A clique version of the Erdős-Gallai stability theorems

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Abstract

Combining a stability result of the Pósa's rotation lemma with a technique of Kopylov in a novel approach, we prove a generalization of the Erdős-Gallai theorems on cycles and paths. This implies a clique version of the Erdős-Gallai stability theorems and also provides alternative proofs for some recent results.

1 Introduction

The well-known Erdős-Gallai theorem [2] states that any n-vertex graph G with more than $(k-1)(n-1)/2$ edges contains a cycle of length at least k. The exact value of this extremal function is obtained by Kopylov [8] and independent by Faudree and Schelp [4]. In [8], Kopylov determined the maximum numbers of edges of a (or a connected) graph which does not contain a path on k vertices, and of a (or a 2-connected) graph which does not contain cycles of length at least k. In order to state Kopylov's results, we introduce the following graphs. For integers $n \geq k \geq 2\alpha$, let $H(n,k,\alpha)$ be the n-vertex graph whose vertex set is partitioned into three sets A, B, C such that $|A| = \alpha, |B| = n-k+\alpha$, $|C| = k-2\alpha$ and whose edge set consists of all edges between A and B together with all edges in $A \cup C$ (see Figure 1, the subgraphs induced by A and C are complete graphs and the subgraph induced by B contains no edge). One may check that the longest path in $H(n,k,\alpha)$ contains k vertices and the longest cycle in $H(n,k,\alpha)$ contains $k-1$ vertices. Given a graph G, denoted by $N_s(G)$ the number of copies of K_s in G. Let $h_s(n,k,\alpha) := N_s(H(n,k,\alpha)) = \binom{k-\alpha}{s} + (n-k+\alpha)\binom{k-\alpha}{s-1}$. In particular, $h_2(n,k,\alpha) = e(H(n,k,\alpha))$.

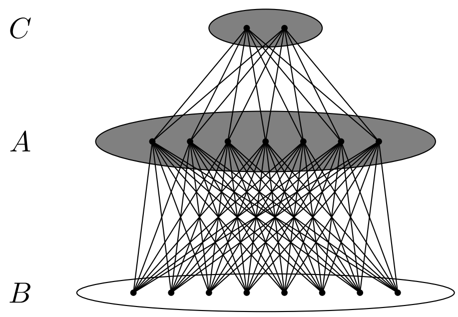

Figure 1. $H(17,16,7)$.

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Throughout the rest of the paper, let $k \geq 5$ be an integer and $\ell = \lfloor (k-1)/2 \rfloor$. Kopylov [8] showed that any n-vertex 2-connected graph G without containing cycles of length at least k has at most $\max\{h_2(n,k,2)), h_2(n,k,\ell\}$ edges. Combined with the results in [5], Füredi, Kostochka, Luo and Verstraëte [6] proved a stability version of Kopylov's theorem, which says that for any 2-connected graph G with $c(G) < k$, if $e(G)$ is close to the above maximum number from Kopylov's theorem, then G must be a subgraph of some well-specified graphs.

Theorem 1.1 (Füredi, Kostochka, Luo and Verstraëte [5,6]). *Let G be an n-vertex 2-connected graph with $c(G) < k$. Then*

$$e(G) \leq \max\{h_2(n,k,\ell-1), h_2(n,k,3)\},$$

unless

- $k = 2\ell + 1$, $k \neq 7$, *and* $G \subseteq H(n,k,\ell)$;
- $k = 2\ell + 2$ *or* $k = 7$, *and* $G - A$ *is a star forest for some* $A \subseteq V(G)$ *of size at most* ℓ;[1] *or*
- $G \subseteq H(n,k,2)$.

Extending Koplov's theorem, Luo [9] proved the following theorem.

Theorem 1.2 (Luo [9]). *Let G be an n-vertex 2-connected graph. If G does not contain cycles of length at least k, then*

$$N_s(G) \leq \max\{h_s(n,k,2)), h_s(n,k,\ell\}.$$

For an integer α and a graph G, the *α-disintegration* of G, denoted by $H(G,\alpha)$, is the graph obtained from G by recursively deleting vertices of degree at most α until that the resulting graph has no such vertex. We also call $H(G,\alpha)$ the α-core of G and denote the order of it by $s(G,\alpha)$, and moreover this core is unique for every α.[2] For a graph G, let $\omega(G)$ be the order of a maximum clique in G. Based on a stability result [11] of the well-known Pósa lemma, we will establish the following theorem.

Theorem 1.3. *Let $n \geq k \geq 5$, $\alpha \geq 0$ and $\beta \geq 2$ be integers. Let G be an n-vertex 2-connected maximal $\mathcal{K}_{k,\alpha}$-free graph with $c(G) < k$. If $\ell - \alpha \geq \beta$ and*

$$N_s(G) > \max\{h_s(n,k,\ell-\alpha), h_s(n,k,\beta)\}, \tag{1}$$

then we have either $\omega(G) > k - \beta$ or $s(G,\alpha) < k - \ell + \alpha$.

Remark. The family of graphs $\mathcal{K}_{k,\alpha}$ will be defined in Section 3. Roughly speaking, the graphs in $\mathcal{K}_{k,\alpha}$ are 2-connected with $c(G) < k$.

Theorem 1.3 will be used to prove the following stability result of Luo's theorem which also can be viewed as a clique version of Theorem 1.1. The graph $Z(n,k,\delta)$ denotes the vertex-disjoint union of a clique $K_{k-\delta}$ and some cliques $K_{\delta+1}$'s, where any two cliques share the same two vertices. In particular, if $\delta = 2$, then $Z(n,k,\delta) = H(n,k,2)$.

Theorem 1.4. *Let G be an n-vertex 2-connected graph with minimum degree $\delta \geq 2$. Let $n \geq k \geq 9$, $s \geq 3$ and $\ell - 1 \geq \delta + 1$.*[3] *If $c(G) < k$ and*

$$N_s(G) > \max\{h_s(n,k,\ell-1), h_s(n,k,\delta+1)\}, \tag{2}$$

then one of the following holds:

- $s \leq \ell + 1$ *and* $G - A$ *is a star forest for some* $A \subseteq V(G)$ *of size at most* ℓ;

[1] A *star forest* is a graph in which every component is a star.
[2] One can see that $H(G,\alpha)$ is unique in G and has minimum degree at least $\alpha + 1$ (if non-empty).
[3] If $5 \leq k \leq 8$, then $\ell \leq 3$. By (2), it follows from Luo's theorem that G contains a cycle of length at least k.

- *$s \le \ell+1$, and $k = 9, 10$ and $G - A$ consists of stars, complete bipartite graphs with one part of size two and at least $\theta(n)$ triangles for some $A \subseteq V(G)$ of size three;*

- *$s = \ell + 1$ and the copies of K_s except at most $\ell + 2$ of them can be divided into two families $\mathcal{A}$ and $\mathcal{B}$ such that each in $\mathcal{A}$ shares only $x, y \in V(G)$ and each in $\mathcal{B}$ shares only $x, z \in V(G)$.*

- *$s = \ell + 2$, k is even and G contains a unique copy of K_s.*

- *G is a subgraph of the graph $Z(n, k, \delta)$;*

- *G is a subgraph of $H(n, k, \delta)$.*

Remark. Theorem 1.3 can be applied to prove Theorem 1.1 and the main results in [10] concerning the stability results of cycles in a 2-connected graph with given minimum degree.

The organization of this paper is as follows. In Section 2, we study a family of graphs in which contains no cycles of length at least k. In Section 3, we prove our main result Theorem 1.3. In Section 4, we show how to use Theorem 1.3 to deduce Theorem 1.4 as well as some main results in [5, 6, 10].

2 Notation and a family of graphs

2.1 Notation

The general notation used in this paper is standard (see, e.g., [1]). For disjoint subsets $A, B \subseteq V(G)$, we denote $G(A, B)$ to be the induced bipartite subgraph of G with parts A, B. Let $E(A, B) = E(G(A, B))$ for short. When defining a graph, we will only specify these adjacent pairs of vertices, that says, if a pair $\{a, b\}$ is not discussed as a possible edge, then it is assumed to be a non-edge. Denote by $N_G(x)$ the set of neighbors of x in G and let $d_G(x)$ be the size of $N_G(x)$. For $U \subseteq V(G)$, let $N_U(x) = N_G(x) \cap U$ and $d_U(x) = |N_U(x)|$. Let $P = x_1x_2 \cdots x_m$ be a path in G and call P and an (x_1, x_m)-path or an x_1-path. For $x \in V(G)$, let $N_P(x) = N_G(x) \cap V(P)$ and $N_P[x] = N_P(x) \cup \{x\}$, with $d_P(x) := |N_P(x)|$. For $x_i, x_j \in V(P)$, we use x_iPx_j to denote the sub-path of P between x_i and x_j. For $S_1, S_2 \subseteq V(G)$, we call P an (S_1, S_2)-*path* if $x_1, x_m \in S$. Moreover, if $S_1 = S_2 = H$, then we call P an H-path, and if $S_1 = \{x_1\}$ and $S_2 = \{x_2\}$, then we call P an (x_1, x_2)-path for simplicity.

2.2 A family of graphs

Let $m \ge k \ge 5$ and $1 \le r \le \ell$ be integers. We now devote the rest of this subsection to the definition of a family of m-vertex graphs $\mathcal{F}(m, k, r)$ [4] introduced in [11]. We divide $\mathcal{F}(m, k, r)$ into the following four classes, namely Types I, II, III and IV. Along the way, we also define some special graphs (see Figures 2, 3, 4 and 5).

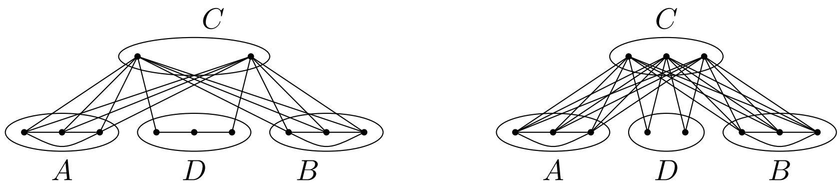
Type I: Let $k = 2\ell + 1$ be odd and $r \le \ell - 1$. Each graph $F \in \mathcal{F}(m, k, r)$ of Type I satisfies:

- F contains a Hamilton path with $|V(F)| = m$ and $c(F) < k$,
- $V(F) = A \cup B \cup C \cup D$,
- $|A| = |B| = r$,
- $F[C]$ is empty with $|C| = \ell - r + 1$,
- $F[D]$ is empty when $|C| \ge 3$ and $F[D]$ is a path on at most $\ell - 1$ vertices when $|C| = 2$,[5]

[4] For the parameter r, roughly speaking we may view it as something close to $\omega(F)$, though its own meaning will be clear in the proof of Lemma 3.1.

[5] An isolated vertex will also be viewed as a (trivial) path in this paper.

- each vertex in A has degree ℓ in $G[A \cup C]$ and each vertex in B has degree ℓ in $G[B \cup C]$, and
- $F[C \cup D]$ is a C-path.



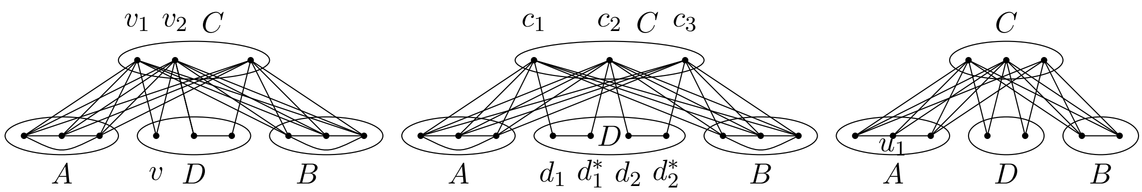
$F \in \mathcal{F}(11, 9, 3)$ with $|C| = 2$ $\qquad$ $F \in \mathcal{F}(11, 11, 3)$ with $|C| = 3$

Figure 2. Graphs of **Type I**.

Type II: Let $k = 2\ell + 2$ be even and $r \leq \ell - 1$. Each graph $F \in \mathcal{F}(m, k, r)$ of Type II satisfies:

- F contains a Hamilton path with $|V(F)| = m$ and $c(F) < k$,
- $V(F) = A \cup B \cup C \cup D$,
- $|A| \in \{r, r+1\}$ and $|B| = r$,
- $F[C]$ is empty with $|C| = \ell - r + 1$,
- $F[D]$ is a path when $|C| = 2$, and $F[D]$ consists of at most two independent edges and some isolated vertices when $|C| \geq 3$ such that one of the following holds:
  - $F[D]$ is empty when $|A| = r + 1$,
  - $F[D]$ contains a unique edge when $|A| = r$, or
  - $F[D]$ consists of two independent edges when $|A| = r = \ell - 2 \geq 2$.
- each vertex in A has degree exactly ℓ in $F[A \cup C]$[6] and each vertex in B has degree exactly ℓ in $F[B \cup C]$, and
- $F[C \cup D]$ is a C-path satisfying that if $|A| = r + 1$ then the end-vertices of $F[C \cup D]$ are adjacent different vertices of A.

In particular, we denote the graph of Type II with $|A| = r = \ell - 2$ and $|D| = 3$ by $F_1(m, k, r)$, the graph of Type II with $|A| = r = \ell - 2$ and $|D| = 4$ by $F_2(m, k, r)$, and the graphs family of Type II with $F[A]$ being a star on three vertices by $\mathcal{F}_3(m, k, 2)$.



$F_1(12, 12, 3)$ $\qquad$ $F_2(13, 12, 3)$ $\qquad$ $F \in \mathcal{F}_3(10, 10, 2)$

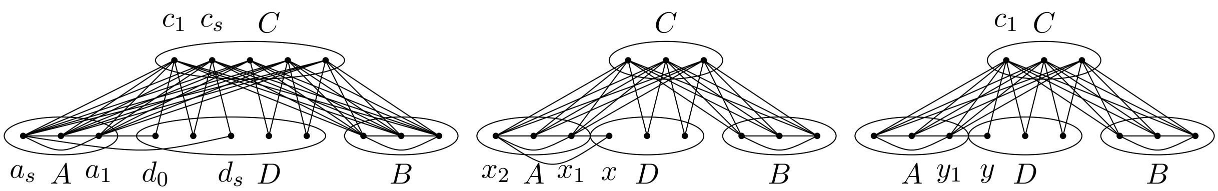
Figure 3. Graphs of **Type II**.

Type III: Let $k = 2\ell + 2$ be even and $r \leq \ell - 1$. Each graph $F \in \mathcal{F}(m, k, r)$ of Type III satisfies:

- F contains a Hamilton path with $|V(F)| = m$ and $c(F) < k$,

[6] Note that if $r = 1$ and $|A| = 2$, then $F[A] = K_2$ (by the fact that F contains a Hamilton path).

