ is a crossing path. Note that $x_p\notin N_H(x_\gamma)$. Consider the neighbors of x_γ and x_m in M_γ (in H). Since $x_p\notin V(H)$ by Claim 3, we have $N_H[x_1]=N_H[x_\gamma]$ for $2\le\gamma\le\ell$ and $\gamma\ne p,p-1$, and hence we have $N_H[x_1]=N_H[x_{p-1}]$ (x_γ is adjacent to x_{p-1}). Let

$$A=\{x_1,\cdots,x_\ell\}\setminus\{x_p\},\ B=\{x_{m-\ell+2},\cdots,x_m\} \text{ and } C=\{x_i,x_j\}=\{x_{\ell+1},x_{m-\ell+1}\}.$$

Hence, combining with Claim 11 and (15), $G[A]$ and $G[B]$ are complete graphs. Recall that $m\ge k+1$. It is easy to check that $G[V(P)]$ gives a copy of $F(m,k,\ell-1)$ of Type III (view x_p as v_2 when $p<s$ or v_{r+1} when $p=s$ and see Figure 4(b) and 4(c) for some hints), a contradiction.

Therefore, we have $x_p\in V(H)$. Since x_1 is adjacent to x_2 and $N_H(x_1)=N_P(x_1)$, we have $p\ge3$. Let

$$A=\{x_1,\cdots,x_\ell\},\ B=\{x_{m-\ell+2},\cdots,x_m\} \text{ and } C=\{x_i,x_j\}=\{x_{\ell+1},x_{m-\ell+1}\}.$$

Then we consider the path $T_\gamma=x_\gamma Px_1x_{\gamma+1}Px_m$ for $2\le\gamma\le\ell$ with $\gamma\ne p-1$. Note that $x_\gamma\in V(H)$ and $N[x_m]=\{x_{\ell+1}\}\cup V(x_{m-\ell+1}Px_m)$. Those paths are crossing H-paths. By Claim 3, we have $N_H(x_\gamma)\subseteq A\cup C$ for $2\le\gamma\le\ell$ and $\gamma\ne p-1$. Combining with $N_H[x_1]=(A\cup C)\setminus\{x_p\}$, we have

$$N_H[x_\gamma]\subseteq A\cup C \text{ for } 1\le\gamma\le\ell \text{ and } \gamma\ne p-1. \tag{16}$$

[9]This simple fact will be used later in the following proofs when $|A\cup C|=\ell+2$.

In particular, if $p \le \ell$, then
$$N_H[x_p] = (A \cup C) \setminus \{x_1\}. \tag{17}$$
We consider the following three cases.

Case A.1. Let $p = i = \ell + 1$. By Claim 11, $x_p = x_i = x_{\ell+1}$ is adjacent to $x_{m-\ell+2}$. Consider the path $Q_1 = x_{p-1}Px_1x_{m-\ell+1}Px_px_{m-\ell+2}Px_m$. Since x_{p-1} is adjacent to x_p and x_m is adjacent to $x_{m-\ell+1}$, $Q_1 \in \mathcal{P}$ is a crossing H-path. By Claim 3, we have $N_H[x_{p-1}] \subseteq A \cup C$. Hence, combining with (16), each vertex of A in $G[A \cup C]$ has degree at least ℓ. Note that both vertices of C are adjacent to A (x_1 is adjacent to $x_j = x_{m-\ell+1}$ and x_{i-1} is adjacent to $x_i = x_{\ell+1}$). Combining with (15), it is easy to check that $G[V(P)]$ gives a copy of $F(m, k, \ell - 1)$ with Type II (recall the definition of Type II), a contradiction.

Case A.2. Let $p \le \ell = i - 1$ and $\ell \ge 4$. By (17), x_p is adjacent to x_{p-2} for $p \ge 4$. Hence, consider the paths $x_{p-1}x_px_{p-2}Px_1x_{p+1}Px_m$ when $p \ge 4$ and $x_{p-1}x_px_{p+1}x_1x_{p+2}Px_m$ when $p = 3$ ($p + 2 = 5 \le \ell + 1 = i$). By $c(G) < k$ and Claim 3, we have $N_H[x_{p-1}] \subseteq A \cup C$, and hence x_{p-1} has degree at least ℓ in $G[A \cup C]$. Combining with (16), each vertex of $A \setminus \{x_{p-1}\}$ has degree at least ℓ in $G[A \cup C]$. Note that x_1 is adjacent to each vertex of C. Thus combining with (15), it is easy to check that $G[V(P)]$ gives a copy of $F \in \mathcal{F}(m, k, \ell - 1)$ of Type II, a contradiction.

Case A.3. Let $p \le \ell$ and $\ell \le 3$. Suppose that $m \ge k + 2$. Then $j = m - \ell + 1 \ge \ell + 5 \ge 2\ell + 2$. Thus $x_1Px_jx_1$ is a cycle of length at least $2\ell + 2 = k$, a contradiction. Now let $m = k + 1$. Then we have $j = m - \ell + 1$. If $\ell \le 2$, then $x_1Px_jx_1$ is a cycle of length at least $m - \ell + 1 = k + 1 - \ell + 1 \ge k$, a contradiction. Let $\ell = 3$. This forces that $k = 8$ and $p = 3$ (see Figure 16.). Note that $x_3 \in V(H)$ and $N_H(x_3) = \{x_2, x_4, x_7\}$. Suppose that $d_P(x_2) \ge 3$. Since $c(G) \le 8$, we have $N_P(x_2) \subseteq A \cup C$. Then G contains a copy of $F \in \mathcal{F}(9, 8, 2)$ ($A = \{x_1, x_2, x_3\}$, $B = \{x_8, x_9\}$, $C = \{x_4, x_7\}$ and x_1x_7 and x_3x_4 are two independent edges) with Type II, a contradiction. Therefore, we have $d_P(x_2) = 2$. It follows from $\ell = 3$ and $x_2 \in V(H)$ that there is a vertex $z \in N_H(x_2) \setminus N_P(x_2)$. Since $c(G) \le 8$, we have $N_H(z) = \{x_2, x_4, x_7\}$. Hence $\{z, x_3, x_1\}$ together with $\{x_2, x_4, x_7\}$ induce a copy of $K_{3,3}$ Since $N_H(x_8) = \{x_4, x_7, x_9\}$ by $c(G) \le 8$, it is easy to check that $G[V(P) \cap \{z\}]$ contains a copy of $K_{3,3}^+$, a contradiction. Moreover, G contains a copy in $F \in \mathcal{F}(8, 8, 1)$ ($m = 9$) with Type III in $K_{3,3}^+$ (the path $zx_2x_3x_4x_8x_9x_7x_1$ with $A' = \{z\}$, $B' = \{x_1\}$ and $C' = \{x_2, x_4, x_7\}$). This completes the proof when k is even and $m \ge k + 1$.

