arXiv:1512.06393v1 [math.CO] 20 Dec 2015

Coloring graphs with two odd cycle lengths

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Abstract

In this paper we determine the chromatic number of graphs with two odd cycle lengths. Let G be a graph and $L(G)$ be the set of all odd cycle lengths of G. We prove that: (1) If $L(G) = \{3, 3+2l\}$, where $l \geq 2$, then $\chi(G) = \max\{3, \omega(G)\}$; (2) If $L(G) = \{k, k+2l\}$, where $k \geq 5$ and $l \geq 1$, then $\chi(G) = 3$. These, together with the case $L(G) = \{3, 5\}$ solved in [14], give a complete solution to the general problem addressed in [14, 3, 8]. Our results also improve a classical theorem of Gyárfás which asserts that $\chi(G) \leq 2|L(G)| + 2$ for any graph G.

Keywords: chromatic number, odd cycle length, 3-colorability, critical graph

Mathematics Subject Classification (2010): 05C15; 05C45

1 Introduction

Only simple graphs are considered. For a graph G, let $\chi(G)$, $\omega(G)$, and $L(G)$ denote the chromatic number of G, the size of maximum cliques in G, and the set of all odd cycle lengths of G, respectively. For notation not defined, we refer the reader to [1].

The study of the relation between $\chi(G)$, $\omega(G)$ and $L(G)$ is a fundamental area in graph theory and has been the subject of extensive research. It is well-known that $\chi(G) = 2$ if and only if $L(G) = \emptyset$. A general upper bound for $\chi(G)$ in terms of the size of $L(G)$ was proposed by Bollobás and Erdős [5], where they conjectured that $\chi(G) \leq 2|L(G)| + 2$ for any G. In [7], Gyárfás confirmed this by showing that if $|L(G)| = k \geq 1$, then $\chi(G) \leq 2k+2$ with equality if and only if some block of G is a K_{2k+2}. If one considers the elements of $L(G)$, then often the value of $\chi(G)$ can be improved. Indeed, in [14] Wang proved that $\chi(G) = 3$ if $L(G) = \{k\}$ for some $k \geq 5$. Kaiser, Rucký and Škrekovski [8] obtained a slight improvement that any proper 3-coloring of an odd cycle of G can be extended to a proper 3-coloring of G, assuming G contains no K_4 and has $|L(G)| = 1$. The problem of determining $\chi(G)$ seems to be much harder for graphs with $|L(G)| = 2$. The case $L(G) = \{3, 5\}$ was resolved by Wang [14], where he proved that if G contains neither K_4 nor W_6 (a wheel on six vertices) then $\chi(G) = 3$, and otherwise $\chi(G) = \max\{4, \omega(G)\}$. In [3], Camacho and Schiermeyer showed that every graph G with $L(G) = \{k, k+2\}$ for $k \geq 5$ satisfies $\chi(G) \leq 4$. The special case $L(G) = \{5, 7\}$ was improved to $\chi(G) = 3$ by Kaiser, Rucký and Škrekovski in [8].

In this paper, we determine $\chi(G)$ for every graph G with $|L(G)| = 2$. Our main theorems are as follows.

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Theorem 1. *Let $l \geq 2$ be an integer. Any graph G with $L(G) = \{3, 3+2l\}$ has $\chi(G) = \max\{3, \omega(G)\}$.*

Theorem 2. *Let $k \geq 5$ and $l \geq 1$ be integers. Any graph G with $L(G) = \{k, k+2l\}$ has $\chi(G) = 3$.*

We point out that these results improve the aforementioned theorem of Gyárfás in the family of graphs considered. Recently, the theorem of Gyárfás was extended to cycles of consecutive odd lengths in a joint paper [10] of the first author. Answering a conjecture of Erdős [6], Kostochka, Sudakov and Verstraëte in [9] proved that every triangle-free graph G with $|L(G)| = k$ satisfies $\chi(G) = O(\sqrt{k/\log k})$. For $L(G)$ in general, the precise value of $\chi(G)$ seems to be out of reach. However, maybe it is possible to determine the maximum integer t such that any triangle-free graph G with $|L(G)| = t$ has $\chi(G) = 3$. The Grötzsch graph and Chvátal graph both have $L(G) = \{5, 7, 9, 11\}$ and $\chi(G) = 4$, which, together with Theorem 2, show that $2 \leq t \leq 3$. It will be interesting to see if $t = 3$.

Let $G = (V, E)$ be a graph, x, y be vertices of G, and H, H' be subgraphs of G. For a subset S of V, by $N_H(S)$ we denote the set of vertices in $V(H)\backslash S$ adjacent to some vertex in S. For $x, y \in V(H)$, the *distance* in H between x and y, denoted by $d_H(x, y)$, is the length of a shortest path in H with endpoints x and y. For a cycle or a path Q, the *length* of Q, denoted by $|Q|$, counts the number of edges in Q. A cycle C is called a *k-cycle* if $|C| = k$. If we draw a cycle C as a circle in the plane, then xCy denotes the path on C from x to y in the clockwise direction. A path P with endpoints x and y is called an *(x, H, y)-path* if $V(P)\backslash\{x, y\} \subset V(H)$, and an *$(H, H')$-path* if $V(P \cap H) = \{x\}$ and $V(P \cap H') = \{y\}$. An *H-bridge* of G is either an edge with two endpoints in $V(H)$ or a subgraph induced by a component D of $G - H$ together with all edges between D and H. For subsets A, B of V, the pair (A, B) is called a *k-separation* of G if $A \cup B = V$, $|A \cap B| = k$, and G has no edges between $A\backslash B$ and $B\backslash A$. A graph G is *k-chromatic* if $\chi(G) = k$, and is *k-critical* if G is k-chromatic but any proper subgraph of G is not. If there is no danger of ambiguity, we often do not distinguish the vertex set and the graph induced by it. And if H consists of a single vertex v, we also often write v instead of H or $\{v\}$ in the above notation.

The organization of this paper is as follows. In Section 2, we prove Theorem 1. In Section 3, we prove Theorem 2, assuming Lemmas 3 and 4. We then complete the proofs of Lemmas 3 and 4 in Sections 4 and 5, respectively.

2 Proof of Theorem 1

Throughout this section, let G be a graph with

$$L(G) = \{3, k\}, \text{ where } k := 3 + 2l \text{ and } l \geq 2. \tag{1}$$

We shall show that $\chi(G) = \max\{3, \omega(G)\}$. It is always fair to assume that G is 2-connected (as otherwise one can consider its 2-connected blocks instead). By (1), observe that $\omega(G) \in \{3, 4\}$. According to the value of ω, we divide the proof of Theorem 1 into two subsections as follows.

2.1 $\omega(G) = 4$

Let X be a K_4 in G with $V(X) = \{x_1, x_2, x_3, x_4\}$. We will need to prove $\chi(G) = 4$. To achieve this, we propose to show that for any component H in $G - X$, any proper

4-coloring of X can be extended to a proper 4-coloring of $G[V(X \cup H)]$.

First we claim that for distinct $x_i, x_j \in V(X)$ there is no (x_i, x_j)-path of even length in G internally disjoint from X. Suppose to the contrary that there is a such path P in G, say from x_1 to x_2. Then $P \cup x_1x_2$ and $P \cup x_1x_3x_4x_2$ are two odd cycles in G with lengths differ by two, a contradiction to (1). This proves the claim.

Suppose that H contains an odd cycle, say C. Since G is 2-connected, there are two (X, C)-paths P_1, P_2, say from $x_1, x_2 \in V(X)$ to $y_1, y_2 \in V(C)$, respectively. Since $|C|$ is odd, there exists a (y_1, y_2)-path Q on C such that $P_1 \cup P_2 \cup Q$ is an even (x_1, H, x_2)-path in G, a contradiction. So, H is bipartite.

Let (A, B) be the bipartition of H. Next we show that no distinct $x_i, x_j \in V(X)$ can be adjacent to the same part in (A, B). Otherwise, by symmetry we may assume that there exist $a \in A \cap N_G(x_1)$ and $a' \in A \cap N_G(x_2)$. Let P be an (a, a')-path of H. As $|P|$ is even, we see $x_1a \cup P \cup a'x_2$ is an even (x_1, H, x_2)-path in G, a contradiction to the claim.

We can then derive that there are at most two vertices in X adjacent to H, say $V(X) \cap N_G(H) \subseteq \{x_1, x_2\}$. Now it is clear that any proper 4-coloring φ of X can be extended to a proper 4-coloring of $G[V(X \cup H)]$, by coloring all vertices of A by the color $\varphi(x_3)$ and all vertices of B by the color $\varphi(x_4)$. The proof of Theorem 1 when $\omega(G) = 4$ is completed. □

2.2 $\omega(G) = 3$

To finish the proof of Theorem 1, it remains to consider a graph G containing no K_4. We are going to prove $\chi(G) = 3$ by the means of contradiction. Let G be a minimal K_4-free graph satisfying (1) but $\chi(G) \geq 4$. So

$$G \text{ is 4-critical, which implies that } \delta(G) \geq 3 \text{ and } G \text{ is 2-connected.}$$

Recall that we write $k = 3 + 2l$, where $k \geq 7$ (as $l \geq 2$).

Our starting point is a result of Voss [12, Theorem 2] (also see [13]) that every K_4-free graph with chromatic number at least 4 contains an odd cycle with at least two diagonals. By this theorem, G contains a k-cycle C with at least two diagonals, as clearly such cycle can not be a triangle. Let $C := v_0v_1 \ldots v_{k-1}v_0$, and $G_0 := G[V(C)]$. (The subscripts will be taken modulo k in the rest of this section.)

In what follows, we will prove a sequence of claims. The first claim shows that the induced subgraph G_0 consists of the k-cycle C and exactly two diagonals. Without loss of generality, we may assume that

Claim 1. $E(G_0) = E(C) \cup \{v_0v_2, v_1v_3\}$.

Proof. For any diagonal v_iv_j of C, there exists a (v_i, v_j)-path P on C such that $P \cup v_iv_j$ forms an odd cycle. Since $L(G) = \{3, k\}$ and $j \notin \{i-1, i+1\}$, we see that $P \cup v_iv_j$ is of length less than k and thus a triangle, implying that $j \in \{i-2, i+2\}$. Without loss of generality let v_0v_2 be a diagonal of C. Consider any other diagonal v_iv_{i+2} of C. If $i \notin \{1, k-1\}$, then there is a $(k-2)$-cycle $(C - \{v_1, v_{i+1}\}) \cup v_0v_2 \cup v_iv_{i+2}$, so $l = 1$, a contradiction. Thus, except v_0v_2, only v_1v_3 or v_1v_{k-1} can be a diagonal of C, and one can easily see that both of them cannot be. This proves Claim 1. ■

We define a proper 3-coloring $\varphi : V(G_0) \to \{1, 2, 3\}$ of G_0 by the following rule:

- Let $S_1 := \{v_3, v_5, ..., v_{k-2}, v_0\}$ and $S_2 := \{v_2, v_4, ..., v_{k-1}\}$.

- Assign $\varphi(v_1) := 3$, and for any $j \in \{1, 2\}$ and $x \in S_j$, assign $\varphi(x) := j$.

Let H be a component in $G - V(G_0)$. The essential idea behind the coming claims is to show that for every H,

$$\varphi \text{ can be extended to a proper 3-coloring of } G[V(G_0 \cup H)]. \tag{2}$$

Note that, if true, this in turn will give rise to a proper 3-coloring of G and complete the proof of Theorem 1.

Claim 2. If there exist $v_i, v_j \in N(u) \cap V(C)$ for some $u \in V(H)$, then $d_C(v_i, v_j) = 1$ or 2. Moreover, $\{v_i, v_j\} \neq \{v_p, v_{p+2}\}$ for any $p \in \{k-1, 0, 1, 2\}$.

Proof. There exists a (v_i, v_j)-path P on C such that $P \cup v_i u \cup u v_j$ forms an odd cycle. As $L(G) = \{3, k\}$, it is easy to see that $d_C(v_i, v_j) = 1$ or 2. Suppose that $\{v_i, v_j\} = \{v_p, v_{p+2}\}$ for some $p \in \{k-1, 0, 1, 2\}$. Then it is easy to check that this will force a 5-cycle in G, a contradiction. ■

Claim 3. We may assume that $|V(H)| \geq 2$.

Proof. Suppose to the contrary that $V(H) = \{u\}$. Claim 2 shows that any two neighbors of u is of distance one or two on C. Since $\delta(G) \geq 3$ and $|C| = k \geq 7$, one can deduce that $N(u) = \{v_i, v_{i+1}, v_{i+2}\}$ for some i. If $v_1 \notin N(u)$, then we can assign $\varphi(u) := 3$ such that (2) holds. So $v_1 \in N(u)$, which means that $i \in \{k-1, 0, 1\}$, contradicting Claim 2. ■

Claim 4. Let $v_i, v_j \in V(C)$. If there are two (v_i, H, v_j)-paths P and Q with lengths differ by one, then $\{v_i, v_j\} \cap \{v_1, v_2\} = \emptyset$, $\{|P|, |Q|\} = \{l+1, l+2\}$, and the (v_i, v_j)-path on C containing $\{v_1, v_2\}$ is of length $l+2$.

Proof. Recall that $k = 2l+3$. Let $p := |P|$ and $q := |Q|$, and assume by symmetry that p is odd. Then $p \geq 3$, implying that $p + q \geq 5$. Let X be the even (v_i, v_j)-path on C such that $C_1 := X \cup P$ forms an odd cycle. Then $C_2 := (C - X) \cup Q$ also is an odd cycle. As $L(G) = \{3, k\}$, $|C_1| + |C_2| = |C| + p + q \in \{6, k+3, 2k\}$. In view of $p + q \geq 5$, we see that $|C_1| + |C_2| = 2k$ and thus $p + q = k$.

We first show that $\{v_i, v_j\} \cap \{v_1, v_2\} = \emptyset$. Suppose not, say $v_i = v_1$. If $v_j \in \{v_0, v_2\}$, then $(C - v_i v_j) \cup P$ is an odd cycle of length more than k, a contradiction. If $v_j = v_3$, then $(C - \{v_1, v_2\}) \cup v_0 v_2 v_1 \cup P$ is an odd cycle of length more than k, a contradiction. So $v_j \in V(C) - \{v_0, v_1, v_2, v_3\}$. Let X, Y be the two (v_1, v_j)-paths on C containing v_0, v_2, respectively. Next we will use $v_0 v_2, X, Y, P$ and Q to find two odd cycles whose lengths differ by two, which of course leads to a contradiction. To this end, we may assume that by symmetry X is even. Note that we also assume p is odd. If $q - p = 1$, then $X \cup P$ and $(X - v_1 v_2) \cup v_2 v_0 v_1 \cup Q$ are two desired odd cycles. Otherwise $p - q = 1$, then $Y \cup Q$ and $(Y - v_0 v_1) \cup v_0 v_2 v_1 \cup P$ are two desired odd cycles. This proves $\{v_i, v_j\} \cap \{v_1, v_2\} = \emptyset$.