- $V(F) = A \cup B \cup C \cup D$,
- $F[A]$ and $F[B]$ are cliques on r vertices,
- $F[C]$ is empty with $|C| = \ell - r + 1$,
- $F[D]$ is empty when $|C| \geq 3$, and $F[D]$ consists of a path and an isolated vertex when $|C| = 2$,
- each vertex in $A \cup B$ is adjacent to each vertex in C, and
- $F[C \cup D]$ consists of at most two vertex-disjoint paths such that one of the following holds:
  - $F[C \cup D]$ consists of a path with distinct end-vertices $d_0, d_s \in D$ and a path with end-vertices in C satisfying that $|D| = \ell - r + 1$ and d_0, d_s is adjacent to exactly one vertex a_1, a_s in A, respectively, with $a_1 \neq a_s$ (denote this family of graphs by $\mathcal{F}_4(m, k, r)$ and denote by c_1 and c_s the neighbours of d_0 and d_s in $F[C \cup D]$ respectively),
  - $F[C \cup D]$ consists of a C-path and an isolated vertex $x \in D$ such that x is adjacent to exactly two vertices x_1, x_2 of A (denote this graph by $F_5(m, k, r)$), or
  - $F[C \cup D]$ is a path with the end-vertex $y \in D$ such that y is an isolated vertex in $F[D]$ and is adjacent to exactly one vertex y_1 in A (denote this graph by $F_6(m, k, r)$).



$F \in \mathcal{F}_4(16, 16, 3)$ $\quad$ $F_5(12, 12, 3)$ $\quad$ $F_6(12, 12, 3)$

Figure 4. Graphs of **Type III**.

Type IV: Let $k = 2\ell + 2$ be even and $r = \ell$. Each graph $F \in \mathcal{F}(m, k, r)$ of Type IV satisfies:

- F contains a Hamilton path with $|V(F)| = m$ and $c(F) < k$,
- $V(F) = A \cup B \cup C$,
- $F[A]$ and $F[B]$ are cliques on $\ell - 1$ vertices, and
- $F[C]$ induces a cycle with three distinct vertices w_1, w_2, w such that $w_1 w_2 \in E(F[C])$, $ww_i \notin E(F[C])$ for $i \in \{1, 2\}$, w_1 is adjacent to each vertex of A, w_2 is adjacent to each vertex of B, and w is adjacent to each vertex of $A \cup B$.

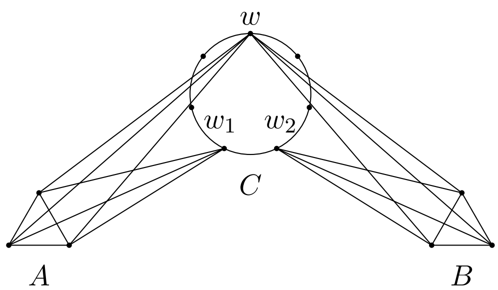


Figure 5. $F \in \mathcal{F}(13, 10, 4)$ of Type IV

We point out that by definition, there is a Hamilton path in each $F \in \mathcal{F}(m,k,r)$ starting from A and ending at B. Also, if k is odd, then all graphs in $\mathcal{F}(m,k,r)$ have Type I. Furthermore, $F_2(k+1,k,\ell-2)$ is the only graph in $\mathcal{F}(m,k,r)$ with $m > k$ and $r \leq \ell-2$, and if $r \geq \ell-1$, there are many graphs in $\mathcal{F}(m,k,r)$ with $m > k$.

Let $S = S(k,a,b)$ be the graph with a partition $V(S) = A \cup B \cup C \cup D$ on k vertices satisfying the following (see Figure 6):

- S contains a Hamilton path with $|V(F)| = k$ and $c(F) < k$,
- $S[A]$ and $S[B]$ are complete graphs with $|A| = a$ and $|B| = b$;
- $S[C]$ is empty with $|C| = c \geq 2$ and $S[D]$ is empty with $|D| = c-1$;
- $a + b + 2c - 1 = k$;
- $S(A,C)$ and $S(B,C)$ are complete bipartite graphs;
- and $S[C \cup D]$ is a C-path on $2c-1$ vertices.

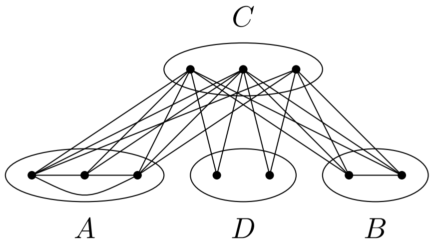


Figure 6. $S(10,3,2)$

We need the following graph $F(k)$ which satisfies all conditions of graph $F(k,k,1)$ with Type II except that it does not contain a Hamiltonian path. For even k, the graph $F(k)$ is obtained by taking a path $P_{2\ell-1}$ on $2\ell-1$ vertices and a disjoint copy of $\overline{K}_3$, and joining each vertex of $\overline{K}_3$ to each vertex of the larger partite set in the unique bipartition of $P_{2\ell-1}$. We denote two vertices of $\overline{K}_3$ by A and the other vertex of $\overline{K}_3$ by B (see Figure 7).

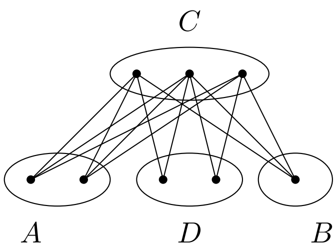


Figure 7. $F(8)$

2.3 Some facts about $\mathcal{F}(m,k,r)$, $S(k,a,b)$ and $F(k)$

We need the following technical propositions.

Proposition 2.1. *Let G be an n-vertex connected graph with a non-edge c_1c_2 and $n \geq 6$. Assume that each vertex except c_1 and c_2 of G has degree $n-2$. Then we have the following.*

(i) For each $ab \in E(G)$, there is a Hamilton (c_1,c_2)-path containing ab.

(ii) For each $v \in V(G) \setminus \{c_1, c_2\}$, there is a path on $n-1$ vertices starting from v, through $V(G) \setminus \{c_1, c_2, v\}$ and ending in $\{c_1, c_2\}$.

Proof. Let $A = V(G) \setminus \{c_1, c_2\}$. Note that each vertex in A has degree $n-2$. We can partition A into A_0, A_1 and A_2 such that each vertex in A_0 is adjacent to both of $\{c_1, c_2\}$ and each vertex of A_i is not adjacent to of c_i for $i = 1, 2$. Since each vertex of A has degree $n-2$, $G[A_1]$ and $G[A_2]$ are complete graphs and $G[A_0]$ is the complement of the graph consisting of $|A_0|/2$ independent edges (clearly, $|A_0|$ is even and if $|A_0| \geq 4$, then there is a Hamilton (u, v)-path in $G[A_0]$ for any $u, v \in A_0$). Moreover, each vertex of A_i is adjacent to each vertex of A_j for $0 \leq i, j \leq 2$. Without loss of generality, assume that $|A_1| \geq |A_2|$.

(i) If $|A_0| = 0$, then $|A_1| \geq 1$ and $|A_2| \geq 1$ by the connectivity of G. It is easy to see that there is a Hamilton path starting from c_1, through all vertices in A_2 and then all vertices in A_1, and ending at c_2. If $|A_0| = 2$, let $A_0 = \{x, y\}$. Note that $|A| \geq 4$. There is a Hamilton path starting from c_1 through $x \in A_0$, all vertices in $A_1 \cup A_2$ and then $y \in A_0$, ending at c_1 and a Hamilton path starting from c_2 through $u \in A_1$, $x \in A_0$, all vertices in $(A_1 \cup A_2) \setminus \{u\}$ and then $y \in A_0$, ending at c_1. If $|A_0| \geq 4$, then for any vertices $u, v \in A_0$, there is Hamilton (u, v)-path in $G[A_0]$. Thus there is a Hamilton path starting from c_1, through all vertices in $A_1 \cup A_2$ and then all vertices in A_0, ending at c_1. It is easy to see that each edge of G is contained in a Hamilton path, whence we finish the proof of (i).

(ii) If $v \in A_0$, then there is a path on $n-1$ vertex starting from v, through all vertices in $A_1 \cup A_2$ and then all vertices in A_0, and ending at c_1. If $v \in A_1$, then there is a path on $n-1$ vertex starting from v, through all vertices in A_1, one vertex in A_0, all vertices in A_2 and then all other vertices in A_0, and ending at c_1. Similarly, there is a path on $n-1$ vertices starting from v ending in $\{c_1, c_2\}$ when $v \in A_2$. □

Proposition 2.2. *Let G be an n-vertex 2-connected graph and a, b be two vertices in G. Then the following holds.*

- *If the longest (a, b) is on at most four vertices, then $G - \{a, b\}$ consists of stars such that each leaf of a star is adjacent to the same one vertex of $\{a, b\}$.*
- *If the longest (a, b) is on at most five vertices, then $G - \{a, b\}$ consists of stars, triangles and complete bipartite graphs with one part of size two.*

Proof. Let X be a component of $G - \{a, b\}$. Suppose that the longest (a, b)-path is on at most four vertices. Let uv be any edge of X. Since G is 2-connected, uv is connected to $\{a, b\}$ by two independent edges. Hence, $G - \{a, b\}$ consists of stars, as otherwise we can easily find an (a, b)-path on five vertices, a contradiction. Moreover, each leaf of X is adjacent to the same one vertex of $\{a, b\}$.

Now assume that the longest (a, b)-path is on at most five vertices. If the longest (a, b)-path though X is on four vertices, then X is a star. Now let $P_3 = z_1 z_2 z_3$ in X such that z_1 and z_3 are adjacent to a and b respectively. Then $X - P_3$ consists of isolated vertices, otherwise there is an (a, b)-path though X on six vertices, a contradiction. Moreover, each isolated vertex of $X - P_3$ is adjacent to $\{a, z_2\}$, $\{b, z_2\}$ or $\{z_1, z_2\}$. If some vertex of $X - P_3$ is adjacent to $\{z_1, z_2\}$, then other vertices of $X - P_3$ is also adjacent to $\{z_1, z_2\}$. Thus X is a complete bipartite graph. If some vertex of $X - P_3$ is adjacent to $\{a, z_2\}$ or $\{b, z_2\}$, then other vertices of $X - P_3$ is also adjacent to $\{a, z_2\}$ or $\{b, z_2\}$. Then X is a star. Finally, if $|X| = 3$, then it is possible that z_1 is adjacent to z_3, whence X is a star or triangle. The proof is complete. □

Recall vertices x, x_i, y, y_1 in those special graphs in $\mathcal{F}(m, k, r)$ (see Figures 3 and 4). For $F \in \mathcal{F}_3(m, k, 2)$ with $F[A] = S_3$, where S_3 is a star in three vertices, we denote by u_1 the center of S_3 (see Figure 3). Let $t = \ell - r + 1 \geq 3$ and $C = \{c_1, c_2, \ldots, c_t\}$. We denote the vertices of $F[C \cup D]$ in the following.

- If $F[C \cup D]$ is a C-path with $F[D]$ empty, then let $F[C \cup D] = c_1 d_1 \ldots c_i d_i c_{i+1} \ldots d_{t-1} c_t$; if $F[C \cup D]$ is a C-path with an unique edge $d_i d_i^*$ in $F[D]$, then let $F[C \cup D] = c_1 d_1 \ldots c_i d_i d_i^* c_{i+1} \ldots d_{t-1} c_t$; if $F = F_2(k+1, k, \ell - 2)$, then let $F[C \cup D] = c_1 d_1 d_1^* c_2 d_2 d_2^* c_3$ (see Figure 3).

- for $F \in \mathcal{F}_4(m,k,r)$, let $P^\alpha = d_0c_1d_1\dots c_sd_s$, $P^\beta = c_{s+1}d_{s+1}c_{s+2}\dots d_{t-1}c_t$ and $F[C \cup D] = P^\alpha \cup P^\beta$ (see Figure 4);
- for $F \in \mathcal{F}_5(m,k,r)$, let $F[C \cup D] = P^\alpha \cup P^\beta$ where $P^\alpha = x$ and $P^\beta = c_1d_1c_2\dots d_{t-1}c_t$ (see Figure 4);
- for $F = F_6(m,k,r)$, let $F[C \cup D] = yc_1d_1c_2\dots d_{t-1}c_t$ (see Figure 4).

In particular, if $F = F_1(k,k,\ell-2)$, then let $F[C \cup D] = v_1vv_2d_2d_2^*c_3$ (see Figure 3).

Proposition 2.3. *For $1 \le r \le \ell-2$ and $m \ge k$, each $F \in \mathcal{F}(m,k,r)$ satisfies the following:*

(i) Let $ab \in E(F)$. If $ab \in \{x_1x_2, a_1c_1, a_sc_s, y_1c_1\} \cup E(\{u_1\}, C)$, or $ab \in E(\{y_1\}, C) \cup \{v_1v, v_2v\}$ with $r \ge 2$, then there is a cycle of length $k-2$ containing ab; otherwise, there is a cycle of length $k-1$ containing ab.

(ii) For each non-edge ab in $A \cup B \cup D$, if $\{a,b\} \subseteq A$, $\{a,b\} \subseteq A \cup \{x\}$, $\{a,b\} \subseteq A \cup \{y\}$, $u_1 \in \{a,b\}$, or $y_1 \in \{a,b\}$ with $r \ge 2$, then there is an (a,b)-path on $k-1$ vertices; otherwise, there is an (a,b)-path on k vertices.

(iii) For each non-edge ab between $A \cup B \cup D$ and C, if ab is between u_1 and C, then there is an (a,b)-path on $k-2$ vertices; if ab is between P^α and P^β, then there is an (a,b)-path on k vertices; otherwise, there is an (a,b)-path on $k-1$ vertices.

(iv) For each non-edge ab in C, if $|A| = r \ge 2$, $|C| = 3$ and k is even, or $ab = c_ic_{i+1}$ is incident with $d_id_i^*$, then there is an (a,b)-path on $k-3$ vertices; if $ab = c_ic_j$ with $i \le s$, $j \ge s+1$ and $F \in \mathcal{F}_4(k,k,r)$, then there is an (a,b)-path on $k-1$ vertices; otherwise, there is an (a,b)-path on $k-2$ vertices.

(v) Suppose that G is 2-connected with $c(G) < k$ and contains a copy of F. Then $G - A \cup B \cup C$ is a star forest.

The Proposition 2.3 can be checked by direction observations. For the reason of completeness, we check Proposition 2.3 case by case.

Proof of Proposition 2.3 (i)**.** Let k be odd and $P = F[C \cup D]$. Note that, for each $1 \le i \le t$, there is a cycle of length $k-1$ starting $\alpha \in A$, through all other vertices of A (in arbitrary order), the sub-path c_1Pc_i of P, all vertices of B (in arbitrary order) and the sub-path c_tPc_{i+1} of P sequentially (each vertex of $A \cup B$ is adjacent to each vertex of C), and ending at α. Clearly, this cycle can contain any edge inside A and B, any edge between $A \cup B$ and C, and any edge in P (choose different i). Thus for each edge ab of F, there is a cycle of length $k-1$ containing ab.

