Intersecting Family

Def: Family $\mathcal{F} \subseteq 2^{[n]}$ is called <u>intersecting</u>, if for any two sets $A, B \subseteq \mathcal{F}$, we have $A \cap B \neq \emptyset$

Q1: How large an intersecting $\mathcal{F}$ can be?

e.g. $\mathcal{F} = \{A \subset [n] : 1 \in A\}$ has size 2^{n-1}. star.

Fact 1: For $\forall$ intersecting $\mathcal{F} \subseteq 2^{[n]}$, $|\mathcal{F}| \leq 2^{n-1}$.

Pf: Consider all pairs $\{A, A^c\}$ where $A^c = [n] \setminus A$. There are exactly 2^{n-1} such pairs. And $\mathcal{F}$ can contain at most 1 set in $\{A, A^c\}$, so $|\mathcal{F}| \leq 2^{n-1}$. #

e.g. When n is odd, $\mathcal{F} = \{A \subseteq [n] : |A| > \frac{n}{2}\}$. $|\mathcal{F}| = 2^{n-1}$.

Q2: How large an intersecting $\mathcal{F} \subset \binom{[n]}{k}$ can be?

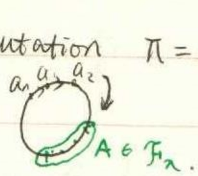
(To avoid triviality, we assume that $n \geq 2k$).

e.g. $\mathcal{F} = \{A \in \binom{[n]}{k} : 1 \in A\}$ is intersecting and of size $\binom{n-1}{k-1}$.

It turns out that $\binom{n-1}{k-1}$ is the maximum.

Erdös-Ko-Rado Thm: For $n \geq 2k$, if $\mathcal{F} \subseteq \binom{[n]}{k}$ is intersecting, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$.

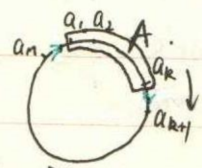
Proof: Take a cyclic permutation $\pi = (a_1, a_2, \cdots a_n)$, and write the numbers clockwise in a circle.
(顺时针)



Let $\mathcal{F}_\pi$ be the set of all $A \in \mathcal{F}$ that appear as k consecutive number in the circle.

claim: $|\mathcal{F}_\pi| \leq k$.

(Pf: Fix $A \in \mathcal{F}_\pi$ and $\pi =$



. call the edge $a_n a_1$, $a_k a_{k+1}$ as the boundary edges of A in π. Let $A = \{a_1 \cdots a_k\}$.

Note that as $n \geq 2k$, any two set in $\mathcal{F}_\pi$ have distinct boundary edges.

For other $B \in \mathcal{F}_\pi - \{A\}$, since $A \cap B \neq \emptyset$, one of the boundary edges of B must be an inner edge of A.

But A has exactly $k-1$ inner edges.

Thus, there are at most $k-1$ sets in $\mathcal{F}_\pi - \{A\}$, so $|\mathcal{F}_\pi| \le k$. \#)

Next, we count the numbers

$$N = \#\text{pairs}(\pi, A) \text{ where } A \in \mathcal{F}_\pi \text{ and } \pi \text{ is cyclic permutation.}$$

By claim, $N = \sum_{\text{cyclic } \pi} |\mathcal{F}_\pi| \le k \cdot (n-1)!$ (there are $(n-1)!$ cyclic permutation)

On the other hand, $N = \sum_{A \in \mathcal{F}} (\#\text{ of cyclic } \pi \text{ s.t. } A \in \mathcal{F}_\pi) = |\mathcal{F}| \cdot k!\,(n-k)!$

π

- $\exists\ k!$ ways for elements in A
- $\exists\ (n-k)!$ ways for other elements.

$\Rightarrow$ fixed A, $\exists\ k!(n-k)!$ many cyclic π.

Now $|\mathcal{F}| = \dfrac{N}{k!(n-k)!} \le \dfrac{k(n-1)!}{k!(n-k)!} = \dfrac{(n-1)!}{(k-1)!(n-k)!} = \dbinom{n-1}{k-1}$. \#

Partially Ordered Sets. 偏序集.

Let X be a finite set.

Def: R is called a <u>relation</u> of X, if $R \subseteq X \times X = \{(x_1, x_2) : x_1, x_2 \in X\}$. ordered.