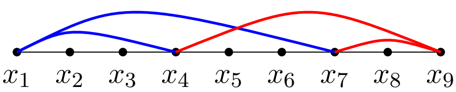


Figure 16. The structure of the crossing path P when $m = k + 1 = 9$

3.2.2 $m = k$.

By Claim 2(iii), we have $\ell(i, j) = 1$ or $\ell(i, j) = 2$. If $\ell(i, j) = 2$ or $d_H(x_1) + d_H(x_k) = 2\ell + 1$, then by Claim 2*$(iv)$, we have
$$V(x_1Px_i) \cup V(x_jPx_k) = U_i^* = V_j^*, N_H(x_1) = N_P(x_1) \text{ and } N_H(x_k) = N_P(x_k). \tag{18}$$
In the following, we only consider the case $\ell(i, j) = 2$, since the case $d_H(x_1) + d_H(x_k) = 2\ell + 1$ is similar to the case after (12). By the definition of s, x_{s-1} and x_s are not adjacent to x_k. Hence, by (18), we have $x_s, x_{s+1} \in N_H(x_1)$. Similarly, we have $x_{t-1}, x_t \in N_H(x_k)$. Let
$$A = V(x_1Px_s), B = V(x_tPx_k) \text{ and } C = \{x_{s+1}, x_{s+3}, \ldots, x_{i-2}, x_i, x_j, x_{j+2}, \ldots, x_{t-2}, x_t\}.$$
By Claim 1(iii), x_1 is not adjacent to x_{s+2}. By Claim 4, x_m is not adjacent to x_{s+2}. Hence, x_1 is adjacent to x_{s+3} by (18). Moreover, consider the minimal crossing pair $(s + 1, s + 3)$. By Claim 4, x_1 is not adjacent to x_{s+4}. Then apply (18) again, x_k is adjacent to x_{s+3} and not adjacent to x_{s+4} by

Claim 4. Repeating the above arguments, x_1 is adjacent to each vertex of $A \cup C$ and x_k is adjacent to each vertex of $B \cup C$. Since $|V(x_1 P x_{s+1})| \leq \ell$ and $x_\gamma \in V(H)$, $R_\gamma = x_\gamma P x_1 x_{\gamma+1} P x_k$ is a crossing H-path for $2 \leq \gamma \leq s$. Hence, x_γ is adjacent all but at most one vertex of $(A \cup C \cup \{x_{j-1}\}) \setminus \{x_\gamma\}$ by Claim 3 (x_γ is not adjacent to $x_{i+1} = x_{j-2}$ since x_k is adjacent to x_i and $c(G) < k$). If x_s is adjacent to x_{j-1}, then $x_{j-1} P x_{s+1} x_k P x_j x_1 P x_s$ is a cycle of length k, a contradiction. Therefore, we have $N_H[x_s] = N_H[x_1]$, and progressively, we have $N_H[x_\gamma] = N_H[x_1]$ for $2 \leq \gamma \leq s$. Similarly, we have $N_H[x_\lambda] = N_H[x_k]$ for $t \leq \lambda \leq k-1$. Hence, $G[A]$ and $G[B]$ are complete graphs with $|A| = |B| = s$, and $G[V(P)]$ gives a copy of $\mathcal{F}(k, k, s)$ with Type II (see Figure 17.), a contradiction.

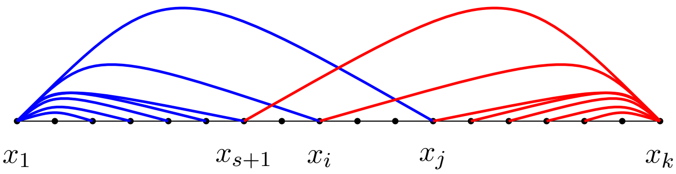


Figure 17. The structure of the crossing path P

Therefore, we may assume that $\ell(i,j) = 1$ and $d_H(x_1) = d_H(x_k) = \ell$. By Claim 2*($iii$), without loss of generality, there exists a vertex $x_p \notin (N_H[x_1] \cup N_H^+(x_k)) \setminus \{x_{i+1}\} = U_i^*$ with $1 \leq p \leq i$, that is

$$V(x_1 P x_i) \cup V(x_j P x_k) = U_i^* \cup \{x_p\}. \tag{19}$$

Hence, we have $x_p \notin N_H(x_1)$ and $x_{p-1} \notin N_H(x_k)$. By the definition of t, we have $i \leq t-2$. Now, subject to previous choices,

$$\text{we choose } P \in \mathcal{P} \text{ such that } |V(x_{s_P+2} P x_{t_P-2}) \cap \{x_p\}| \text{ is as large as possible.} \tag{20}$$

Claim 12. $p \leq s+1$.

Proof. Suppose to the contrary that $p > s+1$, that is $s+2 \leq p \leq t-2$. By the definitions of s and t and (19), we have $x_s, x_{s+1} \in N_H(x_1)$ and $x_{t-1}, x_t \in N_H(x_k)$. Let

$$A = V(x_1 P x_s) \text{ and } B = V(x_t P x_k).$$

We consider the following three cases.