Let k be even. We first consider some specific edges in some $F \in \mathcal{F}(m,k,r)$.

Let $ab = x_1x_2$. Then $F[C \cup D] = \{x\} \cup P^\beta$. Then there is an (x_1,x_2)-path on $k-2$ vertex starting x_1, through c_1Pc_{t-1}, all vertices of B, c_t and all vertices of $A \setminus \{x, x_1, x_2\}$ sequentially, and ending at x_2 (without containing x and one vertex in D). Moreover, there is no cycle of length $k-1$ containing x_1x_2, otherwise there is a cycle of length k (use x_1xx_2 instead of x_1x_2), a contradiction.

Let $ab = a_1c_1$. Then $F[C \cup D] = P^\alpha \cup P^\beta$. Then there is a path on $k-2$ vertex starting from a_1, through $c_1P^1c_s$, all vertices of B, P^β, all vertices of $A \setminus \{a_1\}$ sequentially, and ending at a_1 (without containing d_0 and d_s). Similarly, there is a cycle of length $k-2$ containing a_sc_s. Clearly, there is no cycle of length $k-1$ containing a_1c_1 or a_sc_s, as otherwise we can find a cycle of length at least k.

Let $ab = y_1c_1$. Then $F[C \cup D] = yc_1d_1c_2\dots d_{t-1}c_t$. There is a cycle on $k-2$ vertex starting from c_1, through all vertices of B, c_2Pc_t, $A \setminus \{y_1\}$ and y_1 sequentially, and ending at c_1 (without containing y and another vertex of D). Clearly, there is no cycle of length $k-1$ containing y_1c_1.

Let $ab \in E(\{u_1\}, C)$. Then $F[A] = \alpha u_1\beta$ and $F[C \cup D] = c_1d_1c_2\dots d_{t-1}c_t$. Note that α and β are adjacent to each vertex of C. If u_1 is adjacent to c_i, then there is a cycle of length $k-2$ starting from u_1, through c_iPc_t, all vertices of B, $c_{i-1}Pc_1$ and α sequentially, and ending at u_1 for $i \ge 2$ (starting from u_1, through c_1Pc_{t-1}, all vertices of B, c_t and α sequentially, and ending at u_1 for $i = 1$). Again, we can see that there is no cycle of length $k-1$ in F containing ab.

Let $ab = v_1v$ with $r \geq 2$. Then $F[C \cup D] = v_1vv_2d_2d_2^*c_3$. There is a cycle on $k-2$ vertex starting from $v_1 = a$, through all vertices of B, c_3, all vertices of A, v_2 and $v = b$ sequentially, and ending at v_1 (without containing d_2, d_2^*). Note that this cycle contains v_2v. We find the desired cycle. Note that each component of $F[D]$ except the isolated vertex d_1 contains at least two vertices ($r \geq 2$). Since any cycle containing v_1v or v_2v can at most three components of $F[D]$, there is no cycle of length $k-1$ containing v_1v or v_2v.

Let $ab \in E(\{y_1\}, C)$ with $r \geq 2$. Let $ab = y_1c_i$ for $i \geq 2$ (we have already proved the case $ab = y_1c_1$) and $P = F[C \cup D]$. Then there is a cycle of length $k-2$ starting from y_1, through c_iPc_t, all vertices of B, $c_{i-1}Pc_1$ and all vertices in $A \setminus \{y_1\}$ sequentially, and ending at y_1 for $i \geq 2$.

We now divide the proof into the following two cases.

Case (1). $|A| = r + 1$.

Let $|A| = r+1 \geq 4$. By the definition of $F \in \mathcal{F}(m,k,r)$ with Type II, $F[C \cup D]$ is a path such that the end-vertices of it are adjacent to different vertices of A. Let $2 \leq i \leq t$. Note that each vertex of A has degree at least $r+1$ in $F[A \cup \{c_1, c_i\}]$. Hence, if there are two independent edges, say e_1 and e_2, between A and $\{c_1, c_i\}$, then we can delete edges between A and $\{c_1, c_i\}$ except e_1 and e_2 such that the resulting subgraph of $F[A \cup \{c_1, c_i\}]$ is connected and each vertex in A has degree exactly $r+1$ in $F[A \cup \{c_1, c_i\}]$. Thus, by Proposition 2.1(i), there is a path P^* starting from e_1 through all vertices of A (this path may contain any edge in $F[A]$) and ending at e_2. Hence, there is a cycle C_i of length $k-1$ starting from $c_1P^*c_i$, the sub-path c_iPc_t, all vertices of B, the sub-path $c_{i-1}Pc_1$ sequentially. Therefore, we have the following fact.

Fact. There is a cycle of length $k-1$ containing any two independent edges between $\{c_1, c_i\}$ and A.

Moreover, since c_1 and c_t are adjacent to different vertices of A, there is a cycle of length $k-1$ containing any edge inside $A \cup B$, between A and $\{c_1, c_t\}$, or between B and $\{c_1, c_2, c_{t-1}, c_t\}$ (by symmetry of c_1 and c_t). Let $C^* = C \setminus \{c_1, c_2, c_{t-1}, c_t\}$. Thus it is sufficient to prove that, for each edge between A and $C \setminus \{c_1, c_t\}$, between B and C^*, and inside P, there is a cycle of length $k-1$ contain it. We divide the proof into the following two subcases.

Subcase (1.1). c_i is adjacent to A for each i with $2 \leq i \leq t-1$.

Fix i, suppose that $\alpha \in A$ is adjacent to c_1 and $\beta \in A$ is adjacent to c_i. In the following, we will find cycles of length $k-1$ containing $c_1\alpha$ and cycles of length $k-1$ containing $c_i\beta$. If $\alpha \neq \beta$, then αc_1 and βc_i are two independent edges, and hence by the fact there is a cycle of length $k-1$ containing both of them. Moreover, each edge between c_{i-1} and B, and each edge of c_1Pc_{i-1} and c_iPc_t are contained in this cycle. Let $\alpha = \beta$. We consider the following two cases.

(a). There is a vertex, say γ, of $A \setminus \{\alpha\}$ which is adjacent to c_1. Then αc_i and γc_1 are two independent edges. Hence there is a cycle of length $k-1$ containing αc_i. If there is a vertex $\eta \in A \setminus \{\alpha\}$ which is adjacent to c_i, then similarly, there is a cycle of length $k-1$ containing αc_1. Moreover, these two cycles can contain any edge of c_1Pc_{i-1}, c_iPc_t and between B and c_{i-1} . Now we may assume that there is no vertex in $A \setminus \{\alpha\}$ which is adjacent to c_i. Then each vertex of $A \setminus \{\alpha\}$ is adjacent to each vertex of $C \setminus \{c_i\}$. Thus there are two independent edges αc_1 and γc_t, implying that there is a path P^* starting from αc_1, through all vertices of A, and ending at γc_t. Therefore, there is a cycle of length $k-1$ starting from $c_1P^*c_t$, the sub-path c_tPc_i, all vertices of B, the sub-path $c_{i-1}Pc_1$ sequentially. In summary, for each i with $2 \leq i \leq t$ and each edge between A and c_i, between B and c_{i-1} and inside c_1Pc_{i-1} and c_iPc_t, there is a cycle of length $k-1$ containing it.

(b). There is no vertex in $A \setminus \{\alpha\}$ which is adjacent to c_1. Then each vertex of $A \setminus \{\alpha\}$ is adjacent to each vertex of $C \setminus \{c_1\}$. Thus there is a vertex $\gamma \in A \setminus \{\alpha\}$ such that γc_i and αc_1 are two independent edges. Therefore, by Proposition 2.1(i), there is a cycle of length $k-1$ through $c_1\alpha$, all vertex of $A \setminus \{\alpha\}$, c_iPc_t, all vertices of B, $c_{i-1}Pc_1$. For the edge αc_i, there is a cycle starting from α, through c_iPc_1, all vertices of B, c_tPc_{i+1} and all vertices of $A \setminus \{\alpha\}$, ending at α. Combining with the above cycles, for each i with $2 \le i \le t-1$ and each edge between A and c_i, between B and c_{i-1} and inside c_1Pc_{i-1} and c_iPc_t, there is a cycle of length $k-1$ containing it.

In conclusion, each edge between $\{c_1, c_i\}$ and A, between c_{i-1} and B, and in c_1Pc_{i-1} and c_iPc_t is contained in a cycle of length $k-1$. Therefore, for each edge between $A \cup B$ and C and inside P, there is a cycle of length $k-1$ containing it by choosing different $2 \le i \le t-1$.

Subcase (1.2). c_i is not adjacent to A for some i with $2 \le i \le t-1$.

Then each vertex of A is adjacent to each vertex of $C \setminus \{c_i\}$. Thus by Proposition 2.1(i), there is a path P^* starting from c_1 through all vertices of A and ending at c_j with $2 \le j \le t$ and $j \ne i$. Then we can consider cycle of length $k-1$ starting from $c_1P^*c_j$, the sub-path c_jPc_t, all vertices of B, the sub-path $c_{j-1}Pc_1$ sequentially. Therefore, there exist cycles of length $k-1$ containing any edge between A and C, between B and $C \setminus \{c_{i-1}\}$, inside P (choose different j). For the edges between c_{i-1} and B, by the symmetry of c_{i-1} and c_{i+1} in F, there is a cycle of length $k-1$ containing those edges. Therefore, for each edge between A and C, between B and C, and inside P, there is a cycle of length $k-1$ containing it. We complete the proof for $|A| = r+1 \ge 4$.

Let $|A| = r+1 = 3$. (a). $F[A]$ is a path $\alpha u_1 \beta$. Then α and β are adjacent to each vertex of C. Hence, there is a cycle of length $k-1$ starting from $c_1\alpha u_1\beta c_i$, through the sub-path c_iPc_t, all vertices of B, the sub-path $c_{i-1}Pc_1$ sequentially. Therefore, each edge in F except the edges between u_1 and C is contained in a cycle of length $k-1$. (b). $F[A]$ is a complete graph on three vertices, say α, β and γ. If $e \in F[A]$, say $e = \alpha\beta$, then there are two independent edges between C and $\{\alpha, \gamma\}$. If $e \in F[A, C]$, say $e = \alpha c$, then it is easy to check that there are two independent edges (including e) between C and A. In both of the above cases, similar to the case $|A| = r+1 \ge 4$, we can find a cycle of length $k-1$ containing e. For other edge in F, it is not hard to see that there is a cycle of length $k-1$ containing it.

Let $|A| = r+1 = 2$. Then by the definition of $\mathcal{F}(m,k,r)$, we have $F[A] = K_2$ (recall that each graph in $\mathcal{F}(m,k,r)$ contains a Hamilton path). Similar to the case $F[A] = K_3$, for each edge e in F, there is a cycle of length $k-1$ containing it.

Case (2). $|A| = r$.

Then $F[C \cup D]$ consists of at most two paths. We divide the proof into the following cases.

Subcase (2.1). $F[C \cup D]$ is a path P.

Let $F \in \mathcal{F}(m,k,r)$ with Type II. Then $F[C \cup D] = c_1d_1 \dots c_id_id_i^*c_{i+1} \dots d_{t-1}c_t$. If $|C| = t \ge 4$ or $r = 1$, there is a cycle of length $k-1$ containing each vertex of C and t components of $F[A \cup B \cup D]$. Hence, for each edge in F, we can easily find a cycle of length $k-1$ containing it. If $|C| = 3$ and $r \ge 2$, there is a cycle of length $k-1$ starting from A, through c_1, B and $c_3d_2^*d_2c_2$, and ending in A. Hence, there is a cycle of length $k-1$ containing ab except $ab = v_1v, v_2v$. For $|C| = 3$ and $r = 1$, we have $k = 8$, and hence it is easy to see that each edge of F is contained in a cycle of length $k-1 = 7$.

Let $F \in \mathcal{F}(k,k,r)$ with Type III. Then $F = F_6(k,k,r)$ and $F^* = F - \{y\}$. If $r = 1$, then applying the odd case for $F^* = F - \{y\}$ with $k^* = k-1$, we can easily check that for each edge $e \ne y_1c_1$ in F^*, there is a cycle of length $k' - 1 = k-2$ containing both of e and y_1c_1. Thus each edge of F is contained in a cycle of length $k-1$ (replacing y_1c_1 with y_1yc_1). If $r \ge 2$, then can easily check that for each edge which is not between C and y_1, there is a cycle containing both of e and y_1c_1. As before, we can find the desired cycle.

Subcase (2.2). $F[C \cup D]$ consists of two paths.

Let $F \in \mathcal{F}_4(k,k,r)$. Then $P^\alpha = d_0c_1d_1 \dots c_sd_s$, $P^\beta = c_{s+1}d_{s+1}c_{s+2} \dots d_{t-1}c_t$ and $F[C \cup D] = P^\alpha \cup P^\beta$. First we will show that, for each edge between $A \setminus \{a_s\}$ and $C \setminus \{c_1\}$ there is a cycle of length $k-1$ containing it. For $2 \le i \le s$, there is a cycle C^α of length $k-1$ starting from $a \in A \setminus \{a_s\}$, through $c_iP^\alpha c_sd_s$, a_s, all vertices of $A \setminus \{a, a_1, a_s\}$, the path P^β, all vertices of B, $c_{i-1}P^\alpha c_1d_0$ sequentially, and ending at a_1a (without containing d_{i-1}). For $s+1 \le i \le t$, there is a cycle C^β of length $k-1$ starting from $a \in A \setminus \{a_s\}$, through $c_iP^\beta c_t$ for $i > s+1$ (c_i for $i = s+1$), all vertices of B, the path

$c_{i-1}P^{\beta}c_{s+1}$ ($c_{i+1}P^{\beta}c_t$ for $i = s+1$), all the vertices of $A \setminus \{a, a_1, a_s\}$, a_s and $d_sP^{\alpha}d_0$ sequentially, and ending at a_1a. Hence, we obtained the desired cycle and by symmetry between a_1 and a_s each edge between a_s and $C \setminus \{c_s\}$ is contained in a cycle of length $k-1$. Moreover, each edge in P^{α}, P^{β}, A and B is contained in a cycle of length $k-1$ by choosing different i. In the cycles C^{α} and C^{β}, we can see that each edge between B and $C \cap V(P^{\alpha})$ and between B and $C \cap V(P^{\beta})$ is contained in a cycle of length $k-1$. Therefore, we finish the proof for $F \in \mathcal{F}_4(k,k,r)$.