Let Z be the (v_i, v_j)-path on C containing $\{v_1, v_2\}$. So $|Z| \geq 3$. Let $\{R_1, R_2\} = \{P, Q\}$ such that $C' := R_1 \cup Z$ forms an odd cycle. Then $C'' := R_2 \cup (Z - \{v_0 v_1, v_1 v_2\}) \cup v_0 v_2$ is also an odd cycle. So $|C'| + |C''| = p + q + 2|Z| - 1 = k + 2|Z| - 1 \in \{6, k+3, 2k\}$. As $|Z| \geq 3$, this shows that $|Z| = (k+1)/2 = l+2$ and $|C'| = |C''| = k$, further implying that $|R_1| = l+1$ and $|R_2| = l+2$. Claim 4 is proved. ■

A *book* of r pages, denoted by B_r^*, is a graph consisting of r triangles sharing with one common edge. It was proved in [14, Theorem 8] that every 2-connected non-bipartite graph containing no odd cycles other than 3-cycles is either a K_4 or a book. This leads us to the next claim.

Claim 5. Every non-bipartite block in H is a book B_r^* for some $r \geq 1$.

Proof. Let B be a non-bipartite block in H. Suppose that B contains a k-cycle C', which is disjoint from C. As G is 2-connected, there exist two disjoint paths X, Y from $x, y \in V(C)$ to $x', y' \in V(C')$, respectively and internally disjoint from $C \cup C'$. Let P be an (x, y)-path on C and P' be an (x', y')-path on C' such that $C_1 := P \cup P' \cup X \cup Y$ is an odd cycle. Then $C_2 := (C - P) \cup (C' - P') \cup X \cup Y$ is also odd. But $|C_1| + |C_2| = |C| + |C'| + 2|X| + 2|Y| > 2k$, a contradiction to $L(G) = \{3, k\}$. This shows that B contains no k-cycles and thus $L(B) = \{3\}$. Claim 5 then follows by the theorem of [14] just mentioned and the fact that G is K_4-free. ■

Claim 6. H has at most one non-bipartite block.

Proof. Suppose to the contrary that H has two such blocks, say B_1 and B_2. Let W be a path in H from $w_1 \in V(B_1)$ to $w_2 \in V(B_2)$ internally disjoint from $B_1 \cup B_2$, where w_i is a cut-vertex of H contained in B_i. By Claim 5, B_i is a book and thus contains a triangle, say $T_i := G[\{w_i, x_i, y_i\}]$. Each of $\{x_i, y_i\}$ is either a vertex of degree two in the book B_i or adjacent to such a vertex in $V(B_i) - V(T_i)$; while for each vertex u of degree two in the book B_i, there is a path from u to $V(C)$ internally disjoint from B_i. Hence, by symmetry, we may assume that there exist two disjoint paths P_1, P_2 from x_1, x_2 to $v_i, v_j \in V(C)$, respectively and internally disjoin from $B_1 \cup B_2 \cup W \cup C$.

If one can choose the above P_1, P_2 such that $v_i \neq v_j$, then we can find three (v_i, H, v_j)-paths, namely, $P := P_1 \cup x_1w_1 \cup W \cup w_2x_2 \cup P_2$, $(P - x_1w_1) \cup x_1y_1w_1$ and $(P - \{x_1w_1, x_2w_2\}) \cup x_1y_1w_1 \cup x_2y_2w_2$, with three consecutive lengths, a contradiction to Claim 4. Thus, for all choices of $\{P_1, P_2\}$,

$$P_1, P_2 \text{ intersect } V(C) \text{ at the same vertex, say } v_i. \tag{3}$$

Then we get three cycles of consecutive lengths, which implies that the middle cycle is a k-cycle and so

$$|P_1| + |P_2| + |W| + 3 = 2l + 3. \tag{4}$$

Since G is 2-connected, there exists a path Q from $v_q \in V(C) - \{v_i\}$ to $w \in V(T_1 \cup T_2 \cup W \cup P_1 \cup P_2)$, internally disjoint from $C \cup T_1 \cup T_2 \cup W \cup P_1 \cup P_2$. If $w \in V(P_1 \cup P_2) \cup \{x_1, y_1, x_2, y_2\}$, then we get a contradiction to (3). Thus, $w \in V(W)$. For each $i \in \{1, 2\}$, let $Q_i := Q \cup wWw_i \cup w_ix_i \cup P_i$, then Q_i and $(Q_i - w_ix_i) \cup w_iy_ix_i$ are two (v_q, H, v_i)-paths whose lengths differ by one. Claim 4 then implies that for $i \in \{1, 2\}$, the length of Q_i is

$$|Q| + |wWw_i| + 1 + |P_i| = l + 1.$$

Adding them up, we have

$$|P_1| + |P_2| + |W| + 2|Q| + 2 = 2l + 2,$$

which, compared with (4), shows that $|Q| = 0$, a contradiction. This proves Claim 6. ■

By Claims 5 and 6, let D be the unique non-bipartite block of H (if existing), such that $V(D) = \{x_1, x_2, y_1, ..., y_r\}$ and $E(D) = \{x_1x_2\} \cup \{x_iy_j : 1 \leq i \leq 2, 1 \leq j \leq r\}$. Denote $H' := H - \{x_1x_2\}$ if D exists; otherwise, denote $H' := H$. Therefore,

$$H' \text{ is connected and bipartite.} \tag{5}$$

Let (A, B) be the bipartition of H'. So $\{x_1, x_2\} \subseteq A$ or B.

Claim 7. If $N_H(v_1) \neq \emptyset$, then D does not exist and thus $H = H'$ is bipartite.

Proof. Suppose that $N_H(v_1) \neq \emptyset$ and there is the non-bipartite block D of H. Let T be a triangle in D with $V(T) = \{x_1, x_2, x_3\}$. Since G is 2-connected, there exist two disjoint paths P, Q from $V(C)$ to $V(T)$ internally disjoint from $C \cup T$. Since $N_H(v_1) \neq \emptyset$, by rerouting paths if necessarily, we may assume that P, Q are from $v_1, v_i \in V(C)$ to two vertices, say $x_1, x_2 \in V(T)$, respectively. Then $P \cup x_1x_2 \cup Q$ and $P \cup x_1x_3x_2 \cup Q$ are two (v_1, H, v_i)-paths with lengths differ by one, however it is a contradiction to Claim 4. ∎

Recall the sets $S_1 = \{v_3, v_5, ..., v_{k-2}, v_0\}$ and $S_2 = \{v_2, v_4, ..., v_{k-1}\}$, and the proper 3-coloring φ on G_0.

Claim 8. $N_H(v_1) = \emptyset$.

Proof. We prove this claim by showing that if $N_H(v_1) \neq \emptyset$, then (2) holds. Without loss of generality, assume that there exists $u_1 \in N_G(v_1) \cap A$. By Claim 7, H is bipartite.

We first show that $N_H(S_1) \subseteq B$. Otherwise, there exists $v_iu_i \in E(G)$ for some $v_i \in S_1$ and $u_i \in A$. Let P be a (v_1, H, v_i)-path with even length. If $v_i = v_0$, then $P \cup v_0v_1$ and $P \cup v_1v_3v_2v_0$ are two odd cycles with lengths differ by two, a contradiction. So $v_i \in S_1 - \{v_0\}$. Let X be the (v_3, v_i)-path on C not containing v_1. Note that X is even. Then $v_1v_3 \cup X \cup P$ and $v_1v_0v_2v_3 \cup X \cup P$ are two odd cycles with lengths differ by two, which cannot be.

Next we show that $N_H(S_2) \subseteq A$. Suppose to the contrary that there exists $v_ju_j \in E(G)$ for some $v_j \in S_2$ and $u_j \in B$. Let Q be a (v_1, H, v_j)-path with odd length at least three. If $v_j = v_2$, then $(C - v_1v_2) \cup Q$ is an odd cycle of length at least $k + 2$, a contradiction. So $v_j \in S_2 - \{v_2\}$. Let Y be the (v_3, v_i)-path on C not containing v_1. So Y is odd. Then $v_1v_3 \cup Y \cup Q$ and $v_1v_0v_2v_3 \cup Y \cup Q$ are two odd cycles with lengths differ by two, again a contradiction.

Note that $V(C) = \{v_1\} \cup S_1 \cup S_2$ and G is 2-connected. So $N_H(S_1) \cup N_H(S_2)$ is not empty. This in turn shows that $N_H(v_1) \subseteq A$. Note that H is bipartite. So φ can be extended to a proper 3-coloring of $G[V(G_0 \cup H]$, by simply coloring all vertices in A using color 1 and all vertices in B using color 2. ∎

Claim 9. If there exist distinct $v_p, v_q \in S_i$ for some i adjacent to $u_p \in A, u_q \in B$, respectively, then the (v_p, v_q)-path on C not containing v_1 is of length $l + 1$, any (u_p, u_q)-path in H' is of odd length l (in particular, l is odd), and $|N_G(A) \cap S_i| = |N_G(B) \cap S_i| = 1$.

Proof. By (5), any (u_p, u_q)-path P in H' is of odd length . Let X be the (v_p, v_q)-path on C not containing v_1, and Y be the (v_p, v_q)-path $(C - X - \{v_1\}) \cup v_0v_2$. Both X and Y are even with $|X| + |Y| = k - 1$. Then $X \cup v_pu_p \cup P \cup u_qv_q$ and $Y \cup v_pu_p \cup P \cup u_qv_q$ are two odd cycles, implying that $|X| + |Y| + 2|P| + 4 \in \{6, k + 3, 2k\}$. As $|X| + |Y| = k - 1$ and $|P| \geq 1$, we deduce that $|X| + |Y| + 2|P| + 4 = 2k$. This implies that $|P| = (k - 3)/2 = l$ and $|X| = (k - 1)/2 = l + 1$.

Suppose that there is some $v_j \in N_G(A) \cap S_i - \{v_p\}$. Note that $|C| = 2l + 3$. So we have $d_C(v_j, v_q) = d_C(v_p, v_q) = l + 1$. Since $v_j \neq v_p$, vertices v_p, v_q, v_j must lie on C in cyclic order and thus the (v_p, v_j)-path on C containing v_q is of length $2l + 2$, a contradiction to the definition of S_i. This shows that $|N_G(A) \cap S_i| = 1$ and similarly $|N_G(B) \cap S_i| = 1$, completing the proof. ∎

Claim 10. l is odd.

Proof. Suppose for a contradiction that l is even. By Claim 9, we see $N_H(S_i) \subseteq A$ or B for each i. By the symmetry between A and B, we have two cases (see below) to consider; and we will show that in each case, φ can be extended in such a way that (2) holds. Note that $N_H(v_1) = \emptyset$ by Claim 8.

Suppose $N_H(S_1) \subseteq A$ and $N_H(S_2) \subseteq B$. We may further assume that $x_1, x_2 \in A$ (if they exist). Then we can extend φ onto $G[V(G_0 \cup H)]$, by coloring x_1 using color 3, all vertices of $A - \{x_1\}$ using color 2 and all vertices of B using color 1.

Now we may assume $N_H(S_1) \cup N_H(S_2) \subseteq A$. If $x_1, x_2 \in B$, we can color x_1 by color 1, color $B - \{x_1\}$ by color 2, and color all vertices of A by color 3. Thus, $x_1, x_2 \in A$. If there exist some $i, j \in \{1, 2\}$ such that $x_i \notin N_H(S_j)$, then we can color x_i by color j, color all vertices of $A - \{x_i\}$ by color 3, and color all vertices of B by color $3 - j$. It remains to consider the situation that for each $i \in \{1, 2\}$, there exist $v_p \in S_1$ and $v_q \in S_2$ such that $v_p, v_q \in N(x_i)$. By Claim 2, we see that $d_C(v_p, v_q) = 1$ or 2. As $\{v_p, v_q\} \neq \{v_0, v_2\}$ (by Claim 2) and $v_p \in S_1$, $v_q \in S_2$, it holds that in fact $d_C(v_p, v_q) = 1$. Hence, we may assume that there exist vertices $v_s, v_{s+1} \in N(x_1)$ and $v_t, v_{t+1} \in N(x_2)$. Clearly $s \neq t$ (as, otherwise, G has a K_4). Then $(C - \{v_s v_{s+1}, v_t v_{t+1}\}) \cup (v_s x_1 v_{s+1}) \cup (v_t x_2 v_{t+1})$ forms a $(k+2)$-cycle in G, a contradiction. This proves Claim 10. ■

Claim 11. If there are $v_i \in S_1, v_j \in S_2$ both adjacent to F for some $F \in \{A, B\}$, then $d_C(v_i, v_j) = 1$ and $N_G(v_i) \cap F = N_G(v_j) \cap F = \{u\}$ for some vertex u. Moreover, if such vertex u exists, then it is unique.

Proof. Let P be any path in H' from $u_i \in N_G(v_i) \cap F$ to $u_j \in N_G(v_j) \cap F$. Clearly $|P|$ is even. Let X be the (v_i, v_j)-path on C not containing v_1, and Y be the (v_i, v_j)-path $(C - X - \{v_1\}) \cup v_0 v_2$. Then both X and Y are odd with $|X| + |Y| = k - 1$, thus $X \cup v_i u_i \cup P \cup u_j v_j$ and $Y \cup v_i u_i \cup P \cup u_j v_j$ are two odd cycles. This shows that $|X| + |Y| + 2|P| + 4 \in \{6, k+3, 2k\}$, so $|P| = 0$ or $(k-3)/2 = l$. The latter case contradicts Claim 10, as $|P|$ is even. Hence $|P| = 0$, implying that both $N_G(v_i) \cap F$ and $N_G(v_j) \cap F = \{u\}$ consist of a single vertex, say u. We also have $\{|X|, |Y|\} = \{1, k-2\}$. If $|Y| = 1$, then $\{v_i, v_j\} = \{v_0, v_2\}$, a contradiction to Claim 2. So $|X| = 1$, i.e., $d_C(v_i, v_j) = 1$.