Case 12.1. $x_{p-1} \in N_H(x_1)$ and $x_p \in N_H(x_k)$. Let

$$C = \{x_{s+1}, x_{s+3}, \cdots, x_{p-5}, x_{p-3}, x_{p+2}, x_{p+4}, \cdots, x_{t-3}, x_{t-1}\}.$$

We shall show that x_1 and x_k are adjacent to each vertex of C. It follows from the definition of s that x_k is adjacent to x_{s+1}. By Claim 1(iii) x_1 is not adjacent to x_{s+2}. Since $x_{t-1}, x_t \in N_H(x_k)$, by Claim 4, x_k is not adjacent to x_{s+2}. Next, by (19) x_1 is adjacent to x_{s+3}, and hence by Claim 4, x_1 is not adjacent to x_{s+4}. By (19) again, x_k is adjacent to x_{s+3}. Applying Claim 4 again, x_k is not adjacent to x_{s+4}. Progressively, we can show that x_1 and x_k are adjacent to each vertex of $\{x_{s+1}, x_{s+3}, \cdots, x_{p-5}, x_{p-3}\}$ and are not adjacent to each vertex of $\{x_{s+2}, x_{s+4}, \cdots, x_{p-4}, x_{p-2}\}$. Similarly, x_1 and x_k are adjacent to each vertex of $\{x_{p+2}, x_{p+4}, \cdots, x_{t-3}, x_{t-1}\}$. Thus, x_1 and x_k are adjacent to each vertex of C (see Figure 18.).

Then we consider the paths $T_\gamma = x_\gamma P x_1 x_{\gamma+1} P x_k$ and $S_\lambda = x_\lambda P x_k x_{\lambda-1} P x_1$ for $2 \leq \gamma \leq s$ and $t \leq \lambda \leq k-1$. Since $|V(x_1 P x_s)| = |V(x_t P x_k)| \leq \ell - 1$ and $x_\gamma, x_\lambda \in V(H)$, $T_\gamma, S_\lambda \in \mathcal{P}$ are crossing H-paths. By Claim 3, x_γ is adjacent to all but at most one vertex of $A \cup C \cup \{x_{p-1}, x_p\}$. If x_s is adjacent to x_p, then $x_p P x_k x_{s+1} P x_{p-1} x_1 P x_s x_p$ is a cycle of length at least k, a contradiction. Thus x_s is not adjacent to x_p and hence $N_H[x_s] = N_H[x_1]$. If x_{s-1} is adjacent to x_p, then $x_p P x_k x_{s+1} P x_{p-1} x_1 P x_{s-2} x_s x_{s-1} x_p$ is a cycle of length k (by $N_H[x_s] = N_H[x_1]$, x_{s-2} is adjacent to x_s), a contradiction. Thus x_{s-1} is not adjacent to x_p and hence $N_H[x_{s-1}] = N_H[x_1]$. Repeat the above argument, we have $N_H[x_\gamma] = N_H[x_1]$ for $2 \leq \gamma \leq s$. By symmetry, we have $N_H[x_\lambda] = N_H[x_k]$ for $t \leq \lambda \leq k-1$. Finally, we can see that $x_{p-1} x_1 P x_s x_{t-1} P x_p x_k P x_t x_{s+1} P x_{p-1}$ is a cycle of length k, a contradiction.

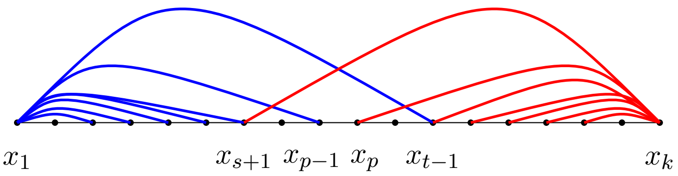


Figure 18. The structure of the crossing path P when $m = k$

Case 12.2. $x_{p-1} \in N_H(x_1)$ and $x_p \notin N_H(x_k)$. Then we have $p < i$ and $x_{p+1} \in N_H(x_1)$. Let

$$C = \{x_{s+1}, x_{s+3}, \cdots, x_{p-1}, x_{p+1}, \cdots, x_{t-3}, x_{t-1}\}.$$

Then the similar proof as (a) shows that x_1 is adjacent to each vertex of C and x_k is adjacent to each vertex of $C \setminus \{x_{p-1}\}$ (see Figure 19.). Note that $d_H(x_1) = d_H(x_k) = \ell$. Then we have $|A| = s$ and $|B| = s+1$. Consider the crossing paths $S_\lambda = x_\lambda P x_k x_{\lambda-1} P x_1 \in \mathcal{P}$ for $t \le \lambda \le k-1$ (since $|V(x_t P x_k)| \le \ell$ and $x_\lambda \in V(H)$, S_λ is a crossing path). By Claim 3 we have $N_H(x_\lambda) \subseteq B \cup C$. Moreover, consider the crossing paths $T_\gamma = x_\gamma P x_1 x_{\gamma+1} P x_k \in \mathcal{P}$ for $2 \le \gamma \le s$ (since $|V(x_1 P x_s)| \le \ell$ and $x_\gamma \in H$, T_γ is a crossing path). Similarly as the proof as in the last paragraph, we have $N_H[x_1] = N_H[x_\gamma] = A \cup C$ for $2 \le \gamma \le s$. Let $A' = B$ and $B' = A$. Note that $x_k \in A'$ is adjacent to x_{s+1} and $x_t \in A'$ is adjacent to x_{t-1}. It is easy to check that $G[V(P)]$ gives a copy in $\mathcal{F}(k, k, s)$ with Type II.