Let $F = F_5(k,k,r)$. Let $F^* = F - \{x\}$. Then the proof for odd $k' = k-1$ shows that for each edge $e \neq x_1x_2$, there is a cycle of length $k-2$ in F^* containing e and x_1x_2. Thus each edge $e \neq x_1x_2$ of F is contained in a cycle of length $k-1$ (replacing x_1x_2 with x_1xx_2). The proof of Proposition 2.3(i) is complete. $\square$

Proof of Proposition 2.3(ii). First let k be odd and $P = F[C \cup D] = c_1d_1c_2d_2 \dots c_{t-1}d_{t-1}c_t$. If the non-edge ab is between A and B, then there is an (a,b)-path on k vertices starting from A, through the path P, and ending in B. If the non-edge ab is between A and D, say $b = d_i \in D$, then there is an (a,b)-path on k vertices starting from A, through the path c_1Pc_i, all vertices of B and the path c_tPc_{i+1}, and ending at $b = d_i$. Similarly, there is an (a,b)-path on k vertices when $a \in B$ and $b \in D$. Let ab be a non-edge in D. Without loss of generality, let $a = d_i$ and $b = d_j$ with $1 \le i \le j \le t-1$. The there is an (a,b)-path on k vertices starting from $a = d_i$, through the path c_1Pc_i, all vertices of A, the path c_jPc_{i+1}, all vertices of B and the path c_tPc_{j+1}, and ending at $b = d_j$. We finish the proof of Proposition 2.3(ii) for odd k.

Now let k be even. We will finish our proof in the following two cases.

Case 1. $|A| = r+1$.

If $\{a,b\} \subseteq A$, then a,b are adjacent to each vertex of C. Moreover, we have $|A| \ge 4$ or $F[A] = \alpha u_1 \beta$. In both cases, we can add ab and delete two independent edges between a,b and C. The resulting graph F' is still in $\mathcal{F}(k,k,r)$. Hence, by Proposition 2.3(i), there is a cycle of length $k-1$ containing ab in F', and hence there is an (a,b)-path on $k-1$ vertices in F. Suppose that $a \in A$, $a \neq u_1$ and $b \in (B \cup D)$. Then by Proposition 2.1(ii) and some observations ($|A| = 2$ and $|A| = 3$), there is a path P^* starting from a, through all vertices of $A \setminus \{a\}$ and ending in $\{c_1, c_t\}$. Without loss of generality, suppose that P^* starts from a and ends at c_t. If $b \in B$, then there is an (a,b)-path on k vertices start from P^*, through P and ending in B. If $b = d_i \in D$, then c_i and c_{i+1} are two neighbours of b in $F[C \cup D]$. Thus there is an (a,b)-path on k vertices start from P^*, through c_tPc_{i+1} and B, ending in c_1Pc_ib. Let $a = u_1$ and $F[A] = \alpha u_1 \beta$. If $b \in B$, then there is an (a,b)-path on $k-1$ vertices starting $a = u_1$, through α and P, and ending in B. If $b = d_i \in D$, then there is an (a,b)-path on $k-1$ vertices starting $a = u_1$, through α, c_1Pc_i and B, and ending in $c_tPc_{i+1}d_i$. Similarly, for $a \in B$ and $b \in D$, there is an (a,b)-path on k vertices (note that there is a Hamilton (c_1,c_t)-path in $F[A \cup \{c_1,c_t\}]$. Now let $a = d_i$ and $b = d_j$ with $1 \le i < j \le t-1$. Since there is a Hamilton (c_1,c_t)-path in $F[A \cup \{c_1,c_t\}]$, there is an (a,b)-path on k vertices start from d_iPc_1, through A, c_tPc_{j+1} and B, and ending in $c_{i+1}Pd_j$.

Case 2. $|A| = r$.

Then $F[C \cup D]$ consists of at most two paths. As the proof of Proposition 2.3(i), we divide the proof into the following cases.

Subcase (2.1). $F[C \cup D]$ is a path P.

Let $F = F_6(k,k,r)$ and $F' = F - \{y\}$. Applying the proof of Proposition 2.3(ii) for odd k, there is an (u,v)-path in F' on $k'-1 = k-2$ (k' is odd) vertices when $u = y_1$, $v \in A$ and an (u,v)-path in F' on $k' = k-1$ when $u = y_1$ and $v \in B \cup D$. Thus if $\{a,b\} \subset A \cup \{y\}$ or $y_1 \in \{a,b\}$ with $r \ge 2$, then there is an (a,b)-path on $k-1$ vertices. If $y_1 \in \{a,b\}$ and $r = 1$, then it is easy to see that there is an (a,b)-path on k vertices. The rest cases are simpler and we omit the proofs.

Subcase (2.2). $F[C \cup D]$ consists of two vertex-disjoint paths.

Clearly, we have $F \in \mathcal{F}_4(k,k,r) \cup \{F_5(k,k,r)\}$. Let $F = F_5(k,k,r)$. If $\{a,b\} \subseteq A \cup \{x\}$, then we can add ab and delete x_1x. Since the resulting graph is still $F = F_5(k,k,r)$, the result follows by applying Proposition 2.3(i). For other no-edge $ab \subseteq (A \cup B \cup D)$, the proof is the same as the case $F = F_6(k,k,r)$. Now we can assume that $F \in \mathcal{F}_4(k,k,r)$. If $a \in A$ and $b \in B$, then there is an (a,b)-path on k vertices through aa_1, $d_0P^{\alpha}d_s$, $A \setminus \{a,a_1\}$, P^{β} and B. Let $a \in A$ and $b = d_i \in D$. If $0 \leq i \leq s$, then there is an (a,b)-path starting $A \setminus \{a_s\}$ (or $A \setminus \{a_1\}$), through $d_0P^{\alpha}c_i$ (or $d_sP^{\alpha}c_{i+1}$), B, P^{β} and $a_sd_sP^{\alpha}d_i$ (or $a_1d_0P^{\alpha}d_i$) on k vertices; if $s+1 \leq i \leq t$, then there is an (a,b)-path through $d_iP^{\beta}c_t$, B, $c_{i-1}P^{\beta}c_{s+1}$, a_1 (or a_s), P^{α} and $A \setminus \{a_1\}$ (or $A \setminus \{a_s\}$) on k vertices. Let $a \in B$ and $b = d_i \in D$. If $0 \leq i \leq s$, then there is an (a,b)-path through $d_iP^{\alpha}d_0$, a_1, $c_{i+1}P^{\alpha}d_s$, $A \setminus \{a_1\}$, P^{β} and B on k vertices; if $s+1 \leq i \leq t$, then there is an (a,b)-path through $d_iP^{\beta}c_{s+1}$, a_1, P^{α}, $A \setminus \{a_1\}$, $c_{i+1}P^{\beta}c_t$ and B on k vertices. Now, let $a = d_i \in D$ and $b = d_j \in D$ with $1 \leq i < j \leq t$. If $j \leq s$, then there is an (a,b)-path through $d_iP^{\alpha}d_0$, a_1, $c_{i+1}P^{\alpha}c_j$, B, P^{β}, $A \setminus \{a_1\}$ and $d_sP^{\alpha}d_j$ on k vertices; if $i \geq s+1$, then there is an (a,b)-path through $d_iP^{\beta}c_{s+1}$, B, $c_iP^{\beta}c_j$, a_1, P^{α}, $A \setminus \{a_1\}$ and $c_tP^{\beta}d_j$ on k vertices; if $i \leq s$ and $j \geq s+1$, then there is an (a,b)-path through $d_iP^{\alpha}d_0$, a_1, $c_jP^{\beta}c_{s+1}$, $A \setminus \{a_1\}$, $d_sP^{\alpha}c_i$, B and $c_tP^{\beta}d_j$ on k vertices. □

Proof of Proposition 2.3(iii)**.** Let k be odd. Then we may assume that $a = c_i \in C$ and $b = d_j \in D$ with $j \neq i-1, i$. Without loss of generality, let $i < j$. Hence there is an (a,b)-path through c_iPc_1, A, $c_{i+1}Pc_j$, B and c_tPd_j on $k-1$ vertices (without containing d_i).

Now, we may assume that k is even. Let ab be a non-edge between A and C. Without loss of generality, let $a \in A$ and $b = c_i \in C$. Then we have $|A| = r+1$. Let $a \neq u_1$. If $2 \leq i \leq t-1$, then a is adjacent to both of $\{c_1, c_t\}$. Hence, we can add ab and delete ac_1 or ac_t such that the resulting graph F' is in $\mathcal{F}(k,k,r)$. If $i = 1$ or $i = t$, then we can add edge ab and delete ac_j with $2 \leq j \leq t-1$ such that the resulting graph F' is in $\mathcal{F}(k,k,r)$. In both of the above cases, the result follows by applying Proposition 2.3(i) to F' (a cycle of length $k-1$ containing ab, then there is an (a,b)-path in F on $k-1$ vertices). If ab is a non-edge between u_1 and C. We add this edge ab and delete an edge between u_1 and C. By Proposition 2.3(i), there is a cycle of length $k-2$ containing ab in the resulting graph, and hence there is an (a,b)-path on $k-2$ vertices in F.

Since each vertex of B is adjacent to each vertex of C, from now, we may assume that ab is a non-edge between C and D with $a \in C$ and $b \in D$. Since c_1 and c_t are adjacent to different vertices of A, by Proposition 2.1(i), for each $F \in \mathcal{F}(m,k,r)$, there is path P^* starting from c_1, through all vertices of A, and ending at c_t. Hence, if $F \notin \mathcal{F}_4(k,k,r)$, we can easily find an (a,b)-path on $k-1$ vertices as k is odd. Let $F \in F_4(k,k,r)$. Note that $F[A \cup V(P^{\alpha})]$ and $F[B \cup V(P^{\beta})]$ are Hamilton graphs (graphs contains a spanning cycle). If $a \in V(P^{\beta})$ and $b \in V(P^{\beta})$, then there is a Hamilton (a,a^*)-path in $F[A \cup V(P^{\alpha})]$ and a Hamilton (b,b^*)-path in $F[B \cup V(P^{\beta})]$ such that a^* is adjacent to b^*, an hence there is an (a,b)-path on k vertices. If $a,b \in V(P^{\alpha})$ or $a,b \in V(P^{\beta})$, then we can easily find an (a,b)-path on $k-1$ vertices. □

Proof of Proposition 2.3(iv)**.** Let $c_i, c_j \in C$ with $1 \leq i < j \leq t$. Note that there is a Hamilton (c_1,c_t)-path in $F[A \cup \{c_1, c_t, w\}]$, where $w = x$ or $w = y$. If $F \notin \mathcal{F}_4(k,k,r)$, then the result follows easily from the fact that each C-path contains at most $|C|-1$ components of $F[A \cup B \cup D]$. Now, we consider the case $F \in \mathcal{F}_4(k,k,r)$. For $c_i, c_j \in V(P^{\alpha})$ or $c_i, c_j \in V(P^{\beta})$, we only consider the case $c_i, c_j \in V(P^{\alpha})$ with $i \leq j$. Other cases can be proved similarly. The desired (c_i,c_j)-path on $k-2$ vertices starts from c_iPd_0, through a_1, P^{β}, B, $c_{i+1}P^{\alpha}c_{j-1}$, $A \setminus \{a_1\}$, ends in $d_sP^{\alpha}c_j$. If $c_i \in V(P^{\alpha})$ and $c_j \in V(P^{\beta})$, then by Proposition 2.3(iii), there is a (c_i,d_j)-path on k vertices containing d_jc_j and hence there is a (c_i,c_j)-path on $k-1$ vertices. □

Proof of Proposition 2.3(v)**.** Let X be a non-trivial component of $G - A \cup B \cup C$.[7] By Proposition 2.3(i), (ii), (iii) and (iv), for any $a,b \in V(F)$ there is an (a,b)-path on at least $k-3$ vertices. Since G is 2-connected with $c(G) < k$, the longest path starting from $a \in V(F)$, though X and ending at $b \in V(F)$ is on at most four vertices. By Proposition 2.2, X is a star forest. Thus $G[X]$ is a star and we finish the proof of Proposition 2.3. □

[7] We say a component is *trivial* if it consists of a unique vertex.

We give the following propositions without proofs, since their proofs are very similar to that of Proposition 2.3.

Proposition 2.4. *Let u, v be two vertices of $S(k, a, b)$ with $a + b \le k - 2$. If u and v are not both in C, then there is a (u, v)-path on at least $k - 1$ vertices, otherwise there is a (u, v)-path on $k - 2$ vertices.*

Proposition 2.5. *Let u, v be two vertices of $F(k)$. If u and v are in $A \cup B \cup D$, then there is an (u, v)-path on $k - 1$ vertices. If u and v are between $A \cup B \cup D$ and C, then there is a (u, v)-path on $k - 2$ vertices. If u and v are in C, then there is an (u, v)-path on $k - 3$ vertices.*

The following proposition shows that if a 2-connected graph G contains a copy of $F \in F(m, k, r)$ or $F(k)$, then we have $\omega(G) < k - \ell + 1$ or $c(G) < k$.

Proposition 2.6. *Let G be a 2-connected graph $\omega(G) \ge k - \ell + 1$. If G contains a copy of $F \in F(m, k, r)$ or $F(k)$, then G contains a cycle of length at least k.*

Proof. Let G contains a copy of $F \in F(m, k, r)$ with $r \le \ell - 2$ and a copy of K_t with $t \ge k - \ell + 1$. Let k be even. Then $\omega(G) \ge k - \ell + 1 \ge \ell + 3$. By Proposition 2.3($v$), for each pair of vertices (a, b), there is an (a, b)-path of length at least $k - 3$ in F. If $|V(K_t) \setminus V(F)| \ge 3$, then there is a cycle of length k by G is 2-connected. If $|V(K_t) \setminus V(F)| = 2$, then $|V(K_t) \cap V(F)| \ge t - 2 \ge k - \ell - 1 \ge 3$. Applying Proposition 2.3 there is an H-path on $k - 2$ vertices in F, where $H = V(K_t) \cap V(F)$ (there is at most one pair (a, b) in $V(F)$ such that the longest (a, b)-path is on $k - 3$ vertices). Hence, there is a cycle of length k. Let $|V(K_t) \setminus V(F)| = 1$. Then $|V(K_t) \cap (A \cup B \cup D)| \ge t - 1 - |C| \ge 2$. From Proposition 2.3, for any $a, b \in A \cup B \cup D$, there is an (a, b)-path of length at least $k - 1$ in F, whence there is a cycle of length k.