- $(x, y) \in R \Leftrightarrow xRy$

Def: A <u>partially ordered set</u> (or <u>poset</u>) is an ordered pair (X, R), where R is a relation of X, satisfying:

(i). R is reflexive (自反的): $(x, x) \in R$ for $\forall x \in X$.

(ii). antisymmetric (反对称的): $\left.\begin{matrix}(x,y) \in R \\ (y,x) \in R\end{matrix}\right\} \Rightarrow x = y$

(iii). transitive (传递的): $(x, y) \in R$, $(y, z) \in R \Rightarrow (x, z) \in R$.

e.g. $(\underset{\substack{\parallel \\ B_n}}{2^{[n]}}, \subseteq)$ is a poset, where "$\subseteq$" stands for the "inclusion" relationship.

→ <u>inclusion-poset</u>

- We often use "$\preceq$" to replace "R" : $(x,y)\in R \Leftrightarrow x \preceq y$.
- "$\prec$" : if $x \prec y$, then $x \neq y$ and $x \preceq y$.

Def: If $x \prec y$, then x is a predecessor of y.

Def: $x \in X$ is a minimal element of $(X, \preceq)$ if there is NO predecessor of x.

Def: x is an immediate predecessor of y, if (i). $x \prec y$, (ii) NO t s.t. $x \prec t \prec y$.
We then express this as $x \lhd y$.

Fact: For any $x, y \in (X, \preceq)$, $x \preceq y$ iff $\exists\, x_1, \cdots x_k$. s.t $x \lhd x_1 \lhd x_2 \lhd \cdots \lhd x_k \lhd y$.
(possibly $k=0$).

Proof : "$\Leftarrow$" ✓

"$\Rightarrow$" $M_{xy} \triangleq \{t : x \prec t \prec y\}$. prove this by induction on $|M_{xy}|$.

Base case : $|M_{xy}|=0$, $x \lhd y$. $(k=0)$

Assume this holds for numbers less than $|M_{xy}|$ and more than 0.

$\forall t \in M_{xy}$, $|M_{xt}|, |M_{ty}| < |M_{xy}|$.

by induction $\exists\, x_1, \cdots x_{k_1}$, s.t. $x \lhd x_1 \cdots \lhd x_{k_1} \lhd t$.

$\exists\, x_{k_1+2}, \cdots x_k$, s.t. $t \lhd x_{k_1+2} \lhd \cdots \lhd x_k \lhd y$.

write t as x_{k_1+1}. so that $x \lhd x_1 \lhd \cdots \lhd x_k \lhd y$. #

One of the nice properties posets have is that we can express posets as diagrams.

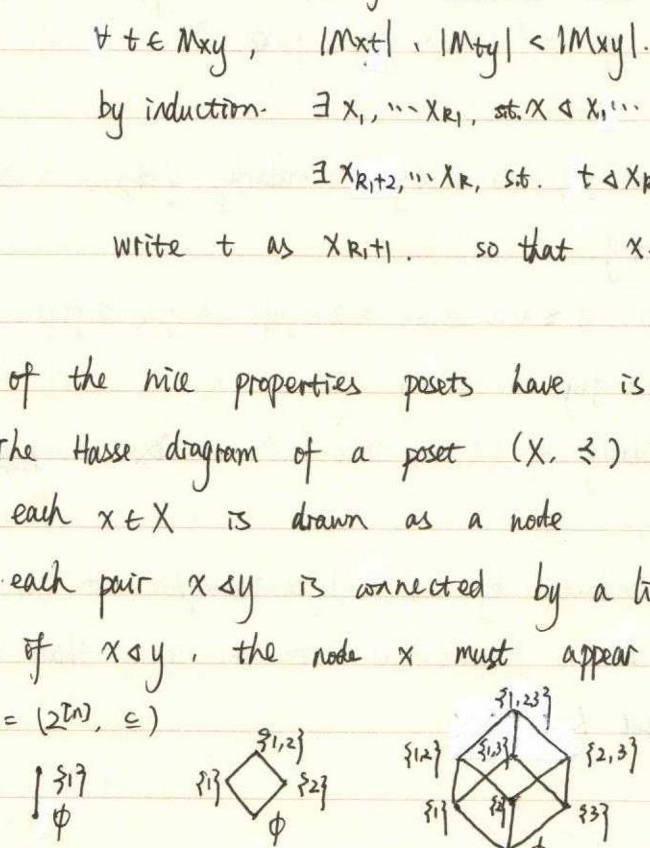
Def: The Hasse diagram of a poset $(X, \preceq)$ is a drawing in the plane. s.t:

(i). each $x \in X$ is drawn as a node

(ii). each pair $x \lhd y$ is connected by a line segment

(iii). if $x \lhd y$, the node x must appear lower than y.

e.g. $\mathcal{B}_n = (2^{[n]}, \subseteq)$



$\mathcal{B}_1$ $\mathcal{B}_2$ $\mathcal{B}_3$.