Let $u \in N_H(v_i) \cap N_H(v_{i+1})$. Suppose there is a vertex $w \in V(H) - \{u\}$ such that $w \in N_H(v_p) \cap N_H(v_{p+1})$ for some p. If $\{v_i, v_{i+1}\} \neq \{v_p, v_{p+1}\}$, then $(C - \{v_i v_{i+1}, v_p v_{p+1}\}) \cup (v_i u v_{i+1}) \cup (v_p w v_{p+1})$ is a $(k+2)$-cycle in G, a contradiction. So $\{v_i, v_{i+1}\} = \{v_p, v_{p+1}\}$. Let P' be a path in H' from u to w, and $P := v_i u \cup P' \cup w v_{i+1}$ be from v_i to v_{i+1} with $|P| \geq 3$. Denote the cycle C' by $(C - \{v_i v_{i+1}\}) \cup P$ if P is odd and $(C - \{v_i v_{i+1}, v_0 v_1, v_1 v_2\}) \cup v_0 v_2 \cup P$ if P is even. As $|P| \geq 3$, in either case C' is an odd cycle of length more than k, a contradiction. This proves Claim 11. ■

By Claim 11, let $u \in N_H(v_p) \cap N_H(v_{p+1})$ (if existing) and $U := \{u\}$; otherwise, let $U := \emptyset$.

Claim 12. $N_H(S_i) \subseteq A \cup U$ and $N_H(S_j) \subseteq B \cup U$ for some $\{i, j\} = \{1, 2\}$.

Proof. By Claim 11, we see that if $N_H(S_i) \nsubseteq U$ for each $i \in \{1, 2\}$, then this assertion follows. Thus, without loss of generality, we assume that $N_H(S_2) \subseteq U$. If $N_H(S_1) \subseteq A \cup U$ or $B \cup U$, then again this assertion follows. So we may assume that there exist $v_i \in N_G(A - U) \cap S_1$ and $v_j \in N_G(B - U) \cap S_1$ (and we will choose distinct v_i, v_j if existing). Suppose $U = \{u\}$, where u is adjacent to both v_p and v_{p+1}. By the symmetry, let us assume that $u \in A$ and $v_p \in S_2$. We then apply Claim 11 to the pair of vertices

v_i, v_p, and it follows that $N_G(v_i) \cap A = \{u\}$, a contradiction to the choice of v_i. So $U = \emptyset$. Then we have $(S_2 \cup \{v_1\}) \cap N_G(H) = \emptyset$ and thus $|S_1 \cap N_G(H)| \geq 2$ (as G is 2-connected), implying that $v_i \neq v_j$ (by the choice). This in turns enables us to apply Claim 9 and conclude that $N_G(A) \cap S_1 = \{v_i\}$ and $N_G(B) \cap S_1 = \{v_j\}$.

By symmetry, let $\{x_1, x_2\} \subseteq A$ (if existing). If v_i is not adjacent to some vertex in $\{x_1, x_2\}$, say x_1, then φ can be extended onto $V(H)$ by coloring x_1 using color 1, all vertices in $A - \{x_1\}$ using color 2 and all vertices in B using color 3. It is clear that (2) holds. So v_i is adjacent to both x_1 and x_2. Since H' is connected, there exists a path P in H' from $w \in N_G(v_j) \cap B$ to some vertex in $\{x_1, x_2\}$, say x_1. By Claim 9, $|P| = l$. Then $v_j w \cup P \cup x_1 v_i$ and $v_j w \cup P \cup (x_1 x_2 v_i)$ are two (v_i, H, v_j)-paths of lengths $l+2$ and $l+3$, respectively, a contradiction to Claim 4. This proves Claim 12. ■

Let $(i, j) = (1, 2)$ in Claim 12. Note that $N_H(v_1) = \emptyset$ (by Claim 8). We show how to extend φ onto $V(H)$ and make (2) hold. By symmetry, let $\{x_1, x_2\} \subseteq A$ (if existing). Suppose that u or $\{x_1, x_2\}$ does not exist, or some $x_i u \notin E(G)$ (say $i = 1$). Then we can color vertices in $\{x_1, u\}$ using color 3, all vertices in $A - \{x_1, u\}$ using color 2 and all vertices in $B - \{x_1, u\}$ using color 1. Hence, we may assume that vertices x_1, x_2, u exist and induce a triangle in H. As G is 2-connected, there is a path P in $G - \{u\}$ from $v_j \in V(C)$ to some vertex in $\{x_1, x_2\}$ (say x_1). Recall that u is adjacent to both v_p and v_{p+1}. By the symmetry between v_p and v_{p+1}, let $v_j \neq v_p$. Then $(v_p u x_1) \cup P$ and $(v_p u x_2 x_1) \cup P$ are two (v_p, H, v_j)-paths with lengths differ by one. By Claim 4, $\{v_p, v_j\} \cap \{v_1, v_2\} = \emptyset$ and the (v_p, v_j)-path X on C containing $\{v_1, v_2\}$ is of length $l+2$. This also shows $v_j \notin \{v_p, v_{p+1}\}$. Then $v_{p+1} u x_1 \cup P$ and $v_{p+1} u x_2 x_1 \cup P$ are two (v_{p+1}, H, v_j)-paths with lengths differ by one as well. By Claim 4 again, the (v_{p+1}, v_j)-path on C containing $\{v_1, v_2\}$ is of length $l+2$, which is a contradiction to $|X| = l+2$. The proof of Theorem 1 is finished. □

3 Proof of Theorem 2

In this section, we shall prove Theorem 2, assuming the following two lemmas whose proofs are postponed to the later sections.

Lemma 3. *Let G be a 4-critical graph with $L(G) = \{k, k+2l\}$, where $k \geq 5$ and $l \geq 1$. Then G is 3-connected.*

Lemma 4. *Let G be a 4-critical graph with $L(G) = \{k, k+2l\}$, where $k \geq 5$ and $l \geq 1$. Then every two odd cycles in G intersect in at least two vertices.*

Like in the proof of Theorem 1, we start the arguments by finding a cycle with certain property. We say a cycle C in G is *non-separating* if $G - V(C)$ is connected. The coming result will be needed in the proof.

Theorem 5 ([11, 2])**.** *Every 3-connected non-bipartite graph contains a non-separating induced odd cycle.*

Now we are ready to prove Theorem 2.

Proof of Theorem 2.(Assuming Lemmas 3 and 4) We prove by contradiction. Suppose it is not true. Then there exists a minimal counterexample graph G, which is

4-critical and clearly non-bipartite with $L(G) = \{k, k+2l\}$, where $k \geq 5$ and $l \geq 1$.

By Lemma 3, G is 3-connected. Then by Theorem 5, G has a non-separating induced odd cycle C such that $H := G - V(C)$ is connected. Moreover, Lemma 4 implies that H is bipartite. Let (A, B) be the bipartition of H. Since $\delta(G) \geq 3$,

$$\text{every vertex on } C \text{ has at least one neighbor in } H. \tag{6}$$

We will need to prove a sequence of claims and then arrive at the final contradiction to conclude this proof.

Claim 1. For any $u \in V(C)$, $N_H(u) \subseteq A$ or B.

Proof. Suppose that some $u \in V(C)$ has two neighbors $a \in A$ and $b \in B$. Since H is connected and bipartite, there is an odd path P from a to b. So $D := ua \cup P \cup bu$ is an odd cycle such that $V(C \cap D) = \{u\}$, contradicting Lemma 4. ■

We can further deduce that

Claim 2. $N_H(C) \subseteq A$ or B. In the rest of this proof, assume that $N_H(C) \subseteq A$.

Proof. We say a vertex $u \in V(C)$ is of *type 0* if $N_H(u) \subseteq A$ and of *type 1* if $N_H(u) \subseteq B$. In view of Claim 1, every vertex on C has type 0 or 1.

Suppose there exist vertices on C of different types. Then we can divide C into paths $P_1, P_2, ..., P_{2s}$ (appearing along a given cyclic order of C) such that $V(C) = \bigcup_{i=1}^{2s} V(P_i)$ and for each $j \in \{0, 1\}$, $V(P_{2i-j})$ consists of vertices of type j. We now define a 3-coloring $\varphi : V(G) \to \{0, 1, 2\}$ as follows: every vertex in A is colored by 1; every vertex in B is colored by 0; and for every $j \in \{0, 1\}$, we alternatively color $V(P_{2i-j})$ using colors $j, 2$ such that the first vertex of the path (along the given cyclic order of C) is colored by j. It is easy to see that φ is a proper 3-coloring of G. This proves Claim 2. ■

Claim 3. $|C| = k$. Denote by $C := x_0x_1x_2 \cdots x_{k-1}x_0$.

Proof. Suppose that $|C| = k + 2l$. Write $C = x_0x_1x_2 \cdots x_{k+2l-1}x_0$. We show that for any i, $N_H(x_i) = N_H(x_{i+2})$. Otherwise, there exist $a_1, a_2 \in A$ with $a_1x_i, a_2x_{i+2} \in E(G)$. There is an (a_1, a_2)-path P with an even length in H. Thus $(C - \{x_{i+1}\}) \cup \{a_1x_i, a_2x_{i+2}\} \cup P$ is an odd cycle of length at least $k + 2l + 2$, a contradiction. Now we can infer that in fact all $N_H(x_i)$ are the same set, implying that there are triangles in G, a contradiction. ■

Claim 4. $|V(H)| \geq 3$.

Proof. Otherwise, $|V(H)| = 1$ or 2. Then by Claim 2, in either case there exists a vertex $u \in V(H)$ which every vertex on C is adjacent to. This implies triangles in G, a contradiction. ■

Claim 5. (1) If there is a vertex $y \in V(H)$ such that $x_iy, x_{i+2}y \in E(G)$, then $l = 1$. (2) If there is a trivial end-block (i.e., an edge) in H, then $l = 1$.

Proof. Set $x_j := x_{i+2}$. Clearly $x_{i+1}y, x_{j+1}y \notin E(G)$, since otherwise there is a triangle. So x_{j+1} has a neighbor $y' \in A - \{y\}$. There is an even (y, y')-path P in H, so $P \cup yx_jx_{j+1}y'$ and $P \cup yx_ix_{i+1}x_jx_{j+1}y'$ are two odd cycles with lengths differ by two. This proves (1).

Suppose $B := yb$ is a trivial end-block in H, where b is the cut-vertex. Since G is 3-connected, y has two neighbors $x_i, x_j \in V(C)$. Since $|C| = k$ is the least odd cycle length, we have $d_C(x_i, x_j) = 2$. By Claim 5(1), we obtain $l = 1$. This proves (2). ■

Claim 6. For any 2-connected end-block D in H, if there are two vertices $x_i, x_{i+2} \in V(C)$ adjacent to D, then $l = 1$.

Proof. Suppose not. By Claim 5(1), assume that $l \geq 2$ and there are distinct $y_i, y_{i+2} \in V(D)$ such that $x_i y_i, x_{i+2} y_{i+2} \in E(G)$. Let R be any (y_i, y_{i+2})-path in H, which must be of length $2l$. This is because $R \cup \{x_i y_i, x_{i+2} y_{i+2}\} \cup (C - \{x_{i+1}\})$ is an odd cycle of length $|R| + k = k + 2l$.

Since D is 2-connected, there are two disjoint (y_i, y_{i+2})-paths P, Q in D such that $|P| = |Q| = 2l$, Then $C' := P \cup Q$ is an even cycle of length $4l$. Write $C' := u_0 u_1 u_2 \ldots u_{4l-1} u_0$ with $u_0 := y_i$ and $u_{2l} := y_{i+2}$. Let $P_1 := x_i u_0$ and $P_2 := x_{i+2} u_{2l}$. As G is 3-connected, there exists a path P_3 from $v \in V(C)$ to $u_j \in V(C') - \{u_0, u_{2l}\}$, internally disjoint from $P_1 \cup P_2 \cup C \cup C'$. Next we aim to show

$$\text{for every path } P_3 \text{ defined as above, } v = x_{i+1} \text{ and } u_j \in \{u_l, u_{3l}\}. \tag{7}$$

By symmetry, assume that $0 < j < 2l$. We draw C' in the plane such that $u_0, u_1, ..., u_{2l-1}$ appear on C' clockwise, and let $Q_1 := u_0 C' u_j$, $Q_2 := u_j C' u_{2l}$ and $Q_3 := u_{2l} C' u_0$ so that $C' = Q_1 \cup Q_2 \cup Q_3$.

To prove (7), we first show $v \in V(C) - \{x_i, x_{i+2}\}$. Otherwise, say $v = x_i$, then either $C_1 := x_i x_{i+1} x_{i+2} \cup P_2 \cup Q_2 \cup P_3$ or $C_2 := (C - \{x_{i+1}\}) \cup P_2 \cup Q_2 \cup P_3$ is odd. If C_1 is odd, then $C_3 := P_1 \cup Q_3 \cup Q_2 \cup P_3$ is also odd with $|C_3| - |C_1| = 2l - 2 \in \{0, 2l\}$, implying that $l = 1$; otherwise C_2 is odd, then $C_4 := (C - \{x_{i+1}\}) \cup P_2 \cup Q_3 \cup Q_1 \cup P_3$ is an odd cycle of length at least $k + 2l + 1$, a contradiction. Now we see P_1, P_2, P_3 are disjoint paths. Since C is odd and $|Q_1| + |Q_2| = 2l = |Q_3|$, there is a (v, x_i)-path L on C such that $C_5 := L \cup P_1 \cup Q_1 \cup P_3$, and $C_6 := L \cup P_1 \cup (Q_2 \cup Q_3) \cup P_3$ are odd. So $|C_6| - |C_5| = |Q_2| + |Q_3| - |Q_1| = 4l - 2|Q_1| \in \{0, 2l\}$. Since $|Q_1| < 2l$, this implies that $|Q_1| = l$ and thus $u_j = u_l$. Lastly, suppose that $v \neq x_{i+1}$, i.e., $v \in V(C) - \{x_i, x_{i+1}, x_{i+2}\}$. By the symmetry between x_i and x_{i+2}, let X be the (v, x_{i+2})-path on C not containing x_i such that $C_7 := X \cup P_2 \cup Q_2 \cup P_3$ is odd. Then $C_8 := X \cup (x_{i+2} x_{i+1} x_i) \cup P_1 \cup Q_1 \cup P_3$ is also odd with $|C_8| - |C_7| = 2$, implying $l = 1$. This proves (7).