Now we may assume that $V(K_t) \subseteq V(F)$. If $|A| = r + 1$, then $\omega(F[A \cup B \cup D]) \le r + 1$. Since $|V(K_t) \setminus C| \ge k - \ell + 1 - |C| \ge r + 2$, $E(K_t)$ contains a non-edge ab incident with $B \cup D$. Hence by Proposition 2.3(ii), there is a path on k vertices containing ab. If $|A| = r$, then $\omega(F[A \cup B \cup D] \setminus \{w\}) \le r$, where $w \in \{x, y, u_1, y_1\}$. Since $|V(K_t) \setminus (C \cup \{w\})| \ge k - \ell + 1 - |C| \ge r + 2$, $E(K_t)$ contains a non-edge ab incident with $B \cup D \setminus \{w\}$. Hence by Proposition 2.3(ii), there is a path on k vertices containing ab. In both of the above cases, there is a cycle of length k in F. For the rest cases, the proposition holds similarly. The proof is complete. $\square$

Next we bound the number of copies of K_s in 2-connected graphs containing a copy of $F \in \mathcal{F}(m, k, r)$. Let

$$f_s(n, k, \alpha) = \binom{k-\ell}{s} + \binom{\ell+1}{s} - \binom{\alpha}{s} + (n - k + \alpha - 1)\binom{\alpha}{s-1}.$$

Lemma 2.7. *Let G be an n-vertex 2-connected graph with $c(G) < k$ and $\alpha \ge 3$. If G contains a copy of $S(k, k - \ell - \alpha, \ell + 1 - \alpha)$[8], then*

$$N_s(G) \le f_s(n, k, \alpha).$$

Moreover, $f_s(n, k, \alpha) \le h_s(n, k, t)$ for any $\alpha \le t \le \ell - 1$.

Proof. Assume that G is an n-vertex 2-connected graph with $c(G) < k$ and contains a copy of $S(k, k - \ell - \alpha, \ell + 1 - \alpha)$. Since $\alpha \ge 3$, by Proposition 2.4, $X = G - V(F)$ is an independent set and each vertex of X is only adjacent to C of $V(F)$, otherwise G contains a cycle of length k, a contradiction. Since the numbers of unlabeled s-cliques inside $A \cup B \cup C$ and unlabeled s-cliques incident with D are at most $\binom{k-\ell}{s} + \binom{\ell+1}{s} - \binom{\alpha}{s}$ and at most $(n - k + \ell - r)\binom{\alpha}{s-1}$ respectively, we have $N_s(G) \le f_s(n, k, \alpha)$.

Let $t \ge \alpha$. Note that

$$\binom{x}{s} - \binom{x-a}{s} = \sum_{i=1}^{s} \binom{x-a}{s-i}\binom{a}{i}$$

[8] $S(k, k - \ell - \alpha, \ell + 1 - \alpha)$ contains a copy of $F \in \mathcal{F}(m, k, r)$ of Type II with $r = \ell + 1 - \alpha$.

for integers x and a with $x \geq a$. If $t = \ell$, then we have

$$\binom{t+1}{s} - \binom{\ell+1}{s} = \binom{k-\ell}{s} - \binom{k-t}{s}.$$

If $t > \ell$, then $\ell + 1 \geq k - t$ (recall $\ell = \lfloor (k-1)/2 \rfloor$), and hence we have

$$\binom{t+1}{s} - \binom{\ell+1}{s} = \sum_{i=1}^{s} \binom{\ell+1}{s-i}\binom{t-\ell}{i} \geq \sum_{i=1}^{s} \binom{k-t}{s-i}\binom{t-\ell}{i} = \binom{k-\ell}{s} - \binom{k-t}{s}.$$

If $t < \ell$, then $t + 1 \leq k - \ell$, and hence we have

$$\binom{\ell+1}{s} - \binom{t+1}{s} = \sum_{i=1}^{s} \binom{t+1}{s-i}\binom{\ell-t}{i} \leq \sum_{i=1}^{s} \binom{k-\ell}{s-i}\binom{\ell-t}{i} = \binom{k-t}{s} - \binom{k-\ell}{s}.$$

Combining the above arguments, we have

$$\binom{t+1}{s} - \binom{\ell+1}{s} \geq \binom{k-\ell}{s} - \binom{k-t}{s}. \tag{3}$$

Therefore, by $t \geq \alpha$, we obtain

$$\begin{aligned} f_s(n,k,\alpha) &= \binom{k-\ell}{s} + \binom{\ell+1}{s} - \binom{\alpha}{s} + (n-k+\alpha-1)\binom{\alpha}{s-1} \\ &\leq \binom{k-t}{s} + \binom{t+1}{s} - \binom{t}{s} + (n-k+t-1)\binom{t}{s-1} = h_s(n,k,t), \end{aligned}$$

where the second inequality follows by (3) and the fact that $(n-k+t-1)\binom{t}{s-1} - \binom{t}{s}$ increases with t when $s \geq 2$. The proof is complete. □

Let E_{n-k+1} be the $(n-k+1)$-vertex graph consisting of $\lfloor (n-k+1)/2 \rfloor$ independent edges. Denote by $G(n,k,3)$ the graph obtained from a disjoint union of $F_2(k+1,k,\ell-2)$ and E_{n-k+1} by joining each vertex of C to each vertex in $D \cup V(E_{n-k+1})$, where C and D are the vertex sets of $F_2(k+1,k,\ell-2))$ in its definition. Denote by $g_s(n,k,3)$ the number of unlabeled s-cliques of $G(n,k,3)$. Recall that $h_s(n,k,r)$ is the number of unlabeled s-cliques of $H(n,k,r)$. Also recall that $F_2(k+1,k,\ell-2)$ is the only graph in $\mathcal{F}(m,k,r)$ with $m > k$ and $r \leq \ell - 2$. We need the following lemma to prove our main theorem.

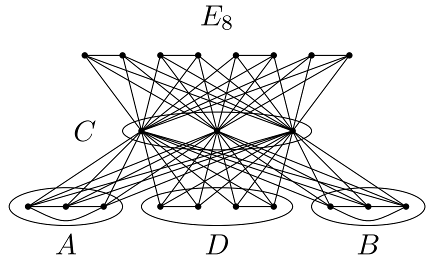


Figure 8. $G(21, 12, 3)$

Lemma 2.8. *Let G be a 2-connected graph on n vertices with $c(G) < k$. Let $m \geq k \geq 9$, $1 \leq r \leq \ell - 2$ and $s \geq 2$. Suppose that G contains a copy of $F \in \mathcal{F}(m,k,r)$. Then the following holds.*

- *If $F \in \{F_1(k,k,\ell-2), F_2(k,k,\ell-2)\}$, then*

$$N_s(G) \leq \min\{g_s(n,k,3), h_s(n,k,4), \ldots, h_s(n,k,\ell)\}.$$

- *If $F \in \mathcal{F}_3(k,k,r)$ with $r=2$ or $F = F_6(k,k,r)$ with $r \geq 2$, then*

$$N_s(G) \leq \min\{h_s(n,k,\ell-r+2), \ldots, h_s(n,k,\ell)\}.$$

- *Otherwise, we have*

$$N_s(G) \leq \min\{h_s(n,k,\ell-r+1), \ldots, h_s(n,k,\ell)\}.$$

Proof. Let $F \in \mathcal{F}(m,k,r)$ with $m \geq k \geq 9$ and $1 \leq r \leq \ell - 2$. Let G be a maximal n-vertex 2-connected graph with $c(G) < k$ containing a copy of F and $X = G - V(F)$. By Proposition 2.3(v), X is a star forest.

First, we consider the case: $|A| = r+1$ or k is odd, i.e., $F[C \cup D]$ is a C-path and C, D are empty sets. By Proposition 2.3, for any two vertices a, b not both in C there is an (a,b)-path on at least $k-1$ vertices, except $a, b \in (\{u_1\} \cup C)$. Moreover, for any two vertices a, b in C or $\{u_1\} \cup C$, there is an (a,b)-path on at least $k-2$ vertices. Since G is 2-connected with $c(G) < k$, it is easy to check that X is an independent set. Moreover, each $x \in X$ is only adjacent to C or $\{u_1\} \cup C$. If $F \notin \mathcal{F}_3(k,k,2)$, then G contains a copy of $S(k, r+1, r, \ell - r + 1)$ by the maximality of G. It follows from Lemma 2.7 that $N_s(G) \leq h_s(n,k,t)$ for any $t \geq \ell - r + 1$. Therefore, we deduce $N_s(G) \leq \min\{h_s(n,k,\ell-r+1), \ldots, h_s(n,k,\ell)\}$. Now let $F \in \mathcal{F}_3(k,k,2)$. Then each vertex of X can only be adjacent to $\{u_1\} \cup C$. If each vertex of X can only be adjacent to C, then Lemma 2.7 implies $N_s(G) \leq \min\{h_s(n,k,\ell-1), \ldots, h_s(n,k,\ell)\}$. If some vertex of X is adjacent to $\{u_1\}$, then $N_s(G) \leq f_s(n,k,\ell) \leq h_s(n,k,\ell)$. Combining the above two cases, we have $N_s(G) \leq h_s(n,k,\ell)$.

Next, we consider the case k is even and $|A| = r$. Let F be of Type III. We consider the following three cases. Case (a.1). $F \in \mathcal{F}_4(k,k,r)$. Let $F[C \cup D] = P^\alpha \cup P^\beta$. By Proposition 2.3, X is an independent set and only adjacent to C. Moreover, each isolated vertex of X can not be adjacent to both of $C \cap V(P^\alpha)$ and $C \cap V(P^\beta)$. Since $a_1 d_0$ and $a_s d_s$ is not contained in any copy of K_s with $s \geq 3$ and they are the only two edges between A and D (by Proposition 2.3), we can easily check that $N_s(G) \leq f_s(n,k,\ell-r+1) \leq h_s(n,k,t)$ for any $s \geq 2$ and $t \geq \ell - r + 1$. Case (a.2). $F = F_5(k,k,r)$. Since G is 2-connected with $c(G) < k$, by Proposition 2.3, it is easy to check that X is an independent set and each $x \in X$ is only adjacent to C or to $\{x_1, x_2\}$. If $x^* \in X$ is adjacent $\{x_1, x_2\}$, then it contributes less copies of K_s. The result follows similarly as before. Case (a.3). $F = F_6(k,k,r)$. Since G is 2-connected with $c(G) < k$, by Proposition 2.3, it is easy to check that X is an independent set and each $x \in X$ is only adjacent to C or to $\{y_1\} \cup C$. The result holds as the case $F \in \mathcal{F}_3(k,k,2)$, that is $N_s(G) \leq \min\{h_s(n,k,\ell-r+2), \ldots, h_s(n,k,\ell)\}$.

Now, we may assume that F is of Type II. Let $|A| = r \leq \ell - 3$. Then we have $|C| = t \geq 4$ and $F[C \cup D]$ is a path such that $F[D]$ contains one edge. Then By Proposition 2.3, $X = G - V(F)$ is a star forest and each vertex of X is only adjacent to C of $V(F)$. Moreover, each edge is only adjacent to $\{c_i, c_{i+1}\}$ (recall that $c_i d_i d_i^* c_{i+1}$ is a sub-path of $F[C \cup D]$) and each leaf of a star is only adjacent to the same one of $\{c_i, c_{i+1}\}$. Note that $|C| \geq 4$. For each edge e of $G[X]$, the number of copies of K_s incident with e is at most $r(s)$, where $r(s) = \binom{4}{s}$ for $s \geq 3$ and $r(s) = 5$ for $s = 2$. For each star S_{t+1} of $G[X]$, the number of copies of K_s incident with it is at most $r'_s(t)$, where $r'_s(t) = 0$ for $s \geq 4$, $r'_s(t) = t+1$ for $s = 3$ and at most $r'_s(t) = 2t+2$ for $s = 2$. Recall that each isolated vertex in $X \cup D$ is incident with $\binom{\alpha}{s-1}$ copies of K_s. Let p be the number of independent edges and $S_{t_1+1}, \ldots, S_{t_q+1}$ be the stars on at least three vertices in $X \cup D$. Thus

$$N_s(G) \leq f_s(n,k,\ell-r+1) - \left(n - 2p - \sum_{i=1}^{q}(t_i+1)\right)\binom{\alpha}{s-1} + pr(s) + \sum_{i=1}^{q} r'_s(t_i) < f_s(n,k,\ell-r+1).$$

Thus the proof of Lemma 2.7 implies $N_s(G) \leq h_s(n,k,t)$ for any $t \geq \ell - r + 1$.

From the above analysis, we now may assume that $|A| = r = \ell - 2$. Then we have $|C| = t = 3$. Since G is 2-connected with $c(G) < k$, by Proposition 2.3, each vertex of $G[D \cup X]$ is not adjacent to $A \cup B$. Moreover, for each star S_α with $\alpha \geq 3$, the center of the star is adjacent to at least two vertices of C and the leaves of S_α are adjacent to the same vertex $x \in C$. Furthermore, for other vertices (isolated vertices and independent edges), each of them is adjacent to all vertices of C. Let p be the number of independent edge, p' be number of isolated vertex and $S_{t_1+1}, \ldots, S_{t_q+1}$ be the stars on at least three vertices in $X \cup D$. Let $q(s) = \binom{3}{s-1}$. Therefore, the number of copies of K_s containing vertices in $X \cup D$ is at most $pr(s) + p'q(s) + \sum_{i=1}^{q} r'_s(t_i) \le \lfloor (n-k-3)/2 \rfloor r(s) + jq(s)$, where $j = 1$ when $n-k+3$ is odd, and $j = 0$ when $n-k+3$ is even. Hence, we have $N_s(G) \le g_s(n,k,3)$. Since $n \ge k$, basic calculations show that $\lfloor (n-k+3)/2 \rfloor \left(\binom{5}{s} - \binom{3}{s}\right) + i\binom{4}{s} \le (n-k+4)\binom{4}{s-1} - \binom{4}{s}$. Let $t \ge 4$. Combining the above arguments, we have

$$\begin{aligned} N_s(G) \le g_s(n,k,3) &= 2\binom{\ell+1}{s} - \binom{3}{s} + \left\lfloor \frac{n-k+3}{2} \right\rfloor \left(\binom{5}{s} - \binom{3}{s}\right) + i\binom{4}{s} \\ &\le \binom{k-t}{s} + \binom{t}{s} + (n-k+4)\binom{4}{s-1} - \binom{4}{s} \\ &\le \binom{k-t}{s} + \binom{t}{s} + (n-k+t)\binom{t}{s-1} - \binom{t}{s} = h_s(n,k,t), \end{aligned}$$

where the third inequality holds from the fact that $(n-k+t)\binom{t}{s-1} - \binom{t}{s}$ increases with t. Thus we finish the proof of this lemma. □

3 Proof of the main result

Pósa proved that if there is an (a,b)-path P on m vertices in a 2-connected graph G, then G contains a cycle of length $\min\{m, d_P(a) + d_P(b)\}$. We need the following main result in [11] which can be viewed as a stability result of Pósa Lemma.