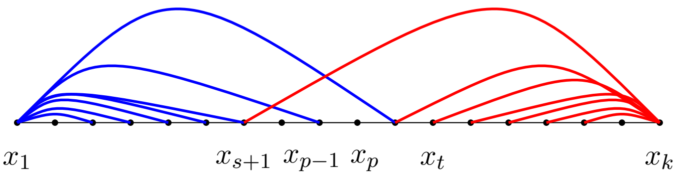


Figure 19. The structure of the crossing path P when $m = k$

Case 12.3. $x_{p-1} \notin N_H(x_1)$. If $x_p \notin N_P(x_k)$, then there is a minimal crossing pair (i', j') of length at least $(p+1)-(p-2)-1 = 2$ ($x_{p-1}, x_p \notin N_P(x_k)$ and $x_{p-1}, x_p \notin N_P(x_k)$), a contradiction. Therefore we have $x_p \in N_P(x_k) = N_H(x_k)$. Let

$$C = \{x_{s+1}, x_{s+3}, \cdots, x_{p-2}, x_p, x_{p+2}, \cdots, x_{t-3}, x_{t-1}\}.$$

As the proofs before, x_k is adjacent to each vertex of C and x_1 is adjacent to each vertex of $C \setminus \{x_p\}$. From $d_H(x_1) = d_H(x_k) = \ell$, we have $|A| = s$ and $|B| = s-1$. Consider the paths $x_\gamma P x_1 x_{\gamma+1} P x_k \in \mathcal{P}$ and $x_\lambda P x_k x_{\lambda-1} P x_1 \in \mathcal{P}$ for $2 \le \gamma \le s$ and $t \le \lambda \le k-1$. As in the previous proofs, we have $N_H[x_\gamma] \subseteq A \cup C$ and $N_H[x_\lambda] = B \cup C$. Note that $x_1 \in A$ is adjacent to x_{t-1} and x_s is adjacent to x_{s+1}. It is easy to check that $G[V(P)]$ gives a copy in $\mathcal{F}(k, k, s-1)$ with Type II, a contradiction. Thus we finish the proof of Claim 12. $\square$

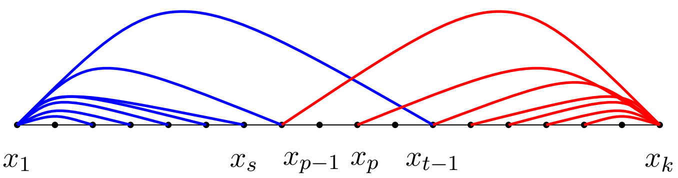


Figure 20. The structure of the crossing path P when $m = k$

By Claim 12, we can assume that $2 \le p \le s+1$, that is, $|V(x_{s+2} P x_{t-2}) \cap \{x_p\}| = 0$. By (19), x_t and x_{t-1} belong to $N_H(x_k)$. By Claim 4, x_k is not adjacent consecutive vertices of $V(x_{s+1} P x_{i_2})$. From the definition of s, we have $p \le i_1$. First, we will show that $i_1 = s+1$. Suppose that $i_1 > s+1$. Then by Claim 4, x_k is not adjacent to x_{s+2}, and hence we have $i_1 \ge s+3$. Moreover, since $2 \le p \le s+1$, by (19), x_1 is adjacent to x_{s+3}, that is $(s+1, s+3)$ is a crossing pair, a contradiction to $i_1 > s+1$.

Claim 13. *$N_H[x_\lambda] = N_H[x_k]$ for $t \le \lambda \le k-1$.*

Proof. Note that $x_\lambda \in V(H)$, x_1 is adjacent to x_{t-1} and $|V(x_tPx_k)| \le \ell$. The path $P^\lambda = x_1Px_{\lambda-1}x_kPx_\lambda$ is a crossing path. By Claim 3, we have $N_H[x_\lambda] \subseteq N_H[x_k] \cup \{x_{p-1}\}$. By our choice of P, we may assume $d_H(x_\lambda) = \ell$. Suppose that x_λ is adjacent to x_{p-1} (see Figure 21.). Case (13.1). $p < s+1$. Note that x_1 is adjacent to x_{p+1}. Consider the path $P^\lambda \in \mathcal{P}$. By the maximality of the number of minimal crossing pairs of P, x_λ is adjacent to each vertex of $V(x_{t-1}Px_k)$ (otherwise the path $P^\lambda \in \mathcal{P}$ has more minimal crossing pairs than P). Hence, x_λ is not adjacent to a vertex $x \in N_H[x_k] \setminus V(x_{t-1}Px_k)$. Note that $N_{P^\lambda}(x_\lambda) \cup \{x\} = N_P(x_k) \cup \{x_{p-1}\}$. Thus the path $P^\lambda = x_\lambda Px_kx_{\lambda-1}Px_1 = y_ky_{k-1}\dots y_2y_1 = y_kP^\lambda y_1$ is a crossing path with a minimal crossing pair $(i', j') = (p-1, p+1)$ satisfying $|V(y_{s'+2}Py_{t'-2}) \cap \{y_{p'}\}| = 1$, where $s' = \min\{h : y_{h+1} \in N_{P'}(y_k)\} = p-1$, $t' = \max\{h : y_{h-1} \in N_{P'}(y_1)\} = t$ and $\{y_{p'}\} = (V(y_1Py_{i'}) \cup V(y_{j'}Py_k)) \setminus (N_{P^\lambda}[y_1] \cup N^+_{P^\lambda}(y_k))$, a contradiction to the choice of P. Case (13.2). $p = s+1$. Note that $i_1 = s+1 = p$. Then x_1 is not adjacent to x_{s+1}. Since P^λ is a crossing path, by Claim 3, x_λ is adjacent to all but at most one vertex of $N_H[x_k] \cup \{x_s\}$. Suppose that x_λ is adjacent to x_s. Then x_λ is adjacent to x_{s+1}, as otherwise $(s, s+3)$ is minimal crossing pair with $\ell(s, s+3) = 2$ in P^λ, a contradiction to our choice of P. Note that $Q_\lambda = x_1Px_\lambda x_kPx_{\lambda+1}$ is a crossing path (it is possible that $x_{\lambda+1} = x_k$). By Claim 3, $x_{\lambda+1}$ is adjacent to all but at most one vertex of $N_H[x_k] \cup \{x_s\}$. Hence $x_{\lambda+1}$ is adjacent to at least one of $\{x_s, x_{s+1}\}$. Note that x_k is adjacent to x_t. We can find a cycle of length k ($x_{\lambda+1}x_sPx_1x_{t-1}Px_{s+1}x_\lambda Px_tx_kPx_{\lambda+1}$ or $x_\lambda x_sPx_1x_{t-1}Px_{s+1}x_{\lambda+1}Px_kx_tPx_\lambda$), a contradiction. Thus x_λ is not adjacent to x_s and hence $N_H[x_\lambda] = N_H[x_k]$ for $t \le \lambda \le k-1$. This completes the proof of the claim. $\square$