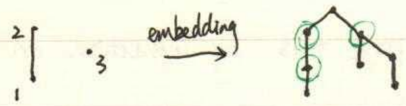
⇒ 立方体!

- a poset $\Leftrightarrow$ a Hasse diagram
- $x \prec y$ if and only if $\exists\ x_1, \cdots x_k$. s.t. $x \lhd x_1 \lhd x_2 \lhd \cdots x_k \lhd y$

  iff $\exists$ a path in Hasse diagram from node x to node y.

  strictly from bottom to top!

Def: Let $(X, \preceq)$ and $(X', \preceq')$ be two posets. A mapping $f: X \to X'$ is an embedding (嵌入) of $(X, \preceq)$ into $(X', \preceq')$, if $\begin{cases} \text{(i). } f \text{ is injective} \\ \text{(ii). } f(x) \preceq' f(y) \text{ iff } x \preceq y. \end{cases}$

e.g. 

Thm: For every poset $(X, \preceq)$, there exists an embedding into the poset $(2^X, \subseteq)$ $= B_X$.

Proof: Define $f: X \to 2^X$ by $f(x) = \{y \in X : y \preceq x\} \in 2^X$.

Verify: (i). f is injective.

If $f(x) = f(y)$. so $x \in f(x) = f(y) \Rightarrow x \preceq y$. similarly $y \preceq x$. $\Rightarrow x = y$.

(ii). $f(x) \subseteq f(y)$ iff $x \preceq y$.

- If $x \preceq y$, if $z \in f(x)$. $z \preceq x \Rightarrow z \preceq y \Rightarrow z \in f(y) \Rightarrow f(x) \subseteq f(y)$.
- If $f(x) \subseteq f(y)$, $x \in f(x) \subseteq f(y) \Rightarrow x \preceq y$

(i) & (ii) $\Rightarrow$ f is an embedding of $(X, \preceq)$ into $(2^X, \subseteq) = B_X$. #

$(X, \preceq) \sim (f(X), \preceq')$

Note that the poset on $f(X)$ induced by $(X', \preceq')$ is isomorphic to $(X, \preceq)$. This Thm tells us that B_n is "universal" in the sense that they contain a copy of every poset!

Def: For poset $(X, \preceq) \triangleq P$,

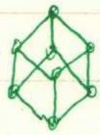
(i). For $x, y \in X$, if $x \prec y$ or $y \prec x$, then x, y are comparable. O.W. incomparable.

(ii). A set $A \subseteq X$ is an antichain (反链) of $(X, \preceq)$, if any 2 elements in A are incomparable.

(iii). A set $B \subseteq X$ is a chain of $(X, \preceq)$, if any 2 elements in B are comparable.

Def:
- $\alpha(P) = \max\limits_{\text{antichain } A} |A|$.
- $w(P) = \max\limits_{\text{chain } B} |B|$

在 Hasse diagram 中若虑？ $\left\{ \begin{array}{l} \alpha \text{ 代表宽度 (width)} \\ w \text{ 代表高度 (height)} \end{array} \right.$

e.g. $B_3 =$  $\alpha = 3$, $\beta = 4$.

Thm: For any poset $P = (X, \preceq)$, $\alpha(P) \cdot w(P) \geq |X|$.

Fact: The set of all minimal elements of P forms an antichain.

Proof: We will inductively define a sequence of sub-posets P_i as following:

Let $\preceq_1 = \preceq$, $P_1 := (X, \preceq_1)$ and Let $M_1 := \{\text{all minimal elements of } P_1\}$.

Now assume $P_1, \cdots, P_{k-1}$ are defined. and let $M_i = \{\text{minimal elements of } P_i\}$ for $1 \leq i \leq k-1$. We use $\preceq_k$ to denote the ordering $\preceq$ restricted (限制) on $X - \bigcup_{i=1}^{k-1} M_i$.