Theorem 3.1 (Ma and Yuan [11])**.** *Let G be a 2-connected graph with $c(G) < k$ and H be the $(\ell-1)$-disintegration of G. If the longest H-path in G has at least k vertices, then G contains a subgraph $F \in \mathcal{F}(m,k,r)$ for some $m \ge k$ and $r \le \ell$.*

Let $\mathcal{H}(k,r)$ be the set of graphs $\bigcup_{m \ge k} \mathcal{F}(m,k,r)$. Let $k \ge 5$ and $\mathcal{K}_{k,0} = \emptyset$. For $1 \le \alpha \le \ell - 2$, let $\mathcal{K}_{k,\alpha}$ be the family of graphs consisting of the following graphs:[9]

- $\bigcup_{r=1}^{\alpha} \mathcal{H}(k,r)$, $\mathcal{H}(k,\ell-1)$ and $\mathcal{H}(k,\ell)$,
- $F_1(k,k,\ell-2)$ and $F_2(k+1,k,\ell-2)$ when $\alpha = \ell - 3$,
- $\mathcal{F}_3(k,k,\alpha+1)$ and $F_6(k,k,\alpha+1)$ when $\alpha \le \ell - 3$, and
- $F(k)$.

The following theorem is the main result of this paper, from which one can derive Theorem 1.4. and some other results (such as the results of [5, 6, 10]), to be discussed in Section 4. Mainly, it says that by forbidding some family $\mathcal{K}_{k,\alpha}$, one can have a good understanding of structural properties of graphs with given circumference and relatively many s-cliques. Now we are ready for the proof of Theorem 1.3.

We note that if α or β is larger, then $\max\{h_s(n,k,\ell-\alpha), h_s(n,k,\beta)\}$ is smaller and presumably the structure of G becomes more complicated. Also we have $\omega(G) \le k-2$ (as otherwise G contains a cycle of length at least k).

For a given family of graphs $\mathcal{F}$, we say a graph G is a *maximal* $\mathcal{F}$-free graph with $c(G) < k$ if, for any non-edge ab of G, $G + ab$ contains either a copy of $F \in \mathcal{F}$ or a cycle of length at least k.

Proof of Theorem 1.3. Let $k \ge 9$, $\alpha \ge 0$ and $\beta \ge 2$, $\ell = \lfloor (k-1)/2 \rfloor$ and $\ell - \alpha \ge \beta$. Let G be an n-vertex 2-connected maximal $\mathcal{K}_{k,\alpha}$-free graph with $c(G) < k$ satisfying (1), that is

$$N_s(G) > \max\{h_s(n,k,\ell-1), h_s(n,k,\delta+1)\}.$$

[9] If k is odd, then $\mathcal{K}_{k,\alpha}$ only contains graphs in $\mathcal{F}(m,k,r)$ with $r \in \{1,\ldots,\alpha,\ell-1\}$ and $m \ge k$.

Thus, if $xy \notin E(G)$, then either $G + xy$ contains a copy of $K \in \mathcal{K}_{k,\alpha}$, or a cycle of length at least k. Recall the definition of $(\ell - 1)$-disintegration of G. Now suppose that $\omega(G) \leq k - \beta$ and $s(G, \alpha) \geq k - \ell + \alpha$. We will finish our proof by contradictions. Let $H = H(G, \ell - 1)$.

Claim. H is a complete graph.

Proof. Suppose not, there is a non-edge ab in H. We prove the claim in the following cases.

Case 1. $G + ab$ contains a cycle of length at least k.

Then, by $a, b \in V(H)$, there is an H-path on at least k vertices. Thus, there exists a longest H-path P on $m \geq k$ vertices. By Lemma 3.1, G contains a copy of $F \in \mathcal{F}(m, k, r)$. If $\alpha = \ell - 2$, then we can see that $F \in \mathcal{K}_{k,\alpha}$, contradicting that G is $\mathcal{K}_{k,\alpha}$-free. Hence, we may suppose $\alpha < \ell - 2$.

Let k be odd. Then G contains a copy of $F \in \mathcal{F}(m, k, r)$ with $\alpha + 1 \leq r \leq \ell - 2$ and hence by Lemma 2.8, we have $N_s(G) \leq \min\{h_s(n, k, \ell - r + 1), \ldots, h_s(n, k, \ell)\} \leq h_s(n, k, \ell - \alpha)$, a contradiction to (1).

Now let k be even and $r \leq \ell - 3$. If $\alpha = 1$, then since G is $\mathcal{K}_{k,\alpha}$-free, G contains a copy of $F \in \mathcal{F}(m, k, r) \setminus (\mathcal{F}_3(k, k, 2) \cup \{F_6(k, k, 2)\})$ with $2 \leq r \leq \ell - 3$. Thus Lemma 2.8 implies $N_s(G) \leq \min\{h_s(n, k, \ell - 1), h_s(n, k, \ell)\} \leq h_s(n, k, \ell - 1)$, a contradiction. If $2 \leq \alpha \leq \ell - 3$, then G contains a copy of $F \in \mathcal{F}(m, k, r) \setminus \{F_6(k, k, \alpha + 1)\}$ with $\alpha + 1 \leq r \leq \ell - 3$. Applying Lemma 2.8, we have $N_s(G) \leq \min\{h_s(n, k, \ell - r + 1), \ldots, h_s(n, k, \ell)\} \leq h_s(n, k, \ell - \alpha)$, a contradiction.

We may assume that $r = \ell - 2$ and $k \geq 10$ is even. Note that $\ell - \alpha \geq 3$. If $\ell - \alpha = 3$, i.e., $\alpha = \ell - 3$, then we have $F \in \mathcal{F}(m, k, \ell - 2) \setminus (\mathcal{F}_3(k, k, 2) \cup \{F_1(k, k, \ell - 2), F_2(k + 1, k, \ell - 2), F_6(k, k, \ell - 2)\})$ (G is $\mathcal{K}_{k,\alpha}$-free). By Lemma 2.8 we have

$$N_s(G) \leq \min\{h_s(n, k, 3), h_s(n, k, 4), \ldots, h_s(n, k, \ell)\} \leq h_s(n, k, 3),$$

a contradiction. Let $\ell - \alpha \geq 4$. Then $F \in \mathcal{F}(m, k, \ell - 2) \setminus (\mathcal{F}_3(k, k, 2) \cup \{F_6(k, k, \ell - 2)\})$. It follows from Lemma 2.8 that

$$N_s(G) \leq \min\{\max\{g_s(n, k, 3), h_s(n, k, 3)\}, h_s(n, k, 4), \ldots, h_s(n, k, \ell)\} \leq h_s(n, k, \ell - \alpha),$$

which is also a contradiction to (1). This completes the proof of Case 1.

If $c(G+ab) \geq k$ or there is an H-path on at least k vertices, then by Case 1, we get a contradiction. Thus, in the following cases, it suffices to show that either $c(G + ab) \geq k$ or there is an H-path on at least k vertices.

Now, suppose that $G+ab$ contains a copy of $F \in \mathcal{K}_{k,\alpha}$. We divide the following proof into following cases basing on the value of r in $\mathcal{F}(m, k, r)$.

Case 2. $G + ab$ contains a copy of $F \in \bigcup_{r=1}^{\alpha} \mathcal{H}(k, r)$.

Let $A \cup B \cup C \cup D$ be a partition of $V(F)$ in Section 2. If $ab \in \{x_1x_2, a_1c_1, a_sc_s, y_1c_1\} \cup E(\{u_1\}, C)$, or $ab \in E(\{y_1\}, C) \cup \{v_1v, v_2v\}$ with $r \geq 2$, then there is a cycle of length $k - 2$ containing ab by Proposition 2.3(*iii*). Thus for each $w_a \in N_H(a) \setminus V(F)$ and each $w_b \in N_H(b) \setminus V(F)$, we have

$$w_a = w_b = w. \tag{4}$$

Otherwise, there is an H-path starting from w_a ending at w_b on k vertices and we are done.

We divide our proof into the following cases.

(2.1) Let $ab = x_1x_2$. First, a and b are not adjacent to any vertex of $(B \cup D) \setminus \{x\}$. Otherwise, by Proposition 2.3(*ii*), we can deduce that $c(G + ab) \geq k$, and hence we are done. Thus by (4) and $|A \cup C| \leq \ell + 1$, there is a unique vertex $w \in N_H[x_1] \setminus V(F)$. Therefore, we have $N_H[x_1] = (A \cup C \cup \{w\}) \setminus \{x_2\}$ implying $C \subseteq H$. Note that each vertex of B has degree ℓ in $G[B \cup C]$. Thus we have $B \subseteq V(H)$, whence there is an H-path starting from w ending in B.

(2.2) Let $ab \in \{a_1c_2, a_sc_s\}$. Without loss of generality, let $ab = a_1c_1$. If $N_H(a_1) \subseteq V(F)$, then $C \subseteq H$ implying $A \cup B \cup C \subseteq H$. Hence, there is an H-path on k vertices and we are done. If there is a vertex $w \in N_H(a_1) \setminus V(F)$, then $N_H(w) \subseteq V(F)$, as otherwise, by Proposition 2.3 there is an H-path on k vertices starting from $w^* \in N_H(w) \subseteq V(F)$ ending at b. Now we have $N_H(w) \cap (V(F) \setminus \{c_1\} \neq \emptyset$, whence by Proposition 2.3 there is an H-path on k vertices starting from w and we are done. The case $ab = a_s c_s$ is similar and omitted.

(2.3) Let $ab = y_1 c_1$, $ab \in E(\{u_1\}, C)$ or $ab \in E(\{y_1\}, C)$ with $r \geq 2$. Now, let $ab \in E(\{u_1\}, C)$. Without loss of generality, let $a = u_1$, $b = c_j$ and u_1 is not adjacent to $c_i \in F$ in F. First we show that there is a vertex w in $N_H(u_1) \setminus V(F)$. Suppose that $N_H(u_1) \subseteq V(F)$. If u_1 is adjacent to c_i in G, then G contains a copy of $F' \in \mathcal{F}_3(k, k, 2)$, a contradiction. Since $|C| = \ell - 1$, we have $|N_H(u_1) \cap (A \cup B \cup D)| \geq \ell - (\ell - 3) \geq 3$. If u_1 is adjacent to $d_{i'} \in H$ and $d_{j'} \in H$, then we can find an H-path $d_{i'} u_1 \alpha c_1 P c_{i'} \beta c_{i'+1} P c_{j'} b_1 b_2 c_t P d_{j'}$ on k vertices. For other cases, we can also find an H-path on k vertices. Therefore, we prove that there exists a vertex $w \in N_H(u_1) \setminus V(F)$. If there is a vertex $w^* \in N_H(w) \setminus V(F)$, then there is an H-path starting from w^* ending at $b = c_j$. Hence we can assume that $N_H(w) \subseteq V(F)$. Then w is adjacent to $V(F) \setminus C$, and hence by Proposition 2.3, there is an H-path on k vertices starting from w ending at $N_H(w) \setminus C$. The rest proofs are essentially the same and be omitted.

(2.4) Let $ab \in \{vv_1, vv_2\}$ with $r \geq 2$. Then we have $|C| = 3$. Without loss of generality, let $ab = vv_1$. Then there is a vertex $w \in N_H(v) \setminus V(F)$. Otherwise, since $|C| \leq \ell$ and v is not adjacent to v_1, we have $N_H(v) \cap (A \cup B \cup D) \neq \emptyset$. It follows from Proposition 2.3(ii) that $c(G + ab) \geq k$ and we are done. Now, it follows from Proposition 2.3(iii) that $C \subseteq N_H(w)$ or there is a vertex $w' \in N_H(w) \setminus V(F)$ (otherwise, there is an H-path on k vertices). Thus, in the former case, we have $A \cup B \cup C \subseteq V(H)$, and hence, there is an H-path on k vertices starting from w, through $v_2 d_2 d_2^* c_3$, B, v_1 and ending at A. In the later case, there is a path on k vertices starting from w', through w, v, v_2, B, v_3 and A, and ending at v_1. We are done in both cases.

Finally, we consider the rest cases. By Proposition 2.3(i), for each edge ab of F, there is a path on $k - 1$ vertices starting from a and ending at b in F except the cases we have already discussed. Hence, we may suppose that

$$N_H(a) \subseteq V(F) \text{ and } N_H(b) \subseteq V(F). \tag{5}$$

Otherwise, there is an H-path on at least k vertices, and we are done. Note that there is no edge in $F[C]$. We can assume that, without loss of generality, $a \in (A \cup B \cup D)$. If $a \in B \cup D$, then by Proposition 2.3(ii), (5) and $|C| \leq \ell - 1$, there is a cycle of length k in $G + ab$ and we are done. Thus we have $a \in A$, and applying Proposition 2.3(ii) again, $N_H(a) \subseteq A \cup C$. Since a is not adjacent to b, $N_H(a) \geq \ell$, $N_H(b) \geq \ell$ and $|A \cup C| \leq \ell + 2$, each vertex in A has degree at least ℓ in $H[A \cup C]$. Thus G contains a copy of $F \in \mathcal{F}(m, k, r)$ with $|A| = r + 1$ or a copy of $F(k)$ (when $|A| = 2$ and $e(H[A]) = 0$). Both are contradictions.

Case 3. $G + ab$ contains a copy of $F \in \mathcal{H}(k, r)$ for $r \in \{\ell - 1, \ell\}$.

Assume that $G + ab$ contains a copy of $F \in \mathcal{H}(k, \ell - 1)$. Let $X = A \cup B \cup C$. Note that c_1 and c_2 are adjacent to different vertices of A (by the definition of $F(m, k, r)$). Since each vertex in $G[X] + ab$ has degree at least ℓ, together with $a, b \in H$, we have $X \subseteq H$. If $ab \in E(F[B])$ or $ab \in E(F(B, C))$, then we can easily find an H-path on k vertices starting from A ending in B. If $ab \in E(F[C \cup D])$, then there exists an H-path on k vertices starting from a (or b), c_1, A, c_2, and ending in B, and we are done. If $ab \in E(F[A])$ with $|A| = \ell - 1$, similarly, we can find an H-path on k vertices. Let $ab \in E(F[A])$ with $|A| = \ell$. If $N_H(a) \subseteq (A \cup C)$ and $N_H(b) \subseteq (A \cup C)$, then clearly, G contains a copy of $F \in \mathcal{H}(k, \ell - 1)$ (note that $\ell \geq 4$), we are done. Let $w \in N_H(a) \setminus (A \cup C)$. Note that there is a cycle of length $k - 1$ containing ab in $F[A \cup B \cup C] + ab$. It is easy to see that there is an H-path starting from $w \in H$ ending at b.

Now, suppose that $G + ab$ contains a copy of $F \in \mathcal{H}(k, \ell)$. Let $X = A \cup B \cup \{w, w_1, w_2\}$. Since the degree of each vertex of $X \setminus \{a, b\}$ in $G[X]$ is at least ℓ, together with $a, b \in H$, we have $X \subseteq H$. Hence, if $ab \notin E(C)$ or $ab = w_1 w_2$, then we can easily find an H-path on at least k vertices. If $ab \in E(C)$ and $ab \neq w_1 w_2$, then there is a path on at least k vertices starting from a (or b), through w, A, $w_1 w_2$ and ending at B (note that $ww_1, ww_2 \notin E(C)$).