Let $u_j = u_l$ and Q_i's be as above. Since $l \geq 2$, $Q_1 - \{u_0, u_l\}$ is not empty. Since G is 3-connected, there is a path R from $r \in V(Q_1) - \{u_0, u_l\}$ to $s \in (C' - Q_1) \cup C \cup P_1 \cup P_2 \cup (P_3 - \{u_l\})$, internally disjoint from $C' \cup C \cup P_1 \cup P_2 \cup P_3$.

If $s \in Q_2 - \{u_l\}$, then $C'' := u_0 Q_1 r \cup R \cup s Q_2 u_{2l} \cup Q_3$ is a cycle of length $4l$, however the path $r Q_1 u_l \cup P_3$ from C'' to C contradicts (7). If $s \in Q_3 - \{u_0, u_{2l}\}$, then $R_1 := u_0 Q_1 r \cup R \cup s Q_3 u_{2l}$, $R_2 := (C' - R_1) \cup R$ are two (y_i, y_{i+2})-paths in H, implying that $4l = |R_1| + |R_2| = |C'| + 2|R| > 4l$, a contradiction. Hence, $s \notin C'$. By (7), we also have $s \notin (C - \{x_i, x_{i+2}\}) \cup (P_3 - \{u_l\})$. Therefore, $s \in \{x_i, x_{i+2}\}$. In either case, let $C_1 := x_{i+1} s \cup R \cup r Q_1 u_l \cup P_3$ and $C_2 := (C - \{x_{i+1} s\}) \cup R \cup r Q_1 u_l \cup P_3$. There is some C_i, which is odd. As C' is even, the cycle $C_i' := C_i \Delta C'$ is also odd. But $|C_i'| - |C_i| = |C'| - 2|r Q_1 u_l| > 4l - 2l = 2l$, a contradiction. The proof of Claim 6 is completed. ■

Claim 7. $l = 1$ and thus $L(G) = \{k, k+2\}$, where $k \geq 5$.

Proof. Suppose to the contrary that $l \geq 2$. By Claims 5 and 6, we see that H is not 2-connected, and all its end-blocks are 2-connected. Let D_1 be an end-block of H, $b \in V(D_1)$ be the cut-vertex of H contained in D_1, and $D_2 := H - V(D_1 - b)$. Since G is 3-connected, there exist $x_i \in V(C)$ and $y_i \in V(D_1 - b)$ such that $x_i y_i \in E(G)$. Let $y_{i-1}, y_{i+1} \in V(H)$ such that $x_{i-1} y_{i-1}, x_{i+1} y_{i+1} \in E(G)$. By Claim 5(1), y_{i-1}, y_{i+1} are distinct, and by Claim

6, $\{y_{i-1}, y_{i+1}\} \nsubseteq V(D_1)$. According to the locations of y_{i-1} and y_{i+1}, we consider the following two cases.

Suppose that exactly one of $\{y_{i-1}, y_{i+1}\}$ is in $D_2 - b$, say $y_{i-1} \in V(D_1)$ and $y_{i+1} \in V(D_2 - b)$. Since G contains no triangles, $y_{i-1} \neq y_i$. Choose $y_{i-2}, y_{i+2} \in V(H)$ such that $x_{i-2}y_{i-2}, x_{i+2}y_{i+2} \in E(G)$. We see that $y_{i-2}, y_{i+2} \in V(D_2 - b)$ (by Claim 6) and are distinct (as, otherwise, G has an odd cycle of length $k-2$). Let P be a (y_i, b)-path in D_1, P_1 a (y_{i-2}, b)-path in D_2, and P_2 a (y_{i+2}, b)-path in D_2. Then by Claim 2, $C_1 := (C - \{x_{i-1}\}) \cup x_{i-2}y_{i-2} \cup P_1 \cup P \cup y_i x_i$ and $C_2 := (C - \{x_{i+1}\}) \cup x_{i+2}y_{i+2} \cup P_2 \cup P \cup y_i x_i$ are two odd cycles with $|P_2| - |P_1| = |C_2| - |C_1| \in \{-2l, 0, 2l\}$. Let P' be a (y_{i-1}, b)-path in D_1. Then $C_3 := (y_{i-1}x_{i-1}x_{i-2}y_{i-2}) \cup P_1 \cup P'$ and $C_4 := (y_{i-1}x_{i-1}x_i x_{i+1}x_{i+2}y_{i+2}) \cup P_2 \cup P'$ are two odd cycles satisfying that

$$|C_4| - |C_3| = 2 + |P_2| - |P_1| \in \{2 - 2l, 2, 2 + 2l\} \cap \{-2l, 0, 2l\}.$$

From the non-empty intersection, one can infer that $l = 1$.

Now assume that $y_{i-1}, y_{i+1} \in V(D_2 - b)$. Let Q be a (y_i, b)-path in D_1, Q_1 a (y_{i-1}, b)-path in D_2, and Q_2 a (y_{i+1}, b)-path in D_2. Consider the odd cycles $C_5 := (y_{i-1}x_{i-1}x_i y_i) \cup Q \cup Q_1$ and $C_6 := (y_{i+1}x_{i+1}x_i y_i) \cup Q \cup Q_2$. We can deduce that $|Q_2| - |Q_1| = |C_6| - |C_5| \in \{-2l, 0, 2l\}$. Since G is 3-connected, there exist $y_j \in V(D_1 - b)$ and $x_j \in V(C) - \{x_i\}$ such that $x_j y_j \in E(G)$. Let Q_3 be a (y_j, b)-path in D_1. If x_j is one of $\{x_{i-1}, x_{i+1}\}$, then we are in the previous case. So x_j is distinct from x_{i-1}, x_{i+1}. Let X be an (x_j, x_{i-1})-path on C and X' be an (x_j, x_{i+1})-path on C such that both $|X|, |X'|$ are odd. By symmetry, let $|X'| - |X| = 2$. Then $C_7 := X \cup x_j y_j \cup Q_3 \cup Q_1 \cup y_{i-1}x_{i-1}$, and $C_8 := X' \cup x_j y_j \cup Q_3 \cup Q_2 \cup y_{i+1}x_{i+1}$ are two odd cycles with

$$|C_8| - |C_7| = |Q_2| - |Q_1| + (|X'| - |X|) \in \{2 - 2l, 2, 2 + 2l\} \cap \{-2l, 0, 2l\},$$

which again implies that $l = 1$. This proves Claim 7. ■

In [8] (see its Theorem 1.2), it was proved that every graph with $L = \{5, 7\}$ has chromatic number 3. By this result, we can assume that $k \geq 7$ in the rest of this section.

Claim 8. *H is not 2-connected.*

Proof. Suppose that H is 2-connected. Note C is the least odd cycle and $\delta(G) \geq 3$. For any two consecutive vertices $x_i, x_{i+1} \in V(C)$, there are distinct $y_i, y_{i+1} \in A$ such that $x_i y_i, x_{i+1}y_{i+1} \in E(G)$. There are 2 disjoint (y_i, y_{i+1})-paths P_1, P_2 in H, which are even. Then for each $i = 1, 2$, $C_i := P_i \cup (y_i x_i x_{i+1} y_{i+1})$ is an odd cycle, implying that $|P_i| \geq k - 3$. Then $C' := P_1 \cup P_2$ forms an even cycle of length at least $2(k-3) \geq 8$, as $k \geq 7$. Since G is 3-connected, there are 3 disjoint paths X_j, $j = 1, 2, 3$, from $u_j \in V(C')$ to $v_j \in V(C)$, internally disjoint with $C \cup C'$. Let C'_i be the (u_{i-1}, u_{i-2})-path of C', containing no u_i, where subscripts are taken mod 3. Assume that $|C'_1| \geq |C'_2| \geq |C'_3|$. So $|C'_1| + |C'_2| - |C'_3| = |C'| - 2|C'_3| \geq |C'| - 2\lfloor \frac{|C'|}{3} \rfloor \geq \lceil \frac{|C'|}{3} \rceil \geq \lceil \frac{8}{3} \rceil = 3$.

Since C is odd and C' is even, there exists a (v_1, v_2)-path P on C such that $C_3 := P \cup X_1 \cup X_2 \cup C'_3$ and $C_4 := P \cup X_1 \cup X_2 \cup (C'_1 \cup C'_2)$ are both odd. However, $|C_4| - |C_3| = |C'_1| + |C'_2| - |C'_3| \geq 3$, contradicting $L(G) = \{k, k+2\}$. This proves Claim 8. ■

Let x be a cut-vertex with $V(H_1 \cap H_2) = \{x\}$ and $H_1 \cup H_2 = H$. For a pair of vertices $\{x_i, x_{i+2}\}$ on C, we say that it is *feasible* (with respect to the cut-vertex x), if $N(x_i) \cap V(H_1 - x) \neq \emptyset$, and $N(x_{i+2}) \cap V(H_2 - x) \neq \emptyset$.

Claim 9. For any cut-vertex x of H, $N(x) \cap V(C) = \emptyset$ and there exists a feasible pair $\{x_i, x_{i+2}\}$.

Proof. If there exist $u, v \in N(x) \cap V(C)$, then u, v are of distance 2 on C, since otherwise there is an odd cycle of length less than k. This shows that $|N(x) \cap V(C)| \leq 2$. Assume, if existing, that $x_0, x_2 \in N(x) \cap V(C)$.

Suppose that there is no feasible pair. We say a vertex $x_j \in V(C)$ is of *type* i if $N_H(x_j) \subseteq V(H_i - x)$ for some $i \in \{1, 2\}$. Then every vertex in C, except x_0 and x_2, must be of certain type. By symmetry, let x_{k-2} be of type 1, then we can infer (in order) that $x_{k-4}, x_{k-6}, ..., x_1, x_{k-1}, x_{k-3}, ..., x_4$ must be all of type 1, and moreover $N_H(x_2) \subseteq V(H_1)$. This shows that $\{x, x_0\}$ is a 2-cut of G separating H_2 and $G - H_2$, but G is 3-connected, a contradiction.

Hence there exist $x_i, x_{i+2} \in V(C)$ and $y \in V(H_1 - x), z \in V(H_2 - x)$ such that $x_i y, x_{i+2} z \in E(G)$. Suppose that $N(x) \cap V(C) \neq \emptyset$. By Claim 3, $x, y, z \in A$. So every (y, z)-path P in H passes through x and thus is of even length at least 4. Then $(C - \{x_{i+1}\}) \cup x_i y \cup P \cup z x_{i+2}$ is an odd cycle of length at least $k + 4$, a contradiction. This proves Claim 9. ■

Claim 10. $|V(H)| = 3$.

Proof. By Claims 8 and 9, there exist a cut-vertex x of H with $N(x) \cap V(C) = \emptyset$ and a feasible pair $\{x_i, x_{i+2}\}$, where $V(H_1 \cap H_2) = \{x\}$ and $H_1 \cup H_2 = H$. Choose vertices $y_1 \in N(x_i) \cap V(H_1 - x)$ and $y_2 \in N(x_{i+2}) \cap V(H_2 - x)$. By Claim 2, $y_1, y_2 \in A$. If there is a (y_1, y_2)-path P in H with length at least 4, then $(C - \{x_{i+1}\}) \cup x_i y_1 \cup P \cup y_2 x_{i+2}$ is an odd cycle of length at least $k + 4$. So all (y_1, y_2)-paths in H are of length 2. This shows that for each $j \in \{1, 2\}$, $y_j x \in E(G)$ and $H - y_j x$ is disconnected. If $|V(H)| \geq 4$, then there is some $|V(H_j)| \geq 3$ and thus y_j is a cut-vertex of H, which is a contradiction to Claim 9. Thus $|V(H)| = 3$. ■

By Claims 8 and 10, let $V(H) = \{x, z_1, z_2\}$ such that $x z_1, x z_2 \in E(G)$ and $z_1 z_2 \notin E(G)$. Claim 9 shows that $N_H(C) \subseteq \{z_1, z_2\}$. So each vertex in $V(C)$ is adjacent to z_1 or z_2, which will force triangles in G. This contradiction completes the proof of Theorem 2. □

It remains to show the proofs of Lemmas 3 and 4, which we leave to Sections 4 and 5, respectively.

4 Proof of Lemma 3

In this section, we establish Lemma 3, which we restate below for the reader's convenience.

Lemma 3. *Let G be*

$$\text{a 4-critical graph with } L(G) = \{k, k + 2l\}, \text{ where } k \geq 5 \text{ and } l \geq 1. \tag{8}$$

Then G is 3-connected.

Clearly every graph G satisfying (8) is 2-connected with $\delta(G) \geq 3$. The following weak version of Lemma 4 will be crucial in the proof of Lemma 3.

Lemma 6. *For any graph G satisfying* (8)*, every two odd cycles intersect.*

Let us first prove Lemma 3, assuming the above lemma.

Proof of Lemma 3. (Assuming Lemma 6) The proof technique is similar to Corollary 4.2 in [8]. Suppose that G is not 3-connected. Then there exists a 2-separator (A, B) of G such that $V(G) = A \cup B$, $A \cap B = \{x, y\}$ and no edges are from $G[A] - \{x, y\}$ to $G[B] - \{x, y\}$. We need a result from [8] (see its Lemma 1.2), which states that for any two vertices v_1, v_2 in a 4-critical graph, there is an odd cycle containing v_1 and avoiding v_2. So for the vertex x and any vertex $u \in A - \{x, y\}$, there is an odd cycle C_1 in G containing u and avoiding x; and for the vertex y and any vertex $v \in B - \{x, y\}$, there is an odd cycle C_2 in G containing v and avoiding y. It is easy to see that $V(C_1) \subseteq A - \{x\}$ and $V(C_2) \subseteq B - \{y\}$, which imply that $V(C_1 \cap C_2) = \emptyset$. However C_1 and C_2 are odd, contradicting Lemma 6. The proof of Lemma 3 is finished. □

In the remainder of this section, we prove Lemma 6. To do so, as $L(G) = \{k, k + 2l\}$, we consider three situations: (i) two $(k+2l)$-cycles; (ii) one $(k+2l)$-cycle and one k-cycle; and (iii) two k-cycles. We will demonstrate each of the situations in a following separated subsection.

The next result will be used several times in this and forthcoming sections.