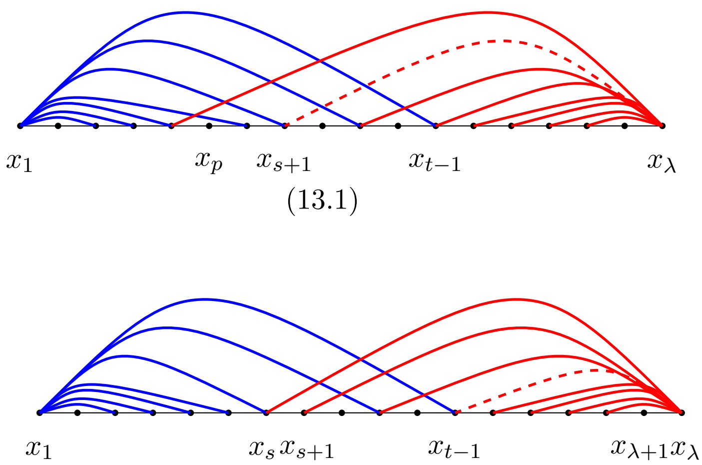


Figure 21. The structure of the crossing path P^λ when $m = k$

We consider the following three cases.

Case B.1. $2 \le p \le s-1$. By (19) we have $\{x_s, x_{s+1}\} \subseteq N_H(x_1)$, and hence by Claim 4, x_1 is not adjacent to any two consecutive vertices of $x_{s+1}Px_{t-1}$. Let $2 \le \gamma \le s$ and $\gamma \ne p, p-1$. Note that $x_\gamma \in V(H)$, x_k is adjacent to x_{s+1} and $|V(x_1Px_{s+1})| \le \ell$. $T_\gamma = x_\gamma Px_1x_{\gamma+1}Px_k$ is a crossing H-path. Let

$$A = V(x_1Px_s), B = V(x_tPx_k) \text{ and } C = \{x_{s+1}, x_{s+3}, \cdots, x_{t-3}, x_{t-1}\}.$$

By (19) we have $\{x_{t-1}, x_t\} \subseteq N_H(x_k)$, hence since x_k is adjacent to x_{s+1}, by Claim 4, x_k is not adjacent to x_{s+2}. Then by (19) again, x_1 is adjacent to x_{s+3}. Repeat the above arguments, we have x_1 is adjacent to each vertex of $(A \cup C) \setminus \{x_p\}$ and x_k is adjacent to each vertex of $B \cup C$. (B.1.1). $x_p \notin V(H)$. Consider the path $T_\gamma = x_\gamma Px_1x_{\gamma+1}Px_k$ (clearly, this path is a crossing path). By Claim 3, we have $N_H[x_\gamma] = N_H[x_1]$. Then we can check that $G[V(x_1Px_s) \setminus \{x_p\}]$ is a complete graph (note that each vertex of $A \setminus \{x_{p-1}, x_p\}$ is adjacent to x_{p-1}). By Claim 13, $G[V(x_tPx_k)]$ is a complete graph. Thus it is easy to check that G contains either a copy of $F(k,k,s)$ with Type III, a contradiction. (B.1.2). $x_p \in V(H)$. Then $p \geq 3$ and $s \geq 4$. For $x_\gamma \in A \setminus \{x_{p-1}, x_p\}$, consider the crossing H-path $T_\gamma = x_\gamma P x_1 x_{\gamma+1} P x_k$. We deduce $N_H[x_\gamma] \subseteq A \cup C$ from Claim 3. Consider the path $T_p = x_p x_1 x_{p+1} P x_k$. Similarly, we have $N_H[x_p] \subseteq A \cup C$ from Claim 3. It follows that x_1 is adjacent to each vertex of $(A \cup C) \setminus \{x_p\}$ and x_p is adjacent to each vertex of $(A \cup C) \setminus \{x_1\}$. Now, consider the path $x_{p-1} x_p x_{p-2} P x_1 x_{p+1} P x_m \in \mathcal{P}$ (clearly, this path is a crossing path). Claim 3 implies $N_H(x_{p-1}) \subseteq A \cup C$. Note that x_1 is adjacent to x_{t-1} and x_s is adjacent to x_{s+1}. Hence, it is easy to check that $G[V(P)]$ gives a copy of $F(k,k,s)$ with Type II, a contradiction.

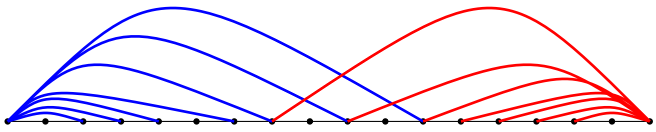


Figure 22. The structure of the crossing path P when $m = k$

Case B.2. $p = s$. Suppose that x_1 is not adjacent to any two consecutive vertices of $x_{j_1} P x_{t-1}$. Then the same proof as in the last paragraph shows that if $x_p \notin V(H)$, $G[V(P)]$ gives a copy in $\mathcal{F}(k,k,s)$ with Type III and if $x_p \in V(H)$, $G[V(P)]$ gives a copy in $\mathcal{F}(k,k,s)$ with Type II, both are contradictions. Therefore x_1 is adjacent to two consecutive vertices of $x_{j_1} P x_{t-1}$. Let λ be the minimum integer such that x_1 is adjacent to both of $\{x_\lambda, x_{\lambda+1}\} \subseteq V(x_{j_1} P x_{t-1})$. By Claim 4, we have $x_s \notin V(H)$. Let $r = \min\{h : h \geq \lambda, x_h \in N_H(x_k)\}$. By Claim 2* and $p = s < i_1$, we have $V(x_\lambda P x_r) \subseteq N_H(x_1)$. Hence, we have $x_r, x_{r-1} \in N_H(x_1)$. It follows from Claim 4 that x_1 is not adjacent to both of any two consecutive vertices of $x_{r+2} P x_{t-1}$. Note that $x_{t-1}, x_t \in N_H(x_k)$. Applying Claim 4 again, x_k is not adjacent to both of any two consecutive vertices of $x_{s+1} P x_{t-1}$.