Case 4. $G+ab$ contains a copy of $F\in\mathcal{F}_3(k,k,\alpha+1)$ ($\alpha=1$) or $F_6(k,k,\alpha+1)$ when $\alpha\le\ell-3$.

Let $F\in\mathcal{F}_3(k,k,\alpha+1)$ with $F[A]=\alpha u_1\beta$ and $B=\{b_1,b_2\}$. If ab is not between u_1 and C, then Proposition 2.3(i) there is an (a,b)-path on $k-1$ vertices. Hence we have $N_H(a)\cup N_H(b)\subseteq V(F)$. Since a and b are not both in C (C is an independent set in F), there is one edge in $G[D]$ or $G[A\cup B,D]$. Thus by Proposition 2.3(ii), we have $c(G+ab)\ge k$. Now, let $ab\in E(\{u_1\},C)$. The rest proofs are essentially the same as case (2.3). We finish the proof for $F\in\mathcal{F}_3(k,k,\alpha+1)$. The proof of the case $F=F_6(k,k,\alpha+1)$ is similar and omitted.

Case 5. $G+ab$ contains a copy of $F_1(k,k,\ell-2)$ or $F_2(k+1,k,\ell-2)$ when $\alpha=\ell-3$.

The proof of this case is essentially the same as the proof of (2.4) and be omitted.

Case 6. $G+ab$ contains a copy of $F(k)$.

If there is an edge in $G[A\cup B]$, then $G+ab$ contains a copy of $F'\in\mathcal{F}(k,k,1)$ with Type II and $|A'|=1$ [10], and we are done by Case 2. Thus, we may assume that $G[A\cup B]$ contains no edge. Note that the edges between $A\cup B$ and C are equivalence. We consider the following two subcases. (6.1) Let $a\in C$ and $b\in D$. If b is adjacent to $A\cup B$, then $G+ab$ contains a copy of $F'\in\mathcal{F}(k,k,1)$ with Type II and $|A'|=1$, and hence we are done. Since $|C|\le\ell$ and a is not adjacent to b, there is a vertex $w\in N_H(b)\setminus V(F)$. If there is a $w'\in N_H(w)\setminus(A\cup B\cup C)$, then there is an H-path on k vertices starting from w' ending at a. Thus, we have $N_H(w)\subseteq V(F)$. Then, it is not hard to see that $c(G+ab)\ge k$, and we are done (since $C\le\ell-1$, w is adjacent to $A\cup B\cup D$). (6.2). Let $a\in C$ and $b\in A\cup B$. As subcase (6.1) b is not adjacent to D, and hence there is a $w\in N_H(b)\setminus V(F)$. If there is a $w'\in N_H(w)\setminus(A\cup B\cup C)$, then there is an H-path on k vertices starting from w' ending at a, otherwise there is a cycle of length k in $G+ab$ as in subcase (6.1). We complete our proof of the claim. $\square$

Let $|V(H)|=m$. Since $s(G,\ell-1)\ge k-\ell+\alpha$ and $\omega(G)\le k-\beta$, we have $k-\ell+\alpha\le m\le k-\beta$. Apply to the graph G the process of $(k-m)$-disintegration. Let $H'=H(G,k-m)$. If $H'=H$, then

$$N_s(G)\le\binom{m}{s}+(n-m)\binom{k-m-1}{s-1}\le\max\{h_s(n,k,\ell-\alpha),h_s(n,k,\beta)\},$$

a contradiction to (1) ($h_s(n,k,a)$ is a convex function in a). If $H'\ne H$, then there exists a vertex $b\in V(H')$ which is not adjacent to a vertex $a\in V(H)$. We divide the proof into the following two cases: Case (a). Adding ab, the obtained graph contains a cycle of length at least k. Then there is a path in G on at least k vertices starting in H and ending in H'. Let $P=xPy$ be a longest such path with $x\in V(H)$ and $y\in V(H')$. Then we have $d_P(a)\ge m-1$ and $d_P(b)\ge k-m+1$. It follows from Pósa's lemma that $c(G)\ge k$, a contradiction. Case (b). Adding ab, the obtained graph contains a copy of $F\in\mathcal{K}_{k,\alpha}$. From Case ($a$), it is sufficient to show that $c(G+ab)\ge k$. Note that H is a complete graph on $m\ge k-\ell+\alpha$ vertices. By Proposition 2.6, we can easily find a cycle of length k in $G+ab$. Thus we complete the proof of Theorem 1.3. $\square$

4 Implications

Theorem 4.1 (Fan [3], Wang and Lv [13], Ji and Ye [7]). *Let G be a 2-connected n-vertex graph with $n\ge 3$. Assume $n-2=r(\ell-1)+t$ where $1\le t\le\ell-1$. If G has an edge uv such that G has no cycle containing uv of length at least $\ell+1\ge 4$, then*

$$N_s(G)\le g_s(n,k)=\begin{cases} r\binom{\ell+1}{s}+\binom{t+2}{s}, & \text{if } s\ge 3;\\ r\binom{\ell-1}{2}+\binom{t}{2}+2(n-2)+1, & \text{if } s=2.\end{cases}$$

We need the following lemma proved in [10].

[10] F' has a vertex partition $A'\cup B'\cup C'\cup D'$.

Lemma 4.2 (Ma and Ning [10])**.** *Let G be a 2-connected n-vertex graph with $c(G) < k$, $\delta(G) = \delta$ and $n \geq k$. If $\omega(G) \geq k - \delta$, then $G = H(n, k, \delta)$ or $G = Z(n, k, \delta)$.*

In the following of this section, we shall use Theorem 1.3 to deduce Theorem 1.4 and equivalent statements of some main results in [5,6,10]. First, we present a lemma concerning the copies of K_s.

Lemma 4.3. *Let G be an n-vertex 2-connected graph with $c(G) < k$ and $n \geq 9$. If G contains a copy of $F \in \mathcal{F}(m, k, r)$ with $r \in \{\ell - 1, \ell\}$, then*

$$e(G) \leq h_2(n, k, \ell - 1).$$

Proof. Let $F \in \mathcal{F}(m, k, \ell - 1)$ and $C = \{c_1, c_2\}$. If $k = 2\ell + 1$, then since G is 2-connected with $c(G) < k$, it is easy to see that the longest path starting from c_1 ending at c_2 is on at most $\ell + 1$ vertices. Since $\ell \geq 4$ and $n \geq k$, by Theorem 4.1,

$$e(G) \leq \frac{(\ell - 2)(n - 2)}{2} + 2n - 3 < \binom{\ell + 2}{2} + (\ell - 1)(n - \ell - 2) = h_2(n, k, \ell - 1),$$

as desired. Let $k = 2\ell + 2 \geq 10$. Note that the longest path starting from c_1 ending at c_2 in G is on $\ell + 2$ vertices (if there is a path starting from c_1 ending at c_2 in G on $\ell + 3$ vertices, then one may easily check that $c(G) \geq k$ by G contains a copy of F, a contradiction). Then by Theorem 4.1 and $\ell \geq 5$, we have

$$e(G) \leq \frac{(\ell - 1)(n - 2)}{2} + 2n - 3 < \binom{\ell + 3}{2} + (\ell - 1)(n - \ell - 3) = h_2(n, k, \ell - 1).$$

Thus we may suppose $k = 10$. If the longest path starting from c_1 and ending in c_2 has at most five vertices, then by Theorem 4.1, we have

$$e(G) \leq \frac{2(n - 2)}{2} + 2n - 3 < 3n - 3 = h_2(n, 10, 3).$$

Thus we may assume that there is a longest path $P = c_1x_1x_2x_3x_4c_2$ in G. Let $G' = G - \{c_1, c_2\}$. Let X be the component of G' contains $\{x_1, x_2, x_3, x_4\}$. Then by Proposition 2.2, X is a star, a triangle or a complete graph with one part of size two. In all of the above cases, the number of edges incident with X in G is at most $3|X|$. For any other component Y of G', since $c(G) < 10$, by Theorem 4.1, the number edges incident with it is at most $3|Y|$. Summing all the above edges, we have $e(G) \leq \binom{6}{2} + 3(n - 6) = 3n - 3 = h_2(n, 10, 3)$. We finish the proof when $F \in \mathcal{F}(m, k, \ell - 1)$.

Let $F \in \mathcal{F}(m, k, \ell)$. Then k is even and $n \geq 10$. Let w, w_1 and w_2 be the vertices of F as in Section 2. Since $c(G) < k$, the longest path starting from w_1 or w_2 through $G - \{w, w_1, w_2\}$ ending at w is on at most $\ell + 1$ vertices and each component of $G - \{w, w_1, w_2\}$ can only be adjacent to w_1, w or w_2, w. Let G_i be the induced subgraph of G containing $\{w, w_i\}$ and all components of $G - \{w, w_1, w_2\}$ which is adjacent to w_i for $i = 1, 2$. Let $n_1 = |V(G_1)|$ and $n_2 = |V(G_2)|$. Then $n = n_1 + n_2 - 1$. Since $n \geq 10$ and $\ell \geq 4$, by Theorem 4.1, we have

$$\begin{aligned} e(G) = e(G_1) + e(G_2) + 1 &\leq \frac{(\ell - 2)(n_1 - 2)}{2} + 2n_1 - 3 + \frac{(\ell - 2)(n_2 - 2)}{2} + 2n_2 - 3 + 1 \\ &= \frac{(\ell - 2)(n - 3)}{2} + 2(n + 1) - 3 - 2 < \binom{\ell + 3}{2} + (\ell - 1)(n - \ell - 3) = h_2(n, k, \ell - 1). \end{aligned}$$

This finishes the proof for $s = 2$. ☐

Define

$$\phi_s(n, k) = \begin{cases} \binom{\ell+2}{s} + (r - 1)\binom{\ell+1}{s} + \binom{t+2}{s}, & n = r(\ell - 1) + t + 3,\ 1 \leq t \leq \ell - 1 \text{ and } k \text{ is even;} \\ r\binom{\ell+1}{s} + \binom{t+2}{s}, & n = r(\ell - 1) + t + 2,\ 1 \leq t \leq \ell - 1 \text{ and } k \text{ is odd.} \end{cases}$$

Let $\mathcal{H}'(k, \ell - 1)$ be the set of graphs in $\mathcal{H}(k, \ell - 1)$ with Type II and $|A| = \ell - 1$.

Lemma 4.4. *Let G be an n-vertex 2-connected graph with $c(G) < k$. If G contains a copy of $F \in \mathcal{H}(k,\ell)$ or $F \in \mathcal{H}(k,\ell-1) \setminus \mathcal{H}'(k,\ell-1)$, then*

$$N_s(G) \leq \begin{cases} h_s(n,k,\ell-1), & \text{if } 3 \leq s \leq \ell \text{ and } n \geq k \geq 11; \\ \phi_s(n,k), & \text{if } s \geq \ell+1. \end{cases}$$

Proof. Since $k \geq 11$, we have $\ell \geq 5$, hence an easy computation implies

$$(\ell-1)\binom{\ell-1}{s-1} \geq \frac{(\ell+1)\ell}{s(\ell-s+1)}\binom{\ell-1}{s-1} = \binom{\ell+1}{s} \tag{6}$$

and

$$(t-2)\binom{\ell-1}{s-1} \geq \binom{t}{s} \text{ for } t \leq \ell+1. \tag{7}$$

Let $F \in \mathcal{F}(m,k,\ell-1)$ and k be odd. Let $n-2 = r(\ell-1)+t$ where $1 \leq t \leq \ell-1$ and $r \geq 2$ are integers. Since G is 2-connected with $c(G) < k$, the longest C-path is on at most $\ell+1$ vertices. If $s \leq \ell$, then applying Theorem 4.1, by (6) and (7) we have

$$\begin{aligned} N_s(G) &\leq r\binom{\ell+1}{s} + \binom{t+2}{s} \\ &\leq \binom{\ell+1}{s} + (r-1)(\ell-1)\binom{\ell-1}{s-1} + \binom{t+2}{s} \\ &\leq \binom{\ell+2}{s} + (r-1)(\ell-1)\binom{\ell-1}{s-1} + \binom{t+1}{s} \\ &\leq \binom{\ell+2}{s} + (n-\ell+2)\binom{\ell-1}{s-1} \\ &= h_s(n,k,\ell-1). \end{aligned}$$

If $s \geq \ell+1$, then since G is 2-connected with $c(G) < k$, we have $s = \ell+1$ and each copy of $K_{\ell+1}$ contain both vertices of C. Thus $N_s(G) \leq \phi_s(n,k)$.

Let $F \in \mathcal{F}(m,k,\ell-1)$ and k be even. Let $n-2 = \ell + r'(\ell-1)+t'$ where $1 \leq t' \leq \ell-1$ and $r' \geq 1$ are integers. Assume that $s \leq \ell$. Let $F \neq F_6(m,k,\ell-1)$. Note that if there is a vertex adjacent to $\{x_1,x_2\}$ (when $F = F_5(m,k,\ell-1)$), then this vertex should be an isolated vertex and contribute less number of copies of K_s in G. We may assume that each vertex of $V(G) \setminus \{A \cup B \cup C\}$ is only connected to C. Then applying Theorem 4.1, by (6), (7) and $F \notin \mathcal{H}'(k,\ell-1)$, we have

$$\begin{aligned} N_s(G) &\leq \binom{\ell+2}{s} + r'\binom{\ell+1}{s} + \binom{t'+2}{s} \\ &\leq \binom{\ell+3}{s} + (r'-1)(\ell-1)\binom{\ell-1}{s-1} + \binom{t'+1}{s} \\ &\leq \binom{\ell+3}{s} + (n-\ell+3)\binom{\ell-1}{s-1} \\ &= h_s(n,k,\ell-1). \end{aligned}$$

Let $F = F_6(m,k,\ell-1)$. If y_1 is adjacent to $B \cup D \cup (V(G) \setminus V(F))$, then by Proposition 2.7, each vertex in $B \cup D \cup (V(G) \setminus V(F))$ can only be adjacent to $\{y_1\} \cup C$. Moreover, each vertex in $D \cup (V(G) \setminus V(F))$ is an isolated vertex. Hence the maximal graph containing G with circumference less than k is $S(k,\ell-1,\ell-2)$. Thus, applying Lemma 4.3, we have $N_s(G) \leq \min\{h_s(n,k,\ell-2), h_s(n,k,\ell-1)\} \leq h_s(n,k,\ell-1)$. Hence, we can suppose that each vertex in $D \cup (V(G) \setminus V(F))$ is only connected to C, whence as before we have $N_s(G) \leq h_s(n,k,\ell-1)$. If $s \geq \ell+1$, then since G is 2-connected with $c(G) < k$, we have $s = \ell+1$ or $s = \ell+2$. Thus $N_s(G) \leq \phi_s(n,k)$.