Theorem 7. ([8, Theorem 3.1]) *Let G be a graph with $|L(G)| = 1$ and C be an odd cycle in G. If G contains no K_4, then any proper 3-coloring of C can be extended to a proper 3-coloring of G.*

4.1 $(k + 2l)$-cycles intersect

We first consider the case of two $(k + 2l)$-cycles and show that it holds even for Lemma 4.

Lemma 8. *For any graph G satisfying* (8)*, every two $(k + 2l)$-cycles intersect in at least two vertices.*

Proof. Suppose to the contrary that there exist two $(k + 2l)$-cycles C_0, C_1 in G with $|V(C_0 \cap C_1)| \leq 1$. Since G is 2-connected, there are two disjoint (C_0, C_1)-paths, say R, S, from $x_0, x_1 \in V(C_0)$ to $y_0, y_1 \in V(C_1)$, respectively. In the case that $V(C_0 \cap C_1) = \{w\}$, we choose $R = \{w\}$. So we always have $|S| \geq 1$. Let X be an (x_0, x_1)-path in C_0 and Y a (y_0, y_1)-path in C_1 such that $C_2 := X \cup Y \cup R \cup S$ is an odd cycle. Then $C_3 := (C_0 \cup C_1 - C_2) \cup R \cup S$ is also an odd cycle. But $|C_2| + |C_3| = 2(|R| + |S|) + |C_0| + |C_1| > 2(k + 2l)$, a contradiction to $L(G) = \{k, k + 2l\}$. This proves the lemma. □

4.2 $(k + 2l)$-cycle intersects with k-cycle

We then consider two odd cycles of different lengths.

Lemma 9. *For any graph G satisfying* (8)*, every k-cycle and $(k + 2l)$-cycle intersect.*

Proof. Suppose to the contrary that there exist some k-cycle C_0 and $(k + 2l)$-cycle C_1 in G with $V(C_0 \cap C_1) = \emptyset$. We will prove three claims, which lead us to contradictions.

(A). For any vertex $u \in V(C_0)$, there is a (u, C_1)-path internally disjoint from $C_0 \cup C_1$.

Proof. Since C_0 is induced (as it is a least odd cycle) and $\delta(G) \geq 3$, for any vertex $u \in V(C_0)$, there exists a neighbor of u not in C_0. Now suppose that (A) fails. Then there exist some $u \in V(C_0)$ and C_0-bridge H such that $u \in V(H)$ and $V(C_1 \cap H) = \emptyset$. Let

$G_0 := G[H \cup C_0]$ and $G_1 := G - V(H - C_0)$. Note that G_1 is a proper subgraph of G. Since G is 4-critical, G_1 has a proper 3-coloring φ. If there is a $(k+2l)$-cycle in G_0, say C_2, then $V(C_1 \cap C_2) \subset V(C_0 \cap C_1) = \emptyset$, a contradiction to Lemma 8. Thus $L(G_0) = \{k\}$. By Theorem 7, the restriction of φ on C_0 can be extended to a proper 3-coloring of G_0. This gives a proper 3-coloring of G, a contradiction to (8). ■

(B). Let R, S be any two disjoint (C_0, C_1)-paths from $x_0, x_1 \in V(C_0)$ to $y_0, y_1 \in V(C_1)$, respectively. Let X be any path from x_0 to x_1 on C_0, and Y be any path from y_0 to y_1 on C_1. Then $|R| + |S| = l$, and $|X| = k + l - |Y|$ or $|Y| - l$.

Proof. Set $C_2 := X \cup Y \cup R \cup S$, and $C_3 := (C_0 \cup C_1 - C_2) \cup R \cup S$. If C_2 is odd, then C_3 is odd, and $|C_2| + |C_3| = 2(|R| + |S|) + 2k + 2l$. Since $L(G) = \{k, k+2l\}$, we can then infer that $|C_2| = |C_3| = k + 2l$, $|R| + |S| = l$ and $|Y| = (k - |X|) + l$. If C_2 is even, repeat the above proof using $X' := C_0 - X$ instead of X. In this case, it holds that $|R| + |S| = l$ and $|Y| = (k - |X'|) + l$, implying that $|Y| = |X| + l$. ■

(C). There are three disjoint (C_0, C_1)-paths.

Proof. Since G is 2-connected, there are two disjoint (C_0, C_1)-paths, say R, S, from $x_0, x_1 \in V(C_0)$ to $y_0, y_1 \in V(C_1)$, respectively. By (B), $|R| + |S| = l$. Let P, Q be the two (x_0, x_1)-paths on C with $|P| \le |Q|$. Since $|C_0| \ge 5$, we have $|Q| \ge 3$. Let x_2 be any vertex in $V(Q) \backslash \{x_0, x_1\}$. We draw C_0 in the planar such that x_0, x_1, x_2 appear on C clockwise. Define $\alpha_i := |x_i C_0 x_{i+1}|$, where subscripts are taken mod 3. By (A), there is an (x_2, C_1)-path, say T, internally disjoint from $C_0 \cup C_1$. Suppose that (C) fails. Then every such T intersects with $R \cup S$.

Let $z \in V(T \cap (R \cup S))$ such that $|x_2 T z|$ is minimal. If $z \in V(R)$, then $R' := x_2 T z \cup z R y_0$ and S are two disjoint (C_0, C_1)-paths. Consider the following paths $x_0 C_0 x_1, x_1 C_0 x_2, x_2 C_0 x_1$ and $x_1 C_0 x_0$. By (B), we obtain that $\alpha_0, \alpha_1, \alpha_0 + \alpha_2, \alpha_1 + \alpha_2 \in \{k + l - |Y|, |Y| - l\}$, where Y is a (y_0, y_1)-path on C_1. This implies that $\alpha_0 = \alpha_1$. If $z \in V(S)$, then by symmetry, we obtain $\alpha_0 = \alpha_2$. Note that x_2 can be picked to be any vertex in $V(Q) \backslash \{x_0, x_1\}$. This shows that for any such x_2, either $|x_1 C_0 x_2|$ or $|x_2 C_0 x_0|$ equals α_0. Thus $|V(Q) \backslash \{x_0, x_1\}| \le 2$, which, together with $|Q| \ge 3$, imply that $|Q| = 3$. Then $|C_0| = 5$ and $|P| = 2$.

Let $C_0 := x_0 a x_1 b c x_0$. Then there exist (c, C_1)-path T_1 and (b, C_1)-path T_2 intersecting with $R \cup S$ and internally disjoint with $C_0 \cup C_1$. By the above proof, any such T_1 (from the direction starting from c) should first intersect with R and any such T_2 (from the direction starting from b) should first intersect with S. So there exist $u \in V(T_1 \cap R)$ and $v \in V(T_2 \cap S)$ such that $c T_1 u \cup u R y_0$ and $b T_2 v \cup v S y_1$ are disjoint. Let Y be a (y_0, y_1)-path on C_1. By (B), $\{|x_0 a x_1|, |x_1 b c x_0|, |bc|\} = \{1, 2, 3\} \subseteq \{k + l - |Y|, |Y| - l\}$, which of course is a contradiction. This proves (C). ■

Hence, there are three disjoint (C_0, C_1)-paths, say P_i's, from $x_i \in V(C_0)$ to $y_i \in V(C_1)$, for $i \in \{0, 1, 2\}$. By (B), $|P_1| + |P_2| = |P_1| + |P_3| = |P_2| + |P_3| = l$, thus $3l = 2(|P_1| + |P_2| + |P_3|)$, implying that l is even.

Observe that the subgraph $C_0 \cup C_1 \cup P_0 \cup P_1 \cup P_2$ is planar. So we can draw it in the plane such that x_0, x_1, x_2 appear on C_0 clockwise and y_0, y_1, y_2 appear on C_1 counterclockwise. Define $\alpha_i := |x_i C_0 x_{i+1}|$ and $\beta_i := |y_{i+1} C_1 y_i|$, where the subscripts are taken modulo 3. So $\alpha_0 + \alpha_1 + \alpha_2 = k$ and $\beta_0 + \beta_1 + \beta_2 = k + 2l$. By (B), for any $i \in \{0, 1, 2\}$, $\beta_i = \alpha_i + l$ or $\beta_i + \alpha_i = k + l$. We discuss all possible cases. If $\beta_i = \alpha_i + l$ for all i, then $\beta_0 + \beta_1 + \beta_2 = k + 3l$, a contradiction. If $\beta_i + \alpha_i = k + l$ for all i, then $\beta_0 + \beta_1 + \beta_2 = 3k + 3l - (\alpha_0 + \alpha_1 + \alpha_2) = 2k + 3l$, a contradiction. If exactly two i's satisfy $\beta_i = \alpha_i + l$, say $i = 0, 1$, then $\beta_0 + \beta_1 + \beta_2 = k + 3l + \alpha_0 + \alpha_1 - \alpha_2$, implying that $k + l = 2\alpha_2$ is even, a contradiction to the facts that k is odd and l is even. So there is exactly one i, say $i = 0$, satisfying $\beta_i = \alpha_i + l$. Then $\beta_0 + \beta_1 + \beta_2 = 2k + 3l + \alpha_0 - \alpha_1 - \alpha_2$. This shows that $k + 2l = k + 3l + 2\alpha_0$, which cannot be. This finishes the proof of Lemma 9. □

4.3 k-cycles intersect

Lastly, we consider two k-cycles and prove the following result, thereby completing the proof of Lemma 3. Our proof is dependent of a well-known result due to Dirac [4] (also see [1, pp.367–368]).

Lemma 10. *For any graph G satisfying* (8), *every two k-cycles intersect.*

Theorem 11. *[4] Let G be a k-critical graph with a 2-vertex cut $\{u, v\}$. Then:*
(1) $uv \notin E(G)$;
(2) $G = G_1 \cup G_2$, where $G_1 \cap G_2 = \{u, v\}$, and $H_1 := G_1 + uv$ is k-critical.

Proof. Suppose to the contrary that there exist two k-cycles C_0, C_1 in G with $V(C_0 \cap C_1) = \emptyset$. The case $l = 1$ was solved in Proposition 4.1 of [8], so we assume that $l \geq 2$. Write $C_0 := x_0 x_1 ... x_{k-1} x_0$ throughout this proof. We divide the proof into a sequence of claims.

Claim 1. For each $i \in \{0, 1\}$ and each vertex $u \in V(C_i)$, there is a (u, C_{1-i})-path P_u in G, internally disjoint from $C_0 \cup C_1$.

Proof. By symmetry, we may only consider vertices in C_0. Suppose to the contrary that there exists $u \in C_0$ and some C_0-bridge H such that $u \in V(H)$ and $V(H \cap C_1) = \emptyset$. Let $G_0 := G[H \cup C_0]$ and $G_1 := G - V(H - C_0)$. As G is 4-critical, G_1 has a proper 3-coloring φ. If G_0 contains a $(k + 2l)$-cycle, say C_2, then $V(C_1 \cap C_2) \subset V(C_0 \cap C_1) = \emptyset$, a contradiction to Lemma 9. Thus $L(G_0) = \{k\}$. By Theorem 7, the restriction of φ on C_0 can be extended to a proper 3-coloring of G_0. This gives a proper 3-coloring of G, a contradiction. ■

Claim 2. Let R, S be any two disjoint (C_0, C_1)-paths from $x_i, x_j \in V(C_0)$ to $y_i, y_j \in V(C_1)$, respectively. Let X be any (x_i, x_j)-path on C_0 and Y be any (y_i, y_j)-path on C_1. Let $t := |R| + |S|$. Then $t \in \{l, 2l\}$. If $t = l$, then $||X| + |Y| - k| = l$ or $||X| - |Y|| = l$; and if $t = 2l$, then $|Y| = |X|$ or $k - |X|$. In particular, when $1 \leq |X| \leq 2$, we have $|Y| \in \{l + |X|, k - l - |X|\}$ if $t = l$, and $|Y| \in \{|X|, k - |X|\}$ if $t = 2l$.

Proof. Let $C_2 := X \cup Y \cup R \cup S$ and $C_3 := (C_0 \cup C_1 - C_2) \cup R \cup S$. Then $|C_2| + |C_3| = 2(|R|+|S|)+|C_0|+|C_1| = 2(|R|+|S|)+2k$. If C_2 is odd, then C_3 is also odd. There are three cases: (a) $|C_2| = k$, $|C_3| = k + 2l$; (b) $|C_2| = k + 2l$, $|C_3| = k$; and (c) $|C_2| = |C_3| = k + 2l$. In each case, we can infer that $|Y| = (k - |X|) - l$ and $t = l$; $|Y| = (k - |X|) + l$ and $t = l$; $|Y| = k - |X|$ and $t = 2l$, respectively. Otherwise, C_2 is even. Then we can repeat the above proof by using $X' := C_0 - X$ instead of X. Similarly, we have $t \in \{l, 2l\}$, and it is a routine matter to verify other quantities. The result when $1 \leq |X| \leq 2$ easily follows by the facts that $l \geq 2$ and $1 \leq |Y| \leq k - 1$. ■

Claim 3. Let P_1, P_2, P_3 be three disjoint (C_0, C_1)-paths of G, with $|P_1| \leq |P_2| \leq |P_3|$. Then one of the followings holds:
(1) $|P_1| = |P_2| = |P_3| = l/2$; (2) $|P_1| = |P_2| = |P_3| = l$; (3) $|P_1| = |P_2| = l/2, |P_3| = 3l/2$.

Proof. Set $|P_1| = a, |P_2| = b, |P_3| = c$. By Claim 2, each of $a+b, a+c$ and $b+c$ must be in $\{l, 2l\}$. Consider the vector $(a+b, a+c, b+c)$, which cannot be $(l, l, 2l)$. Therefore, the vector only can be $(l,l,l), (l, 2l, 2l)$ or $(2l, 2l, 2l)$, which gives that $(a,b,c) = (\frac{l}{2}, \frac{l}{2}, \frac{l}{2}), (\frac{l}{2}, \frac{l}{2}, \frac{3l}{2})$ or (l,l,l), respectively. This proves Claim 3. ∎

Claim 4. There exist two disjoint (C_0, C_1)-paths from two consecutive vertices $x_i, x_{i+1} \in V(C_0)$ to $V(C_1)$.