Let

$$A = V(x_1 P x_{s-1}) \cup V(x_\lambda P x_{r-1}), \quad B = V(x_t P x_k)$$

and

$$C = \{x_{s+1}, x_{s+3}, \cdots, x_{\lambda-2}, x_r, x_{r+2}, \cdots, x_{t-3}, x_{t-1}\}.$$

Similar to previous proofs, x_1 is adjacent to each vertex of $A \cup C$ and x_k is each vertex of $B \cup C$ Consider the crossing H-path $x_\gamma P x_1 x_{\gamma+1} P x_k \in \mathcal{P}$ for $\gamma \in [2, s-2] \cup [\lambda, r-1]$ (it is easy to check that those paths are crossing paths). By $x_s \notin V(H)$ and Claim 3, we have $N_P[x_1] = N_P[x_\gamma]$. Similarly, $N_P[x_k] = N_P[x_\lambda]$ for $t+1 \leq \lambda \leq k-1$. Note that $|A| = |B| = k - t + 1$ ($d_P(x_1) = d_P(x_k) = k$ and $N_P(x_1) \cap N_P(x_k) = \{x_{s+1}, x_r, x_{r+2}, \cdots, x_{t-3}, x_{t-1}\}$). Moreover, $G[A]$ and $G[B]$ are complete graphs. It is easy to check that $G[V(P)]$ gives a copy of $F(k,k,k-t+1)$ with Type III, a contradiction.

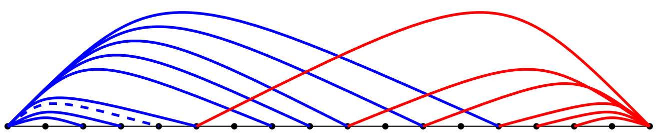


Figure 23. The structure of the crossing path P when $m = k$

Case B.3. $p = s + 1$. Note that $x_p \in N_H(x_k)$. Then we have $x_p \in V(H)$ and $p \geq 3$ by x_1 is adjacent to x_2. First, we show that x_1 is not adjacent to any two consecutive vertices of $V(x_{j_1} P x_t)$. Suppose to the contrary that x_1 is adjacent to both of $\{x_\lambda, x_{\lambda+1}\}$ for some $\lambda \geq j_1$. Consider the path $T_{s-1} = x_{s-1} P x_1 x_s P x_k$ (clearly, T_{s-1} is a crossing H-path). It follows from Claim 3 that x_{s-1} is adjacent to at least one of $\{x_\lambda, x_{\lambda+1}\}$. Hence, $P'_1 = x_s P x_\lambda x_{s-1} P x_1 x_{\lambda+1} P x_k \in \mathcal{P}$ (x_{s-1} is adjacent to x_λ) or $P'_2 = x_s P x_\lambda x_1 P x_{s-1} x_{\lambda+1} P x_k \in \mathcal{P}$ (x_{s-1} is adjacent to $x_{\lambda+1}$) is a crossing H-path on k vertices ending at x_k.

For P'_1, since x_k is not adjacent to both of $\{x_1, x_{\lambda-1}\}$, we have $\{x_{\lambda+1}, x_\lambda\} \subseteq V(P'_1) \setminus N^{+1}_{P'_1}(x_k)$. By Claim 3, x_s is adjacent at least one of $\{x_\lambda, x_{\lambda+1}\}$. Then $x_1 P x_s x_\lambda P x_{s+1} x_k P x_{\lambda+1} x_1$ (x_s is adjacent to x_λ) or $x_1 x_\lambda P x_{s+1} x_k P x_{\lambda+1} x_s P x_1$ (x_s is adjacent to $x_{\lambda+1}$) is a cycle of length k, a contradiction. For P'_2, since x_k is not adjacent to both of $\{x_1, x_{\lambda-1}\}$, we have $\{x_2, x_\lambda\} \subseteq V(P'_2) \setminus N^{+1}_{P'_2}(x_k)$. By Claim 3, x_s is adjacent at least one of $\{x_\lambda, x_2\}$. Note that x_1 is adjacent to x_s and x_{s-1} is adjacent to $x_{\lambda+1}$. Then $x_s x_\lambda P x_{s+1} x_k P x_{\lambda+1} x_{s-1} P x_1 x_s$ (x_s is adjacent to x_λ) or $x_s x_2 P x_{s-1} x_{\lambda+1} P x_k x_{s+1} P x_\lambda x_1 x_s$ (x_s is adjacent to x_2) is a cycle of length k, a contradiction. Therefore x_1 is not adjacent to any two consecutive vertices of $x_{j_1} P x_{t-1}$.

Let

$$A = \{x_1, x_2, \cdots, x_s\}, B = \{x_t, x_{t+1}, \cdots, x_k\} \text{ and } C = \{x_{s+1}, x_{s+3}, \cdots, x_{t-3}, x_{t-1}\}.$$

Similar to the previous proofs, we have $N_H[x_1] = (A \cup C) \setminus \{x_{s+1}\}$ and $N_H[x_k] = B \cup C$. Consider the H-path $x_\gamma P x_1 x_{\gamma+1} P x_k$ (it is easy to check that it is a crossing path). By Claim 3, we have $N_H[x_\gamma] \subseteq A \cup C$ for $2 \leq \gamma \leq s$. Note that $x_s \in A$ is adjacent to x_{s+1} and $x_1 \in A$ is adjacent to x_{t-1}. It is easy to check that $G[V(P)]$ gives a copy of $F(k, k, s-1)$ with Type II, a contradiction. This completes the proof of Theorem 1.2. $\square$

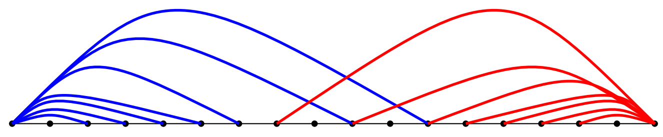


Figure 24. The structure of the crossing path P when $m = k$

4 Proof of Theorem 1.3 for odd k

We need the following theorem proved by Fan [4].