Let $P_k := (X - \bigcup_{i=1}^{k-1} M_i, \preceq_k)$

We proceed this until $X = M_1 \cup \cdots \cup M_t$. (disjoint union)

By fact, M_i is an antichain of P_i. But P_i is induced on $X - \bigcup_{j=1}^{i-1} M_j$ by P

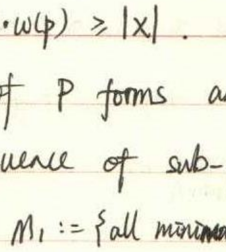
$\Rightarrow$ so M_i is an antichain of P.

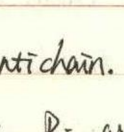
$\Rightarrow |M_i| \leq \alpha(P)$. for $i = 1, 2, \cdots t$. And $|X| = \sum_{i=1}^{t} |M_i| \leq t \cdot \alpha(P)$.

To finish the proof, we show: $t \leq w(P)$.

It suffices to show:

$\exists$ a chain $x_1 \prec x_2 \prec \cdots \prec x_t$ where $x_i \in M_i$.



$\Leftarrow$  去掉一个 M_i 就是去掉一层，最多有 $w(P)$ 层. 故 $t \leq w(P)$.

a chain: $x_1 \prec x_2 \prec \cdots \prec x_t \Rightarrow t \leq w(P)$.

claim: $\forall x \in M_{k+1}$ and $\forall k$, $\exists y \in M_k$, s.t. $y \prec x$.

(Pf: M_{k+1} contains the minimal elements of $(X - M_1 \cup \cdots \cup M_k, \preceq)$

so $x \in M_{k+1}$ is a minimal elements in $X - M_1 \cup \cdots \cup M_k$ but $x \notin M_k$.

$\Rightarrow$ x is not minimal in $X - M_1 \cup \cdots \cup M_{k-1}$.

Thus, there must be an element $y \in M_k$, s.t. $y \prec x$. #)

$\Rightarrow$ By claim, $w(p) = \max|\text{chain}| \geq t$.

$\Rightarrow$ $|X| \leq t\,\alpha(p) \leq \alpha(p) \cdot w(p)$. #

小结: · Poset $(X, \preceq)$: reflexive, antisymmetric, transive.

· Hasse diagram : B_3,  ; $x \lessdot y$: x is an immediate predecessor of y

· chain : $A = x_1 \prec x_2 \prec \cdots \prec x_t$. $\Leftrightarrow$ any pair of A is comparable.

Antichain : $B = \{y_1, y_2, \cdots y_s\}$ any pair is incomparable.

$w(X, \preceq) = \max\limits_{\text{chain } A} |A|$, $\alpha(X, \preceq) = \max\limits_{\text{antichain } B} |B|$, & $w(p)\alpha(p) \geq |X|$.

Def: Given a sequence $(x_1, \cdots x_n)$. $x_i \in \mathbb{R}$, a subsequence is $(x_{i_1}, x_{i_2}, \cdots x_{i_t})$ $i_1 < i_2 < \cdots < i_t$. It is monotone if $x_{i_1} \leq x_{i_2} \leq \cdots \leq x_{i_t}$ or $x_{i_1} \geq x_{i_2} \geq \cdots \geq x_{i_t}$.

Thm (Erdős - Szekeres): Any sequence $(x_1, x_2, \cdots x_{n^2+1})$ contains a monotone subsequence of length $n+1$.

Proof: Define a poset $P = ([n^2+1], \preceq)$ as follows: $i \preceq j$ iff $i \leq j$ & $x_i \leq x_j$.

Clearly, it satisfies the 3 requirements of poset (reflexive, antisymmetric, transice)

By Thm, $\alpha(p)w(p) \geq n^2+1$ then either $\alpha(p)$ or $w(p) \geq \underbrace{n+1}_{= \lceil\sqrt{n^2+1}\rceil}$.

Case 1: $w(P) \geq n+1$.

So P has a chain $A = i_1 \prec i_2 \prec \cdots \prec i_{n+1}$. $\Leftrightarrow \begin{cases} i_1 < i_2 < \cdots < i_{n+1} \\ x_{i_1} \leq x_{i_2} \leq \cdots \leq x_{i_{n+1}} \end{cases}$.

$\Rightarrow$ we find an increasing subsequence of X.