Proof. Since G is 2-connected, there are two disjoint (C_0, C_1)-paths P_1, P_2, say from $x_i, x_j \in V(C_0)$ to $y_i, y_j \in V(C_1)$, respectively. We choose P_1, P_2 such that $d_{C_0}(x_i, x_j)$ is minimal. It is enough to show that $d_{C_0}(x_i, x_j) = 1$. Suppose to the contrary that there exists some vertex $x_m \in V(X) - \{x_i, x_j\}$, where X is the shorter (x_i, x_j)-path on C_0. By Claim 1, there exists an (x_m, C_1)-path Q, which is internally disjoint from $C_0 \cup C_1$. If Q is disjoint from some P_i, then P_i, Q is a pair of disjoint (C_0, C_1)-paths with a shorter distance on C_0, a contradiction. So Q intersects $P_1 \cup P_2$. Let $z \in V(Q) \cap V(P_1 \cup P_2)$ be the vertex such that $|x_m Q z|$ is the minimum, say $z \in P_1$. Then $x_m Q z \cup z P_1 y_1$ together with P_2 form a pair of disjoint (C_0, C_1)-paths with $d_{C_0}(x_m, x_j) < d_{C_0}(x_i, x_j)$, a contradiction. ∎

Let P_i, P_{i+1} be two disjoint (C_0, C_1)-paths from consecutive $x_i, x_{i+1} \in V(C_0)$ to some $y_i, y_{i+1} \in V(C_1)$, respectively. Let P_{i+2} be a path from x_{i+2} to $z \in V(P_i \cup P_{i+1}) - \{x_i, x_{i+1}\}$ internally disjoint from $P_i \cup P_{i+1} \cup C_0 \cup C_1$. Let $t := |P_i| + |P_{i+1}|$. Then the followings hold.

Claim 5. (1) If $z \in V(P_{i+1})$, then $k = l+3$, $|x_{i+1} P_{i+1} z| = t - l + 1$ and $|P_{i+2}| = 2l + 1 - t$. (2) If $z \in V(P_i)$, then $|x_i P_i z| = |P_{i+2}| = 1$ or $l+1$; in the latter case, we have $z = y_i$.

Proof. Without loss of generality, let $i = 0$ and P_0, P_1 be from x_0, x_1 to $y_0, y_1 \in V(C_1)$, respectively. By Claim 2, $t \in \{l, 2l\}$. Let Y be a (y_0, y_1)-paths of C_1.

First consider $z \in V(P_1)$. Let $P_2' := P_2 \cup z P_1 y_1$ and $C_2 := x_1 P_1 z \cup P_2 \cup x_1 x_2$. Let $s := |P_0| + |P_2'|$. By Claim 2, if $s = l$ then $|Y| \in \{l+2, k-l-2\}$; and if $s = 2l$ then $|Y| \in \{2, k-2\}$. Similarly, if $t = l$ then $|Y| \in \{l+1, k-l-1\}$; and if $t = 2l$ then $|Y| \in \{1, k-1\}$. As a consequence, t, s cannot both be $2l$ (as, otherwise, we can obtain $k = 3$, a contradiction). If $t = s = l$, then $\{l+1, k-l-1\} \cap \{l+2, k-l-2\} \neq \emptyset$ implies that $k = 2l+3$; moreover, $x_1 P_1 z$ has the same length as P_2, implying that C_2 is odd. Thus $|x_1 P_1 z| + |P_2| \geq k - 1 = 2l + 2$, contradicting the fact that $|x_1 P_1 z| + |P_2| \leq |P_1| + |P_2'| \leq s + t = 2l$. Hence, $\{t, s\} = \{l, 2l\}$, and in this case, we can always get $k = l + 3$. Let $r := \min\{|x_1 P_1 z|, |P_2|\}$ and $r' := \max\{|x_1 P_1 z|, |P_2|\}$. Note that $|C_2| = 1 + (r' - r) + 2r = 1 + |t - s| + 2r = k - 2 + 2r$ is odd. If $|C_2| = k + 2l$, then $r = l + 1 \leq \min\{t, s\} = l$, a contradiction. So $|C_2| = k$ and thus $r = 1$, $r' = 1 + l$. The left part is easy to check. This proves (1).

Now suppose $z \in V(P_0)$. Let $P_2' := P_2 \cup z P_0 y_0$ and $s := |P_1| + |P_2'|$. By Claim 2, $s, t \in \{l, 2l\}$. If $s \neq t$, then by Claim 2, $|Y| \in \{1, k-1\} \cap \{l+1, k-l-1\}$. This implies $k = l+2$. Let $r := \max\{|x_0 P_0 z|, |P_2|\}$ and $r' := \min\{|x_0 P_0 z|, |P_2|\}$ with $r - r' = |s - t| = l$. Then $(x_0 x_1 x_2) \cup P_2 \cup x_0 P_0 z$ is an odd cycle with length $2 + l + 2r' = k + 2r'$, implying that $r' = l$ and $r = 2l$. This is a contradiction to $r < \max\{s, t\} = 2l$. So $s = t$. Then $|x_0 P_0 z| = |P_2|$, implying that $C_3 := x_0 P_0 z \cup P_2 \cup (C_0 - \{x_1\})$ is an odd cycle with length $k - 2 + 2|P_2|$. Thus $|P_2| = 1$ or $l+1$. In the later case, C_3 is of length $k + 2l$ and by Lemma 9, we must have $z = y_0$. This proves (2). ∎

Claim 6. There exist three disjoint (C_0, C_1)-paths from consecutive $x_i, x_{i+1}, x_{i+2} \in V(C_0)$ to $V(C_1)$.

Proof. By Claim 4, we may assume that there exist two disjoint (C_0, C_1)-paths P_0, P_1 from x_0, x_1 to $y_0, y_1 \in V(C_1)$, respectively. Write $t := |P_0| + |P_1| \in \{l, 2l\}$ (by Claim 2). For each $i \in \{2, k-1\}$, let P'_i be the (x_i, C_1)-path from Claim 1. Suppose that each P'_i intersects $P_0 \cup P_1$. Let $z \in V(P'_2) \cap V(P_0 \cup P_1)$ be such that $P_2 := x_2 P'_2 z$ is the shortest. Similarly, let $w \in V(P'_{k-1}) \cap V(P_0 \cup P_1)$ be such that $P_{k-1} := x_{k-1} P'_{k-1} w$ is the shortest.

We show that P_2 and P_{k-1} are internally disjoint. Suppose not. Then there exists $x \in V(P_2 \cap P_{k-1})$ such that $xP_{k-1}x_{k-1}, xP_2x_2$ and xP_2z are internally disjoint. At this point, one actually need not to distinguish between xP_2z and $xP_{k-1}w$, and thus we may assume, without loss of generality, that $z \in V(P_1)$. Let $C_2 := x_2P_2x \cup xP_{k-1}x_{k-1} \cup (C_0 - \{x_0, x_1\})$, and $C_3 := x_2P_2x \cup xP_{k-1}x_{k-1} \cup (x_{k-1}x_0x_1x_2)$. If C_2 is odd, as $V(C_2 \cap C_1) = \emptyset$, by Lemma 9 we infer that $|C_2| = k$. Let $P_3 := xP_2z \cup zP_1y_1$, $P_4 := (x_2x_1x_0) \cup P_0$, and $P_5 := x_{k-1}x_0 \cup P_0$. Note that $\{P_3, P_4\}$ and $\{P_3, P_5\}$ are both pairs of disjoint (C_1, C_2)-paths. By Claim 2, $|P_3| + |P_4|, |P_3| + |P_5| \in \{l, 2l\}$, where $|P_5| = |P_4| + 1$. This implies $l = 1$, a contradiction. Hence C_3 is odd. As $V(C_3 \cap C_2) = \emptyset$, we infer that $|C_3| = k$. Note that P_0, P_3 are disjoint (C_1, C_3)-paths, and $P_0, P_2 \cup zP_1y_1$ are disjoint (C_0, C_1)-paths. Since P_3 is a subpath of $P_2 \cup zP_1y_1$, we have $|P_0| + |P_3| = l$ and $|P_0| + |P_2 \cup zP_1y_1| = 2l$, implying $|x_2P_2x| = l$. By Claim 5, $k = l + 3$. Then $|C_3| > l + 3 = k$, a contradiction. Therefore indeed P_2 and P_{k-1} are internally disjoint.

Next we discuss the locations of z, w. First assume that both $z, w \in V(P_j)$, say $j = 1$. By Claim 5(1), we have $k = l + 3$, $|x_1P_1z| = t - l + 1$ and $|P_2| = 2l + 1 - t$; and by Claim 5(2), we get $|x_1P_1w| = |P_{k-1}| = 1$ or $l + 1$. Let $C_4 := P_2 \cup P_{k-1} \cup wP_1z \cup (x_2x_1x_0x_{k-1})$. If $t = l$ and $|x_1P_1w| = |P_{k-1}| = 1$, then $|x_1P_1z| = 1$ (implying $z = w$) and $|P_2| = l + 1 = k - 2$, implying $|C_4| = k + 2$ and thus $l = 1$, a contradiction. If $t = l$ and $|x_1P_1w| = |P_{k-1}| = l + 1$, then $|x_1P_1z| = 1$, $|P_2| = l + 1$ and $w = y_1$ (by Claim 5(2)), implying that $|C_4| = 2(l + 1) + l + 3 = k + 2l + 2$, a contradiction. Hence $t = 2l$. So $|x_1P_1z| = l + 1$ and $|P_2| = 1$. If $|x_1P_1w| = |P_{k-1}| = 1$, then $w \in x_1P_1z$ and $|wP_1z| = l$, implying $|C_4| = l + 5 = k + 2$, a contradiction; otherwise $|x_1P_1w| = |P_{k-1}| = l + 1$, then $w = z$, also implying $|C_4| = l + 5 = k + 2$, a contradiction.

Suppose $z \in V(P_1)$ and $w \in V(P_0)$. By Claim 5, $k = l+3$, $|x_1P_1z| = |x_0P_0w| = t-l+1$ and $|P_2| = |P_{k-1}| = 2l + 1 - t$. Then $(C_0 - \{x_0x_{k-1}, x_1x_2\}) \cup x_1P_1z \cup P_2 \cup x_0P_0w \cup P_{k-1}$ is an odd cycle of length $(k - 2) + 2(l + 2) = k + 2l + 2$, a contradiction.

Lastly we consider $z \in V(P_0)$ and $w \in V(P_1)$. By Claim 5, $|P_2| = |x_0P_0z| \in \{1, l + 1\}$ and $|P_{k-1}| = |x_1P_1w| \in \{1, l+1\}$. Then $(C_0 - \{x_0x_{k-1}, x_1x_2\}) \cup x_1P_1w \cup P_{k-1} \cup x_0P_0z \cup P_2$ is an odd cycle of some length s, where $s \in \{k + 2, k + 2l + 2, k + 4l + 2\}$. Note that each case yields a contradiction. This completes the proof of Claim 6. ∎

In the rest of this proof, we write $C_1 = u_0u_1...u_{k-1}u_0$. By Claim 6, we may assume that there exist three disjoint (C_0, C_1)-paths P_0, P_1, P_2 from consecutive $x_0, x_1, x_2 \in V(C_0)$ to $y_0, y_1, y_2 \in V(C_1)$, respectively. In view of Claim 3, we can get

$$(|P_0|, |P_1|, |P_2|) \in \left\{ (\frac{l}{2}, \frac{3l}{2}, \frac{l}{2}), (\frac{3l}{2}, \frac{l}{2}, \frac{l}{2}), (\frac{l}{2}, \frac{l}{2}, \frac{3l}{2}), (l, l, l), (\frac{l}{2}, \frac{l}{2}, \frac{l}{2}) \right\}.$$

Define Y_i be the path of C_1 from y_i to y_{i+1} not containing y_{i+2}, where the indices are taken modulo 3. Let $\beta_i := |Y_i|$ such that $\beta_0 + \beta_1 + \beta_2 = k$. In the following, without loss of generality, we draw C_0, C_1 on the plane such that x_i's appear on C_0 clockwise, and u_j's appear on C_1 anticlockwise.

Claim 7. $(|P_0|, |P_1|, |P_2|) = (\frac{l}{2}, \frac{l}{2}, \frac{l}{2})$ and thus l is even.

Proof. First suppose $(|P_0|, |P_1|, |P_2|) = (\frac{l}{2}, \frac{3l}{2}, \frac{l}{2})$. Note that l is even. By Claim 2, $\beta_0, \beta_1 \in \{1, k-1\}$ and $\beta_2 \in \{l+2, k-l-2\}$. As $\beta_1 + \beta_2 + \beta_3 = k$, we must have $\beta_0 = \beta_1 = 1$ and thus $\beta_2 = k-2$ is odd, a contradiction.

Next, we assume that $(|P_0|, |P_1|, |P_2|) = (\frac{3l}{2}, \frac{l}{2}, \frac{l}{2})$. Note that l is even. By Claim 2, $\beta_0 \in \{1, k-1\}$, $\beta_1 \in \{l+1, k-l-1\}$ and $\beta_2 \in \{2, k-2\}$. Note that $\beta_0+\beta_1+\beta_2 = k$. Clearly $\beta_0 = 1$. If $\beta_2 = k-2$, then β_1 is odd and thus $\beta_1 = l+1$, implying $\beta_0 + \beta_1 + \beta_2 = k+l$, a contradiction. Therefore $\beta_0 = 1$ and $\beta_2 = 2$. So β_1 is even and thus $\beta_1 = k-l-1$. This shows that $k = k-l+2$, implying $l = 2$. Now we have $|P_0| = 3, |P_1| = |P_2| = |Y_0| = 1$, $|Y_1| = k-3$ and $|Y_2| = 2$. Without loss of generality, let $y_0 = u_0, y_1 = u_1$ and $y_2 = u_{k-2}$. By Claim 1, there exists an (x_3, C_1)-path. So there exists a path P_3 from x_3 to $z \in V(P_0 \cup P_1 \cup P_2 \cup C_1)$ internally disjoint from $P_0 \cup P_1 \cup P_2 \cup C_0 \cup C_1$. We consider the location of z. If $z \in V(P_0)$, then $P_3 \cup zP_0u_0$ and P_2 are two disjoint (C_0, C_1)-paths from x_3, x_2 to u_0, u_{k-2}, respectively. By Claim 2, $2 = |Y_2| \in \{1, k-1\}$ or $\{l+1, k-l-1\}$. Since $l = 2$, the only possibility is $2 = k-3$. Thus $k = 5$ and $|P_3 \cup zP_0u_0| + |P_2| = l = 2$, implying that $z = u_0$ and $|P_3| = 1$. Then $P_3 \cup P_0 \cup (x_0x_1x_2x_3)$ is an odd cycle of length 7, a contradiction. So, $z \in V(C_1) - \{u_0\}$ (as $|P_1| = |P_2| = 1$) and P_3 is disjoint from P_0. By Claim 2, we see that $|P_0| + |P_3| \in \{2, 4\}$ and so $|P_3| = 1$. If $z = u_1$, then $P_3 \cup (C_1 - \{u_0u_1\}) \cup P_0 \cup (x_0x_1x_2x_3)$ is an odd cycle of length $k+6$, a contradiction. If $z = u_{k-2}$, then G has a triangle on $\{x_2, x_3, z\}$, a contradiction. If $z = u_{k-1}$, then $P_3 \cup (C_1 - \{u_{k-1}u_{k-2}\}) \cup P_2 \cup x_2x_3$ is an odd cycle of length $k+2$, a contradiction to $l = 2$. Thus $z \in V(Y_1) - \{u_1, u_{k-2}\}$, then by Claim 2, $|zY_1u_{k-2}| \in \{3, k-3\}$. Since $|zY_1u_{k-2}| < |Y_1| = k-3$, we obtain that $|zY_1u_{k-2}| = 3$. Then $P_3 \cup zY_1u_1 \cup P_1 \cup (x_1x_2x_3)$ is an odd cycle of length $(k-6)+4 = k-2$, a contradiction. By symmetry, we can prove $(|P_0|, |P_1|, |P_2|) \neq (\frac{l}{2}, \frac{l}{2}, \frac{3l}{2})$.