Theorem 4.1 (Fan [4]). *Let G be an n-vertex 2-connected graph and ab be an edge in G. If the longest path starting from a and ending at b in G has at most r vertices, then $e(G) \leq \frac{(r-3)(n-2)}{2} + 2n - 3$. Moreover, the equality holds if and only if $G - \{a, b\}$ is a vertex-disjoint union of copies of K_{r-2}.*

The graph $Z(n, k, \delta)$ denotes the vertex-disjoint union of a clique $K_{k-\delta}$ and some cliques $K_{\delta+1}$'s, where any two cliques share the same two vertices. It is easy to check that $\omega(Z(n, k, \delta)) = k - t + 1$ and $\delta(Z(n, k, \delta)) = t - 1$. Recall the definition of $H(n, k, t-1)$. We can also see that $\omega(H(n, k, t-1)) = k - t + 1$ and $\delta(H(n, k, t-1)) = t - 1$. The following Lemma 4.2 is proved by Yuan [14].

Lemma 4.2 (Yuan [14]). *Let G be a 2-connected n-vertex graph with $c(G) < k$ and $n \geq k \geq 5$. If $\omega(G) \geq k - t + 1$ and $\delta(G) \geq t - 1$, then $G = H(n, k, t-1)$ or $G = Z(n, k, t-1)$.*

A cycle C is *locally maximal* in a graph G if there is no cycle C' in G such that $|E(C')| > |E(C)|$ and $|E(C') \cap E(C, G - C)| \leq 2$. We will prove Lemma 4.2 by a result of Ma and Ning (see Lemma 4.4 in [10]).

Lemma 4.3. *Let G be a 2-connected non-Hamilton graph on n vertices with $\delta(G) \geq t - 1$ and C be a locally maximal cycle in G of length $c \leq k - 1$. If the clique number of $G[C]$ is at least $k - t + 1$, then $G = H(n, k, t-1)$ or $G = Z(n, k, t-1)$.*

Proof of Lemma 4.2. Let $n \geq k \geq 5$. Let G be a 2-connected n-vertex graph with $\omega(G) \geq k - t + 1$ and $\delta(G) \geq t - 1$. Suppose that $G \notin \{H(n, k, t-1), Z(n, k, t-1)\}$. We will show that $c(G) \geq k$. Let G' be an edge-maximal counter-example. That is $G' \notin \{H(n, k, t-1), Z(n, k, t-1)\}$ is a 2-connected n-vertex graph with $\omega(G') \geq k - t + 1$, $\delta(G') \geq t - 1$ and adding any edge to G' will create a cycle of length at least k. Thus we may take a maximal clique K_ℓ in G' with $\ell \geq k-t+1$ and a longest path $P = x_1x_2\dots x_m$ starting from $x_1 \in V(K_\ell)$ ending at $x_m \in V(G')\setminus V(K_\ell)$ with $m \geq k$. Thus by the choice of P, we have $d_P(x_1) \geq d_{V(K_\ell)}(x_1) \geq k-t$ and $d_P(x_m) \geq \delta(G') \geq t-1$. Since $c(G) < k$, by Lemma 3.1, there is a cycle of length $k-t+t-1 = k-1$ containing $V(K_\ell) \subseteq N_P[x_1]$. Clearly, this cycle C is local maximal and the clique number of $G[C]$ is at least $k-t+1$. Applying Lemma 4.3, $G' = H(n,k,t-1)$ or $G' = Z(n,k,t-1)$, a contradiction. The proof is complete. □

Proof of Theorem 1.4. Let $k = 2\ell+1 \geq 5$. Let G be a maximal (in the sense that if we add any edge into G, then the resulting graph contains a cycle of length at least k) n-vertex 2-connected graph with $c(G) \leq k$ and

$$e(G) \geq \max\{h(n,k,3), h(n,k,\ell-1)\}. \tag{21}$$

Let $H = H(G,\ell)$ be the $(\ell-1)$-disintegration of G. First H is non-empty, otherwise $e(G) \leq \binom{\ell-1}{2} + (n-\ell+1)(\ell-1) < \binom{\ell+2}{2} + (n-\ell-2)(\ell-1) = h(n,k,\ell-1)$, a contradiction to (21).

Claim. H is a complete graph.

Proof. Suppose not, there is a non-edge ab in H. Then by the maximality of G, $G+ab$ contains a cycle of length $m \geq k$, i.e., there is an H-path in G on at least k vertices. Take a longest H-path in G. Then by Theorem 1.2, G contains a copy of $F \in \mathcal{F}(m,k,r)$ of Type I. We refer $V(F)$ to the sets A, B, C as in Section 2.2.