Let $F \in \mathcal{F}(m,k,\ell)$. Then k is even and $n \geq 10$. As in the proof of Lemma 4.3, let G_i be the induced n_i-vertex subgraph of G containing $\{w, w_i\}$ and all components of $G - \{w, w_1, w_2\}$ which is adjacent to w_i for $i = 1, 2$. Let $n_i - 2 = r_i(\ell - 1) + t_1$ for $i = 1, 2$ and $n_1 + n_2 = n + 1$. Since $n \geq 10$ and $\ell \geq 4$, it follows from Theorem 4.1 that

$$\begin{aligned}
N_s(G) &= N_s(G_1) + N_s(G_2) + \binom{3}{s} \\
&\leq r_1\binom{\ell+1}{s} + \binom{t_1+2}{s} + r_2\binom{\ell+1}{s} + \binom{t_2+2}{s} + \binom{3}{s} \\
&\leq r^*\binom{\ell+1}{s} + \binom{t^*+2}{s} + \binom{3}{s} \\
&\leq \binom{\ell+3}{s} + (n-\ell+3)\binom{\ell-1}{s-1} \\
&= h_s(n,k,\ell-1),
\end{aligned}$$

where $n - 3 = r^*(\ell - 1) + t^*$ with $1 \leq t^* \leq \ell - 1$. This finishes the proof of the lemma. □

First, we can derive a more general result concerning the number of cliques from Theorem 1.3. We need the following family of graphs $\mathcal{G}(n,k)$ introduced in [5] (see Fig. 1 in [5]). The n-vertex graphs in $\mathcal{G}(n,k)$ consist of four types $\mathcal{G}_1(n,k)$, $\mathcal{G}_2(n,k)$, $\mathcal{G}_3(n,k)$ and $\mathcal{G}_4(n,k)$.

- $\mathcal{G}_1(n,k) = \{H(n,k,\ell)\}$;
- Each $G \in \mathcal{G}_2(n,k)$ is defined by a partition $V(G) = A \cup B \cup J$, $|A| = t$ and a pair $a_1 \in A$, $b_1 \in B$ such that $G[A] = K_t$, $G[B]$ is the empty graph, $G(A,B)$ is a complete bipartite graph and for every $c \in J$ one has $N(c) = \{a_1, b_1\}$.
- Every member of $G \in \mathcal{G}_3(n,k)$ is defined by a partition $V(G) = A \cup B \cup J$, $|A| = t$ such that $G[A] = K_t$, $G(A,B)$ is a complete bipartite graph, and $G[J]$ has more than one component, all components of $G[J]$ are stars with at least two vertices each, there is a 2-element subset A' of A such that $N(J) \cap (A \cup B) = A'$, for every component S of $G[J]$ with at least 3 vertices, all leaves of S are adjacent to the same vertex $a(S)$ in A' and any other vertex of J is adjacent to each vertex of A';
- Each member of $\mathcal{G}_4(n,k)$ ($k = 10$) has a 3-vertex set A such that $G[A] = K_3$ and $G - A$ is a star forest such that if a component S of $G - A$ has more than two vertices then all its leaves are adjacent to the same vertex $a(S)$ in A and any other vertex of $G - A$ is adjacent to each vertex of A.

Now, we give a more delicate version of Theorem 1.4.

Theorem 4.5. *Let G be an n-vertex 2-connected graph with minimum degree $\delta \geq 2$. Let $n \geq k \geq 9$, $s \geq 3$ and $\ell - 1 \geq \delta + 1$. If $c(G) < k$ and*

$$N_s(G) > \max\{h_s(n,k,\ell-1), h_s(n,k,\delta+1)\}, \tag{8}$$

then one of the following holds:

- *G is a subgraph of a graph in $\mathcal{G}(n,k)$.*
- *$s \leq \ell+1$, and $k = 9, 10$ and $G - A$ consists of at least $\theta(n)$ triangles, stars and complete bipartite graphs with one part of size two for some $A \subseteq V(G)$ of size at most 3;*
- *$s = \ell + 1$ and the copies of K_s except at most $\ell + 2$ of them can be divided into two families $\mathcal{A}$ and $\mathcal{B}$ such that each in $\mathcal{A}$ shares only $x, y \in V(G)$ and each in $\mathcal{B}$ shares only $x, z \in V(G)$.*

- $s = \ell + 2$, k *is even and* G *contains a unique copy of* K_s.

- G *is a subgraph of the graph* $Z(n, k, \delta)$;

- G *is a subgraph of* $H(n, k, \delta)$.

Proof. Suppose that G is $\mathcal{K}_{k,1}$-free. Let $J \supseteq G$ be a maximal $\mathcal{K}_{k,1}$-free with $c(J) < k$. Suppose that $s(J, \ell - 1) \leq k - \ell$. Then

$$N_s(J) \leq (n - k + \ell)\binom{\ell - 1}{s - 1} + \binom{k - \ell}{s} = h_s(n, k, \ell - 1)$$

contradicting (8). Thus we have $s(J, \ell-1) \geq k-\ell+1$. Clearly, $N_s(J) > \max\{h_s(n, k, \ell-1), h_s(n, k, \delta+1)\}$ and $\delta(J) \geq \delta$. Applying Theorem 1.3 with $\alpha = 1$ and $\beta = \delta + 1$, we have $\omega(J) \geq k - \delta$. Then by Lemma 4.2, we have $J = Z(n, k, \delta)$ or $J = H(n, k, \delta)$, that is, G is a subgraph of the graph $Z(n, k, \delta)$ or G is a subgraph of $H(n, k, \delta)$.

Now we may assume that G contains a copy of $F \in \mathcal{K}_{k,1}$. We divide the rest of proof into the following cases.

Case 1. G contains a copy of $F \in \mathcal{H}(k, 1) \cup \{F(k)\}$.

Let k be odd. Then G contains a copy of $F \in \mathcal{H}(k, 1)$, that is $F = F(k, k, 1)$. Then by Proposition 2.3, it is easy to check that G is a subgraph of $H(n, k, \ell)$. Moreover, we have $s \leq \ell + 1$, otherwise G does not contain a copy of K_s, contradicting (8). Let k be even. If G contains a copy of $F \in \mathcal{H}(k, 1)$. If F is $F(k, k, 1)$ with Type II and $|A| = 2$, then by Proposition 2.6 we deduce that each vertex of $V(G) \setminus (A \cup B \cup C)$ is an isolated vertex and is only adjacent to C, that is G is a subgraph of $H(n, k, \ell)$. If F is $F(k, k, 1)$ with Type II and $|A| = 1$, then Proposition 2.6 implies $G - (A \cup B \cup C)$ consists of star forest. Let c_i and c_{i+1} be the vertices in C which are adjacent to the unique edge in $F[D]$. Moreover, since G is 2-connected with $c(G) < k$, each isolated vertex of $G - (A \cup B \cup C)$ can only be adjacent to C, each non-trivial star is only adjacent to $\{c_i, c_{i+1}\}$ and each leaf of a star on at least three vertices is adjacent to only one of c_i and c_{i+1}. Thus G is a subgraph of $\mathcal{G}_3(n, k)$. If F is $F_6(k, k, 1)$ with Type III, applying Proposition 2.6 again, $G - (A \cup B \cup C)$ consists isolated vertices. Moreover, if the isolated vertex is adjacent to y_1, then it can only be adjacent to $\{y_1, c_1\}$, otherwise the isolated vertex is only adjacent to C. Thus G is a subgraph of $\mathcal{G}_2(n, k)$. Moreover, we have $s \leq \ell + 2$, otherwise G does not contain a copy of K_s, contradicting (8). Furthermore, if $s = \ell + 2$, then G contains at most one copy of K_s, hence by (8) there is a unique copy of K_s in G.

Now we may assume that $F = F(k)$. Suppose that $G - V(F)$ contains an edge xy. Then by Proposition 2.5, xy can only be adjacent to $u_i, u_j \in C$ with $1 \leq i < j \leq \ell$, whence, we find a path starting from $a_1 \in A$, through $u_1 P u_i$, xy, $u_j P u_\ell$, B and $u_{j-1} P u_{i+1}$, and ending at $a_2 \in A$, that is, G contains a copy of $F(k, k, 1)$ with Type II, and we are done by previous argument. Thus we can assume that $G - V(F)$ consists of isolated vertices. Applying Proposition 2.5 again, each isolated vertex can be only adjacent to at most one vertex of $A \cup B \cup D$ and vertices of C. If each isolated vertex only be adjacent to C, then clearly G is a subgraph of $H(n, k, 1)$, and we are done. Assume that there is an isolated vertex be adjacent to $A \cup B \cup D$. If the isolated vertex w is adjacent to $a_1 \in A$ and $u_i \in C$, then there is a path starting from $a_2 \in A$, through $u_1 P u_{i-1}$, $a_1 w u_i$ and $u_{i+1} P u_\ell$, and ending at B. Thus G contains a copy of $F(k, k, 1)$ with Type II, hence we are done. By symmetry between the vertices in A and the vertices in B, if w is adjacent to $a_1 \in A$ and $u_i \in C$, then G also contains a copy of $F(k, k, 1)$ with Type II. Suppose that w is adjacent to $v_i \in D$ and $u_j \in C$. If v_i is adjacent to u_j in F, then G contains a copy of $F(k, k, 1)$ with Type II; if u_i is not adjacent to u_j (without loss of generality suppose $i < j$), then a path starting from $a_1 \in A$, through $u_1 P u_i$, $w v_j P u_\ell$, B and $u_{i-1} P u_{i+1}$, and ending at $a_2 \in A$, implying that G contains a copy of $F(k, k, 1)$ with Type II. In both cases, we are done by previous argument.

Case 2. G contains a copy of $F_6(k, k, 2)$ or $F \in \mathcal{F}_3(k, k, 2)$. Then by Proposition 2.3, each vertex of $V(G) \setminus (A \cup B \cup C)$ is an isolated vertex and can be only adjacent to $\{u_1\} \cup C$ or $\{y_1\} \cup C$. If there is some vertex in $V(G) \setminus (A \cup B \cup C)$ adjacent to u_1 or y_1, then we can see that $V(G) \setminus (\{w\} \cup C)$ ($w = u_1$ or $w = y_1$) contains an independent edge and isolated vertices, and hence G is a subgraph of $H(n, k, \ell)$. If each vertex of $V(G) \setminus (A \cup B \cup C)$ is only adjacent to C, then by Lemma 2.8 we have $N_s(G) \leq f(n, k, \ell - 1) \leq h(n, k, \ell - 1)$, contradicting (8).

Case 3. G contains a copy of $F_1(10, 10, 2)$ or $F_2(11, 10, 2)$. Then by Proposition 2.3, $G - (A \cup B \cup C)$ is a star forest. Moreover, in $G - (A \cup B \cup C)$, each isolated vertex and each independent edge are only be adjacent to C; each leaf of a star on at least three vertices is adjacent to the same one vertex of C. Therefore, G is a subgraph of a graph in $\mathcal{G}_4(n, k)$.

Case 4. G contains a copy of $F \in \mathcal{H}(k, \ell - 1) \cup \mathcal{H}(k, \ell)$.

Let k be odd. Then $F \in \mathcal{H}(k, \ell - 1)$. Let $\ell \geq 5$, i.e., $k \geq 11$. If $s \leq \ell$, then by Lemma 4.3 $N_s(G) \leq h_s(n, k, \ell - 1)$, contradicting (8). Let $s = \ell + 1$. Since G is 2-connected with $c(G) < k$, each copy of K_s shares two common vertices in C of F. If $s \geq \ell + 2$, then it follows from Theorem 4.1 that $N_s(G) \leq 0$, contradicting (8). Let $k = 9$. Then the longest C-path in G is on at most five vertices. By Lemma 2.1, $G - C$ consist of stars, triangles and complete bipartite graphs with one part of size two. Therefore, by (8), $G - C$ consist of $\theta(n)$ triangles.

Let k be even. If $F = F(m, k, \ell)$ and $k \geq 11$, then by Lemma 4.4 we have $N_s(G) \leq h_s(n, k, \ell - 1)$, a contradiction. If $F = F(m, k, \ell - 1)$ with Type II and $|A| = \ell + 1$ or with Type III, then each vertex of $V(G) - A \cup B \cup C$ is only connected to C (if some vertex of $V(G) - A \cup B \cup C$ is adjacent to y_1 or $\{x_1, x_2\}$, then G contains less copies of K_s). If $s \leq \ell$, by Lemma 4.4 $N_s(G) \leq h_s(n, k, \ell - 1)$, contradicting (8). Let $s = \ell + 1$. Since G is 2-connected with $c(G) < k$, each copy of K_s not in A shares two common vertices in C of F. Note that there are at most $\ell + 2$ copies of K_s in A. We are done for $s = \ell + 1$. If $s \geq \ell + 2$, then we can see that there is a unique copy of $K_{\ell+2}$. The case $k = 10$ is similar to $k = 9$ and be omitted. Suppose that $F = F(m, k, \ell - 1)$ with Type II and $|A| = \ell$. If there is a vertex in $V(G) - A \cup B \cup C$ adjacent to $A \cup B$, or the longest C-path is on at most $\ell + 1$ vertices, then the result follows as before by applying Lemma 4.4. Thus there is a C-path on $\ell + 2$ vertices. We may delete A, the resulting graph G' is still 2-connected with $c(G') < k$. Clearly, we have $N_s(G') \geq \max\{h_s(n - \ell + 1, k, \ell - 1), h_s(n - \ell + 1, k, \delta + 1)\}$. Applying the previous proofs, we are done, until there is no longest C-path is on at most $\ell + 2$ vertices. But when $n < 3\ell$, there is no longest C-path is on at most $\ell + 2$ vertices, and hence we can finish our proof by repeating the previous arguments. □

Now we have the following immediate corollary, which can imply some of the main results in [5, 6, 10].

Corollary 4.6. *Let G be an n-vertex 2-connected graph with $c(G) < k$ and minimum degree $\delta(G) = \delta$. Let $k \geq 9$* [11] *and $\ell - 1 \geq \delta + 1$. If*

$$e(G) > \max\{h_2(n, k, \ell - 1), h_2(n, k, \delta + 1)\},$$

then one of the following holds:

- *G is a subgraph of a graph in $\mathcal{G}(n, k)$;*
- *G is a subgraph of $Z(n, k, \delta)$;*
- *G is a subgraph of $H(n, k, \delta)$.*

Proof. The proof is the same as the proof of Theorem 4.5 by applying Lemma 4.3. □

Acknowledgement. The authors would like to thank Alexandr Kostochka and Ruth Luo for helpful discussions at early stage of this study and Qingyi Huo for his careful reading on a draft.

[11] we include the case $k = 9, 10$ which are not deal with in [10].

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