Lastly, we suppose $(|P_0|, |P_1|, |P_2|) = (l, l, l)$. So $|P_i| + |P_j| = 2l$. By Claim 2, $\beta_0, \beta_1 \in \{1, k-1\}$ and $\beta_2 \in \{2, k-2\}$. It is easy to see that $\beta_0 = \beta_1 = 1$ and $\beta_2 = k-2$. Without loss of generality, let $y_i = u_i$ for $i \in \{0, 1, 2\}$. By Claim 1, there exists an (x_3, C_1)-path internally disjoint from $C_0 \cup C_1$. So there exists a path P_3 from x_3 to $z \in V(P_0 \cup P_1 \cup P_2 \cup C_1)$ internally disjoint from $P_0 \cup P_1 \cup P_2 \cup C_0 \cup C_1$. Similarly as the above analysis, we consider four cases. If $z \in V(P_0)$, then $P_3 \cup zP_0u_0$ and P_1 are two disjoint (C_0, C_1)-paths from x_3, x_1 to u_0, u_1, respectively. However, this is a contradiction to Claim 2, as $|P_3 \cup zP_0u_0| + |P_1|$ is larger than l and thus equals $2l$, which implies $\beta_0 \in \{2, k-2\}$. If $z \in V(P_1)$, by Claim 5(2), $|x_1P_1z| = |P_3|$. Then $(x_0x_1x_2x_3) \cup P_3 \cup zP_1y_1 \cup (C_1 - Y_0) \cup P_0$ is an odd cycle of length $3+2l+(k-1) = k+2l+2$, a contradiction. If $z \in V(P_2)$, by Claim 5(1), we get $k = l+3$, $|x_2P_2z| = l+1 > |P_2| = l$, a contradiction. Lastly, we consider $z \in V(C_1) - \{u_0, u_1, u_2\}$. In this case, P_1, P_2, P_3, P_4 are four disjoint (C_0, C_1)-paths. By Claim 3, $|P_3| = l$. By Claim 2, we see that $z = y_3$. Then $(C_1 - \{y_0y_1, y_2y_3\}) \cup \{x_0x_1, x_2x_3\} \cup P_0 \cup P_1 \cup P_2 \cup P_3$ is an odd cycle of length $k+4l$, a contradiction. This proves Claim 7. ■

Claim 8. $L(G) = \{5, 9\}$, and any two disjoint 5-cycles C_0, C_1 in G induce a Petersen graph $G[V(C_0 \cup C_1)]$.

Proof. Note that l is even. By Claim 2, $\beta_0, \beta_1 \in \{l+1, k-l-1\}$ and $\beta_2 \in \{l+2, k-l-2\}$. Since $\beta_0 + \beta_1 \neq k$, we have $\beta_0 = \beta_1$ and thus β_2 must be odd, so $\beta_2 = k-l-2$. Since $2(l+1)+(k-l-2) > k$, we have $\beta_0 = \beta_1 = k-l-1$ and thus $2(k-l-1)+(k-l-2) = k$.

We then get

$$k = \frac{3l}{2} + 2, \tag{9}$$

which implies that $\beta_0 = \beta_1 = \frac{l}{2} + 1$ and $\beta_2 = \frac{l}{2}$. Observe that $\frac{l}{2}$ is odd. Applying Claim 1 for x_3, we see that there is a path, say P_3, from x_3 to $z \in V(P_0 \cup P_1 \cup P_2 \cup C_1)$, internally disjoint from $P_0 \cup P_1 \cup P_2 \cup C_0 \cup C_1$.

We show that $z \in V(C_1) - \{y_0, y_1, y_2\}$. Suppose not, we consider three cases that $z \in V(P_i)$ for $i = 0, 1, 2$. If $z \in V(P_0)$, then $P_3 \cup zP_0y_0$ and P_2 are two disjoint (C_0, C_1)-paths from x_3, x_2 to y_0, y_2, respectively. Let $t = |P_3 \cup zP_0u_0| + |P_2|$. By Claim 2, if $t = l$, then $\beta_0 + \beta_1 \in \{l+1, k-l-1\} = \{l+1, \frac{l}{2}+1\}$. However, $\beta_0 + \beta_1 = l + 2$, a contradiction. If $t = 2l$, then $\beta_0 + \beta_1 \in \{1, k-1\} = \{1, \frac{3l}{2}+1\}$, and thus $\frac{3l}{2} + 1 = l + 2$, implying $l = 2$. So, $k = 5$ and $L(G) = \{5, 9\}$. And $P_0 := x_0y_0 \in E(G)$, $z = y_0$. But $C_2 := P_3 \cup P_0 \cup (x_0x_1x_2x_3)$ gives a 7-cycle, yielding a contradiction. If $z \in V(P_1)$, by Claim 5(2), $|P_3| = |x_1P_1z|$, and then, $P_3 \cup zP_1y_1 \cup Y_0 \cup P_0 \cup (x_0x_1x_2x_3)$ is an odd cycle of length $\frac{l}{2} + \frac{l}{2} + 1 + \frac{l}{2} + 3 = k + 2$. This implies that $l = 1$, a contradiction. If $z \in V(P_2)$, then by Claim 5(1), we get $k = l+3$. Together with (9), this yields that $(l, k) = (2, 5)$. By Claim 2, we obtain $|P_2| = 1$, $|P_3| = 3$ and $z = y_2$. However, $P_3 \cup zC_1y_0 \cup P_0 \cup (x_0x_1x_2x_3)$ is an odd cycle of length 11, a contradiction. Therefore, indeed, $z \in V(C_1) - \{y_0, y_1, y_2\}$ and thus P_0, P_1, P_2, P_3 are pairwise disjoint paths.

By Claim 3, $|P_3| \in \{\frac{l}{2}, \frac{3l}{2}\}$. Let $y_0 = u_0$, $y_1 = u_{1+l/2}$, $y_2 = u_{l+2}$, and Y be a (y_2, z)-path on C_1. First suppose that $|P_3| = \frac{3l}{2}$. By Claim 2, we deduce that $|Y| \in \{1, k-1\}$, which implies $z = u_{l+1}$ or u_{l+3}. If $z = u_{l+3}$, then $P_3 \cup y_0C_1z \cup P_0 \cup (x_0x_1x_2x_3)$ is an odd cycle of length $\frac{3l}{2} + \frac{l}{2} - 1 + \frac{l}{2} + 3 = k + l$, a contradiction; if $z = u_{l+1}$, then $P_3 \cup zC_1y_1 \cup P_1 \cup (x_1x_2x_3)$ is also an odd cycle of length $\frac{3l}{2} + \frac{l}{2} + \frac{l}{2} + 2 = k + l$, again a contradiction. Thus, $|P_3| = \frac{l}{2}$. From Claim 2, we infer that $z = y_1$ (this cannot happen) or $z = u_1$. Then $(C_0 - \{x_1, x_2\}) \cup P_3 \cup y_0z \cup P_0$ is an odd cycle of length $k + l - 2$. This shows that $l = 2$ and by (9), we have $L(G) = \{5, 9\}$. We then see that C_0, C_1 are both 5-cycles, and $x_0u_0, x_1u_2, x_2u_4, x_3u_1 \in E(G)$. Consider the path P_4 from x_4 to $w \in V(P_0 \cup P_1 \cup P_2 \cup C_1)$, internally disjoint from $P_0 \cup P_1 \cup P_2 \cup C_0 \cup C_1$. By the symmetry between x_3 and x_4, similarly as above, we can derive that $|P_4| = 1$ and $w = u_3$. In view of $L(G) = \{5, 9\}$, now it is easy to verify that $G[V(C_0 \cup C_1)]$ induces a Petersen graph. ■

Claim 9. Let $H := G[V(C_0 \cup C_1)]$. For any $u \notin V(H)$, there are 3 disjoint paths from u to H.

Proof. Otherwise, there exists a 2-separation (G_1, G_2) such that $G = G_1 \cup G_2$, $G_1 \cap G_2 = \{a, b\}$, $\{u\} \subset G_1$, and $H \subset G_2$. Since G is 4-critical, by Theorem 11, $ab \notin E(G)$ and $H_1 := G_1 + (ab)$ is 4-critical. First we claim that there is a 5-cycle D in G_2 which is disjoint with $\{a, b\}$. This can be deduced from an easy observation that Petersen graph has a 5-cycle disjoint from any two prescribed nonadjacent vertices of it. Next we show that $G_1 - a$ contains a 5-cycle. Note that $G_1 - a = H_1 - a$ is 3-chromatic (since H_1 is 4-critical). So, there is an odd cycle in $G_1 - a$, say D'. If D' is a 9-cycle, then D and D' are disjoint 5-cycle and 9-cycle in G, a contradiction to Lemma 9. Thus D' is a 5-cycle. Observe that there is at most one edge connecting D and D'. So $G[V(D \cup D')]$ cannot be a Petersen graph, a contradiction to Claim 8. This proves Claim 9. ■

Now we are ready to finish the proof of Lemma 10. If $G = H$, then G is 3-colorable, a contradiction. Thus, there exists $u \notin V(H)$. By Claim 9, there are 3 disjoint paths, say Q_1, Q_2, Q_3, from u to $v_1, v_2, v_3 \in V(H)$, respectively. Since G contains no triangle, we may assume that $v_1v_2 \notin E(G)$. Observe that there always exists (v_1, v_2)-paths of lengths 2,3,4,5 in the Petersen graph H for any nonadjacent vertices v_1, v_2. These paths, together with the path $P_1 \cup P_2$ (internally disjoint from H), form two odd cycles of lengths differ by two, a contradiction to $L(G) = \{5, 9\}$. This final contradiction proves Lemma 10. □

Putting Lemmas 8, 9 and 10 together, we now complete the proof of Lemma 6.

5 Proof of Lemma 4

We devote this section to the proof of Lemma 4, which asserts that for any graph G satisfying (8), every two odd cycles in G intersect in at least two vertices.

In view of Lemma 8, it is enough to consider two cases: (i) one $(k+2l)$-cycle and one k-cycle; and (ii) two k-cycles. To this end, we need the following two lemmas.

Lemma 12. *For any graph G satisfying* (8)*, every k-cycle and $(k+2l)$-cycle intersect in at least two vertices.*

Lemma 13. *For any graph G satisfying* (8)*, every two k-cycles intersect in at least two vertices.*

5.1 Proof of Lemma 12.

Let C_0 be a k-cycle and C_1 be a $(k+2l)$-cycle. Suppose that C_0, C_1 intersect in at most one vertex. By Lemma 6, we can assume that $V(C_0 \cap C_1) = \{o\}$.

Claim 1. For any $u \in C_0 - \{o\}$, there is a P_u from u to $u' \in C_1 - \{o\}$, internally disjoint from $C_0 \cup C_1$.

Proof. Suppose to the contrary that there exist $u \in C_0 - \{o\}$ and some C_0-bridge H such that $u \in V(H)$ and $V(H \cap C_1) \subseteq \{o\}$. Let $G_0 := G[H \cup C_0]$ and $G_1 := G - (H - C_0)$. Then G_1 has a proper 3-coloring φ. By Lemma 8, we have $L(G_0) = \{k\}$. Then Theorem 7 shows that the restriction φ on C_0 can be extended to a proper 3-coloring of G_0. Now this gives rise to a proper 3-coloring of G, a contradiction. ■

Claim 2. Let P be a (C_0, C_1)-path from $x \in V(C_0)$ to $y \in V(C_1)$ and disjoint from o. Let X be an (o, x)-path on C_0, and Y be an (o, y)-path on C_1. Then $|P| = l$, and $|Y| = (k - |X|) + l$ or $|X| + l$.

Proof. Let $C_2 := X \cup Y \cup P$ and $C_3 := (C_0 \cup C_1 - C_2) \cup P$. If C_2 is odd, $|C_2| + |C_3| = 2|P| + k + (k + 2l)$, which implies $|C_2| = |C_3| = k + 2l$ and $|P| = l$. In this case, $|Y| = (k - |X|) + l$. If C_2 is even, then $X' \cup Y \cup P$ is an odd cycle, where $X' := C_0 - X$. Similarly, we have $|P| = l$, and $|Y| = (k - |X'|) + l = |X| + l$. ■

We write $C_0 = ox_1x_2 \cdots x_{k-1}o$ and $C_1 = oy_1y_2 \cdots y_{k+2l-1}o$. For any $x_i \in C_0 - \{o\}$ and (x_i, C_1)-path P_i from Claim 1, we denote by x'_i the vertex in $V(P_i \cap C_1)$. Claim 2 implies that for any i, $|P_i| = l$ and $x'_i \in \{y_{l+k-i}, y_{l+i}\}$.

Claim 3. $l \geq 2$.

Proof. Suppose that $l = 1$. Set $i := \frac{k-1}{2}$. Note that $|P_i| = |P_{i+1}| = 1$ (so P_i, P_{i+1} are disjoint) and by Claim 2, $x'_i, x'_{i+1} \in \{y_{i+1}, y_{i+2}\}$. If $x'_i = x'_{i+1}$, then there is a triangle, a contradiction. By symmetry, assume that $x'_i = y_{i+1}$ and $x'_{i+1} = y_{i+2}$. Then $(C_1 - y_{i+1}y_{i+2}) \cup P_i \cup x_ix_{i+1} \cup P_{i+1}$ is an odd cycle of length $k + 2l + 2$, a contradiction. ■

Claim 4. P_1 and P_{k-1} are disjoint.