Let $r \leq \ell-2$. Then $3 \leq |C| \leq \ell$. Note that for any two vertices $x, y \in V(F)$, there is an (x,y)-path on at least $k-2$ vertices in F and if $x \notin C$, then there is an (x,y)-path on at least $k-1$ vertices in F (see Figure 2.). Since G is 2-connected and $c(G) < k$, each vertex of $G - V(F)$ is an isolated vertex. Moreover, each vertex of $G-V(F)$ can only be adjacent to C of $V(F)$. Hence, if $r=1$, i.e., $|C| = \ell$, then G is a subgraph of $H(n,k,\ell)$ (Theorem 1.4 (b)). Now we may assume $3 \leq |C| \leq \ell-1$. Then we have $\ell \geq 4$ implying $2\binom{\ell+1}{2} + \binom{\ell-1}{2} < \binom{\ell+2}{2} + (\ell-1)^2$. Therefore,

$$\begin{aligned}
e(G) &\leq e(G[V(F)]) + (n-k)|C| \\
&= e(G[A\cup C]) + e(G[B\cup C]) - e(G[C]) + e(G[C, V(F)\setminus(A\cup B\cup C)] + (n-k)|C| \\
&\leq 2\binom{\ell+1}{2} - \binom{|C|}{2} + |C|(|C|-1) + (n-k)|C| \\
&\leq 2\binom{\ell+1}{2} + \binom{\ell-1}{2} + (n-k)(\ell-1) \\
&< \binom{\ell+2}{2} + (\ell-1)^2 + (n-k)(\ell-1) \\
&= h(n,k,\ell-1),
\end{aligned}$$

contradicting (21).

Now, let $r = \ell-1$. Then $|C| = 2$. Note that for any two vertices $x, y \in V(F)$ with $x \notin C$, there is an (x,y)-path on at least $k-1$ vertices in F. Since G is 2-connected and $c(G) < k$, each vertex of $G - V(F)$ only connected to C of F by a path and the longest C-path is on at most $\ell+1$ vertices. If $k \geq 9$, i.e., $\ell \geq 4$, then $\ell^2 + 3\ell/2 < \binom{\ell+2}{2} + (\ell-1)^2$ and $(\ell+2)/2 \leq \ell-1$. It follows from Theorem 4.1 that

$$\begin{aligned}
e(G) &\leq 2n-3+(n-2)(\ell-2)/2 \\
&= \ell^2 + 3\ell/2 + (n-2\ell-1)(\ell+2)/2 \\
&< \binom{\ell+2}{2} + (\ell-1)^2 + (n-k)(\ell-1) \\
&= h(n,k,\ell-1),
\end{aligned}$$

contradicting (21). If $k=7$, then the longest C-path is on at most four vertices. Hence we can easily see that after deleting C the resulting graph is a star forest (Theorem 1.4 (e)). If $k=5$, then $\ell=2$, and hence the longest C-path is on at most 3 vertices. Thus we can see that G is a subgraph of $H(n,5,2)$. The proof of the claim is complete. □

Let $|V(H)| = m$. If $m = k-2$, then since $c(G) < k$ and G is 2-connected, each vertex of $G - V(H)$ is adjacent to the same two vertices of H, and hence $G = H(n, k, 2)$ (Theorem 1.4 (a)). If $m = \ell + 1$, then $e(G) \leq \binom{\ell+1}{2} + (n - \ell - 1)(\ell - 1) < \binom{\ell+2}{2} + (n - \ell - 2)(\ell - 1) = h(n, k, \ell - 1)$, a contradiction to (21).

So we may assume $\ell + 2 \leq m \leq k - 3$, i.e., $3 \leq k - m \leq \ell - 1$. Let $H' = H(G, k - m)$ be the $(k - m + 1)$-disintegration of G. If $H' = H$ is complete, then $e(G) \leq \binom{m}{2} + (n - m)(k - m) = h(n, k, k - m) \leq \max\{h(n, k, 3), h(n, k, \ell - 1)\}$, where the last inequality holds since $h(n, k, a)$ is a convex function in a. By (21), we have $e(G) = h(n, k, 3)$ or $e(G) = h(n, k, \ell - 1)$. If $e(G) = h(n, k, 3)$, then $m = k - 3$. Moreover, (21) implies that H' is obtained by deleting vertices with degree three one by one. Therefore, $\omega(G) \geq m = k - 3$ and $\delta(G) = 3$. Applying Lemma 4.2, $G = H(n, k, 3)$ (Theorem 1.4 (c)). If $e(G) = h(n, k, \ell - 1)$, then $\omega(G) \geq m = k - \ell + 1 = \ell + 2$ and $\delta(G) = \ell - 1$. Applying Lemma 4.2, $G = H(n, k, \ell - 1)$ (Theorem 1.4 (d)).

Now, we can assume that $H' \neq H$. Then there exists a vertex $b \in (V(H') \setminus V(H))$ and a vertex $a \in V(H)$ such that a is not adjacent to b. Then by the maximality of G, there is a longest path P on at least $m \geq k$ vertices starting from H and ending at H'. Let $u \in H$ and $v \in H'$ be two end-vertices of P. By the choice of P, $d_P(u) + d_P(v) \geq d_H(u) + d_{H'}(v) \geq (m - 1) + (k - m + 1) = k$. Applying Lemma 3.1, there is a cycle of length at least $\min\{m, k\} \geq k$, a contradiction. The proof of Theorem 1.4 is complete. □

5 Conclusion

In [9], Luo determined the maximum size of cliques with given size in a 2-connected graph with $c(G) < k$. This is viewed as a clique version of the Erdős-Gallai Theorem. To conclude this paper, we would like to propose the following conjecture. This (if true) would give a clique version of Theorem 1.3 and implies a clique version of results in [10]. Let $N_s(G)$ denote the number of copies of K_s in G.

Conjecture 5.1. *Let G be a 2-connected graph on n vertices and let ab be an edge in G. Let $r \geq 4$ and $s \geq 3$ be integers, and let $n - 2 = x(r - 3) + t$ for some $0 \leq t \leq r - 4$. If $N_s(G) > x\binom{r-1}{s} + \binom{t+2}{s}$, then there is a cycle on at least r vertices containing the edge ab.*

This also can be viewed as a clique version of Theorem 4.1 of Fan [4].

Remark. Very recently, Ji and Ye [7] confirm this conjecture. Based on their result, we will get a stability result of Luo's theorem in a forthcoming paper [11] .

Acknowledgement. The authors would like to thank the anonymous referees for many helpful and constructive suggestions to an earlier version of this article. The authors also thanks Alexandr Kostochka and Ruth Luo for helpful discussions at early stage of this study and Qingyi Huo for his careful reading on a draft.

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