Proof. Suppose that P_1 and P_{k-1} intersect. Let $z \in V(P_1 \cap P_{k-1})$ such that $|x_{k-1}P_{k-1}z|$ is minimal. Let $R := x_1P_1z$, $S := x_{k-1}P_{k-1}z$ and $T := zP_1x'_1$. By Claim 2, $|R| + |T| = l = |S| + |T|$, thus $|R| = |S|$. Then $C_2 := (C_0 - \{o\}) \cup R \cup S$ is an odd cycle of length $k - 2 + 2|R| \in \{k, k + 2l\}$, implying that $|R| = 1$ or $l + 1$. Since $|R| \le |P_1| = l$, we have $|R| = 1$. If $z \ne x'_1$, then C_2 and C_1 are disjoint, a contradiction to Lemma 6. So $z = x'_1$. Then R is a (C_0, C_1)-path, which is disjoint from o. By Claims 2 and 3, $|R| = l \ge 2$, again a contradiction. ■

By Claims 2 and 4, $\{x'_1, x'_{k-1}\} = \{y_{l+1}, y_{k+l-1}\}$. Let Y be an (x'_1, x'_{k-1})-path on C_1 with length $2l + 2$. Then $(C_0 - \{o\}) \cup P_1 \cup P_{k-1} \cup Y$ is an odd cycle of length $k + 4l$, a contradiction. This proves Lemma 12. □

5.2 Proof of Lemma 13.

We prove by contradiction. By Lemma 6, we may assume that there exist two k-cycles C_0, C_1 in G with $V(C_0 \cap C_1) = \{o\}$. Write $C_0 = ox_1x_2 \dots x_{k-1}o$ and $C_1 = oy_1y_2 \dots y_{k-1}o$.

Claim 1. Let P be any (C_0, C_1)-path from $x \in V(C_0)$ to $y \in V(C_1)$ and disjoint from o. Then $|P| = l$ or $2l$.

Proof. We choose X as an (o, x)-path on C_0, and Y as an (o, y)-path on C_1, such that $C_2 := X \cup Y \cup P$ is odd. Thus $C_3 := (C_0 \cup C_1 - C_2) \cup P$ is odd. Since $|C_2| + |C_3| = 2k + 2|P| \in \{2k + 2l, 2k + 4l\}$, we have $|P| \in \{l, 2l\}$. ■

Claim 2. For any $x_i \in C_0 - \{o\}$, there is a P_i from x_i to $x'_i \in C_1 - \{o\}$, internally disjoint from $C_0 \cup C_1$. Similarly, for any $y_j \in C_1 - \{o\}$, there is a path Q_j from y_j to $y'_j \in C_0 - \{o\}$, internally disjoint from $C_0 \cup C_1$.

Proof. By symmetry, consider vertices x_i in $C_0 - \{o\}$. Suppose that there exists some C_0-bridge H such that: $x_i \in V(H - \{o\})$ and $V(H \cap C_1) \subseteq \{o\}$. Let $G_0 =: G[H \cup C_0]$ and $G_1 := G - (H - C_0)$ such that $G = G_0 \cup G_1$. Note that G_1 is a proper subgraph of G and thus has a proper 3-coloring φ. By Lemma 12, we know $L(G_0) = \{k\}$. Then Theorem 7 ensures that the restriction φ on C_0 can be extended to a proper 3-coloring of G_0. Thus G is 3-colorable, a contradiction. ■

The next claim summarizes the possible locations of x'_i and y'_j, whose proof is straightforward and omitted.

Claim 3. If $|P_i| = 2l$, then $x'_i \in \{y_{k-i}, y_i\}$; if $|P_i| = l$, then $x'_i \in \{y_{i-l}, y_{i+l}, y_{k-i-l}, y_{k-i+l}\}$. Similarly, if $|Q_j| = 2l$, then $y'_j \in \{x_j, x_{k-j}\}$; if $|Q_j| = l$, then $y'_j \in \{x_{j-l}, x_{j+l}, x_{k-j-l}, x_{k-j+l}\}$.

In particular, for each $i \in \{1, k-1\}$, we have the following: if $|P_i| = 2l$, then $x'_i \in \{y_1, y_{k-1}\}$; if $|P_1| = l$, then $x'_i \in \{y_{1+l}, y_{k-l-1}\}$. Similarly, if $|Q_1| = 2l$, then $y'_i \in \{x_1, x_{k-1}\}$; if $|Q_1| = l$, then $y'_i \in \{x_{1+l}, x_{k-l-1}\}$.

For convenience, we draw C_0, C_1 on the plane such that x_i's appear on C_0 in the clockwise direction and y_i's appear on C_1 in the counterclockwise direction.

Claim 4. Let P_i, P_j be (C_0, C_1)-paths, from $x_i, x_j \in V(C_0)$ to $x'_i, x'_j \in V(C_1)$, respectively, where $i < j$. Let X be the (x_i, x_j)-path on C_0 not containing o. Then the following hold:
(1) If $|P_i| = l$ or $|P_j| = l$, then P_i, P_j are internally disjoint.
(2) Suppose that $|P_i| = |P_j| = 2l$. If $|X|$ is odd, then P_i and P_j are internally disjoint. Furthermore, if $x'_i = x'_j$, then $i = 2l$. Therefore, when $\{i, j\} = \{1, 2\}$ or $\{1, k-1\}$, P_i, P_j are disjoint.

Proof. (1) Suppose that $|P_i| = l$ and P_i, P_j intersect on some vertex not in C_1. Let $w \in V(P_i \cap P_j) - V(C_1)$ such that $|wP_jx_j|$ is minimal. Let $P := x_iP_iw \cup wP_jx_j$, $C_2 := X \cup P$, and $C_3 := (C_0 - X) \cup P$. Since C_2 and C_1 are disjoint, C_2 is even. So C_3 is odd, and since $V(C_1 \cap C_3) = \{o\}$, we infer that $|C_3| = k$ by Lemma 12. But $wP_ix'_i$ is a (C_1, C_3)-path, disjoint from o, with the length less than l, a contradiction to Claim 2.

(2) Suppose that there exists $w \in V(P_i \cap P_j) - V(C_1)$. Choose w such that $|x_jP_jw|$ is minimal. Let $P = x_iP_iw \cup wP_jx_j$. If $|P|$ is even, then $X \cup P$ is an odd cycle disjoint from C_1, a contradiction to Lemma 6. So $|P|$ is odd, then $C_2 := P \cup (C_0 - X)$ is also odd. As $V(C_2 \cap C_1) = \{o\}$, we infer that $|C_2| = k$. Note that $wP_ix'_i$ is a (C_2, C_1)-path with length less than $2l$. Thus $|wP_ix'_i| = l$, and $|x_iP_iw| = |x_jP_jw| = l$. This implies P is even, a contradiction.

Suppose $V(P_i \cap P_j) = \{x'_i\}$. Then $C_3 := X \cup P_i \cup P_j$ is an odd cycle. As $V(C_3 \cap C_1) = \{x'_i\}$, we infer that $|C_2| = |X| + 4l = k$. Note that oC_0x_i, x_jC_0o are two (C_2, C_1)-paths disjoint from x'_i. Hence $i = |oC_0x_i| = |x_jC_0o| = 2l$. This proves (2). ■

Claim 5. $|P_1| = |P_{k-1}| = |Q_1| = |Q_{k-1}| = l$.

Proof. Suppose not. By symmetry, assume that P_1 is of length $2l$ and from x_1 to y_1. Note that P_1 can also be viewed as Q_1. Suppose that $|P_{k-1}| = 2l$ or $|Q_{k-1}| = 2l$ (let us say $|P_{k-1}| = 2l$). By Claim 4, P_1 is disjoint from P_{k-1}, and thus $x'_{k-1} = y_{k-1}$ (because $x'_{k-1} \in \{y_1, y_{k-1}\}$). Then $(C_1 - \{o\}) \cup P_1 \cup P_{k-1} \cup (x_1ox_{k-1})$ is an odd cycle of length $k + 4l$, a contradiction. So $|P_{k-1}| = |Q_{k-1}| = l$, where $x'_{k-1} \in \{y_{l+1}, y_{k-l-1}\}$ and $y'_{k-1} \in \{x_{l+1}, x_{k-l-1}\}$.

Suppose that $x'_{k-2} = y_{l+1}$ or $y'_{k-2} = x_{l+1}$. By symmetry, let P_{k-1} be from x_{k-1} to y_{l+1}. Then $P_1 \cup x_1C_0x_{k-1} \cup y_{l+1}C_1y_1 \cup P_{k-1}$ is an odd cycle of length $k + 4l - 2$, implying $l = 1$. So P_{k-1} is from x_{k-1} to y_2. If $Q_{k-1} = y_{k-1}x_2$, then $(x_1oy_{k-1}) \cup Q_{k-1} \cup x_2C_0x_{k-1} \cup P_{k-1} \cup y_2y_1 \cup P_1$ is an odd cycle of length $k + 2l + 2$, a contradiction. So $Q_{k-1} = y_{k-1}x_{k-2}$, but then $x_1C_0x_{k-2} \cup Q_{k-1} \cup (y_{k-1}ox_{k-1}) \cup P_{k-1} \cup y_2y_1 \cup P_1$ is an odd cycle of length $k + 2l + 2$, again a contradiction. Hence we may assume that P_{k-1} is from x_{k-1} to y_{k-l-1}, and Q_{k-1} is from y_{k-1} to x_{k-l-1}.

If $k \neq l + 2$, then $y_{k-l-1} \neq y_1$. Let X be the (y_1, y_{k-l-1})-path on C_1 containing o and with $|X| = l + 2$. Then $P_1 \cup X \cup P_{k-1} \cup (C_0 - \{o\})$ is an odd cycle of length $k + 4l$, a contradiction. Thus $k = l + 2$, and now P_{k-1} is from x_{k-1} to y_1, and Q_{k-1} is from y_{k-1} to x_1. Note that $l \geq 3$ is odd. Consider the path P_2 from Claim 2. If $|P_2| = l$, then by Claim 3, $x'_2 \in \{y_{2-l}, y_{l+2}, y_{k-2-l}, y_{k-2+l}\}$, contradicting the facts that $k = l + 2$ and $l \geq 3$. So $|P_2| = 2l$. Then $x'_2 \in \{y_2, y_{k-2}\}$, and P_2 is internally disjoint with P_1 or P_{k-1} (by Claim 4). If $x'_2 = y_{k-2}$, then $P_2 \cup (C_1 - \{o, x_1\}) \cup P_{k-1} \cup (C_0 - \{o, y_{k-1}\})$ is an odd cycle of length $k + 4l - 4 > k + 2l$ (as $l \geq 3$), a contradiction. So $x'_2 = y_2$. Then $(C_1 - y_1y_2) \cup P_1 \cup x_1x_2 \cup P_2$ is an odd cycle of length $k + 4l$, a contradiction. The proof of this claim is complete. ■

By Claims 3 and 4, for any $i, j \in \{1, k-1\}$, P_i and Q_j are disjoint. Moreover, P_1, P_{k-1} (and Q_1, Q_{k-1}) are internally disjoint.

Claim 6. P_1, P_{k-1} share the endpoint in C_1, or Q_1, Q_{k-1} share the endpoint in C_0.

Proof. Otherwise, $P_1, P_{k-1}, Q_1, Q_{k-1}$ are pairwise disjoint. By symmetry, assume that $y'_1 \in y'_{k-1}C_0o$. Let X be the (x'_1, x'_{k-1})-path on C_1 not containing o. Then $x_1C_0y'_{k-1} \cup Q_{k-1} \cup (y_{k-1}oy_1) \cup Q_1 \cup y'_1C_0x_{k-1} \cup P_{k-1} \cup X \cup P_1$ is an odd cycle of length $k + 4l$, a contradiction. ■

Claim 7. $l = 1$, $N(x_1) \cap N(x_{k-1}) \cap \{y_2, y_{k-2}\} \neq \emptyset$ and $N(y_1) \cap N(y_{k-1}) \cap \{x_2, x_{k-2}\} \neq \emptyset$.

Proof. By Claim 6, assume by symmetry that Q_1, Q_{k-1} are from y_1, y_{k-1} to x_{l+1} respectively. Then $C_2 := Q_1 \cup Q_{k-1} \cup (C_1 - \{o\})$ is an odd cycle of length $k - 2 + 2l$, which intersects C_0 only on x_{l+1}. By Lemma 12, $|C_2| = k$ and thus $l = 1$. Suppose P_1, P_{k-1} can be chosen to be disjoint, say $P_1 = x_1y_{k-2}$ and $P_{k-1} = x_{k-1}y_2$. But then the cycle $(C_0 - \{o\}) \cup P_1 \cup (y_{k-2}y_{k-1}oy_1y_2) \cup P_{k-1}$ is an odd cycle of $k + 2l + 2$, a contradiction. This proves the claim. ■

By symmetry, we may assume that $P_1 = x_1y_2$, $P_{k-1} = x_{k-1}y_2$, $Q_1 = y_1x_2$ and $Q_{k-1} = y_{k-1}x_2$. Let $C_2 := (C_0 - \{o\}) \cup (x_1y_2x_{k-1})$. Note that $|C_2| = |C_1| = k$ and $V(C_2 \cap C_1) = \{y_2\}$. We then can treat C_2, y_2 as the new C_0, o, and thus all previous claims hold for C_1 and C_2. In particular, by Claim 7, we have $N(y_1) \cap N(y_3) \cap \{x_2, x_{k-2}\} \neq \emptyset$. If $y_1x_2, y_3x_2 \in E(G)$, as $Q_{k-1} = y_{k-1}x_2$, G has an odd cycle $(C_1 - \{o, y_1, y_2\}) \cup (y_{k-1}x_2y_3)$ of length $k-2$; otherwise $y_1x_{k-2}, y_3x_{k-2} \in E(G)$, then, as $Q_1 = y_1x_2$, G has an odd cycle $(C_0 - \{o, x_1, x_{k-1}\}) \cup (x_{k-2}y_1x_2)$ of length $k-2$, a contradiction. Lemma 13 now is proved. This, together with Lemmas 8 and 12, complete the proof of Lemma 4. □

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