Case 2: $\alpha(P) \geq n+1$.

So P has an antichain $B = \{j_1, j_2, \cdots j_{n+1}\}$. We can assume that $j_1 < j_2 < \cdots < j_{n+1}$. Since any pair of B is incomparable, $x_{j_1} > x_{j_2} > \cdots > x_{j_{n+1}}$.

$\Rightarrow$ we find a decreasing subsequence of X. #
(strictly)

Exercise: A generalization of Erdös-Szekeres:

Any sequence of length $\underline{kl+1}$ contains either an increasing subseq. of length $k+1$ or a strictly decreasing subseq. of length $l+1$.

HW: $\exists$ a sequence of length $\underline{kl}$ such that NO such monotone subseq. exist!

The Pigeonhole Principle 鴿巢原理.

The Pigeonhole Principle: Let X be a set with at least $1 + \sum_{i=1}^{k}(n_i - 1)$ elements, and let $X_1, X_2 \cdots X_k$ be disjoint sets forming a partition of X, Then, $\exists\, i$, s.t $|X_i| \geq n_i$.

Fact 1: Any graph G with n vertices has 2 vertices of the same degree.

Pf: The degrees are from 0 to $n-1$. But vertices of degree $n-1$ and vertices of degree 0 can not occur in the same graph. So the degrees of all n vertices has $\leq n-1$ values. By P-P, $\exists$ 2 vertices of the same degree. #

Exercise: Find a graph with only 2 vertices of the same degree.

eg: $n=2$  $n=3$  $n=4$  $n=5$  ...

$n=$奇，加一个0度点，$\because n=$偶，加一个点并且让这个点与所有原来的点相邻.

Suppose we want to pick a set $S \subseteq [2n]$ s.t $\overset{(*)}{\text{no numbers in } S \text{ divides another}}$.

Q: How large can $|S|$ be? You can pick up $S=\{n+1, n+2, \cdots 2n\}$.

Fact 2: For any such set S, $|S| \le n$. satisfying $(*)$.

Pf: For any odd k, define $S_k = \{k\cdot 2^i : \text{if } k\cdot 2^i \in [2n]\}$.

so $S = S_1 \cup S_3 \cup \cdots \cup S_{2n-1}$. (a disjoint union of n sets.)

Suppose $|S| \ge n+1$. By P-P, $\exists k$, s.t $|S_k| \ge 2$.

Then we have $a, b \in S_k$ s.t $a|b$, a contradiction! #

· Rational Approximation:

Fact 3: For $\forall x \in \mathbb{R}$ and $\forall$ natural number n, there is a rational number where $1 \le q \le n$ such that $|x - \frac{p}{q}| < \frac{1}{nq}$.

Pf: Let $x \ge 0$, $\{x\} := x - \lfloor x \rfloor$ be the fractional part of x.

For any $i = 1, 2, \cdots n+1$, consider $\{ix\} \in [0,1)$.

Partition $[0,1)$ into n subintervals $[0,\frac{1}{n}) \cup [\frac{1}{n}, \frac{2}{n}) \cup \cdots \cup [\frac{n-1}{n}, 1)$.

Since we have $n+1$ number $\{ix\}$ for $1 \le i \le n+1$,

by P-P, there are 2 numbers say $\{ix\}, \{jx\}$ belonging to the same subinterval.

Let $\{jx\} > \{ix\}$, then $\{|j-i|x\} = \{jx\} - \{ix\} < \frac{1}{n}$ as they belong to the same subinterval of length $\frac{1}{n}$. Let $q = |j-i|$, then $qx = p + \varepsilon$ where $1 \le q \le n$ & $0 \le \varepsilon = \{qx\} < \frac{1}{n}$

$\Rightarrow x = \frac{p}{q} + \frac{\varepsilon}{q} \Rightarrow |x - \frac{p}{q}| = |\frac{\varepsilon}{q}| \le \frac{1}{nq}$. #

- **Thm (Erdös-Szekeres)**: (n^2+1)-sequence $\Rightarrow$ $(n+1)$-subseq. monotone.

Proof 2 (By P-P): Let $(x_1, x_2, \cdots x_{n^2+1})$ be the sequence.

For any x_i, let t_i be the size of <u>maximum increasing subseq.</u> starting at x_i.

(反证) We may assume that $1 \le t_i \le n$ for $\forall i$.

But we have n^2+1 elements x_i's in total

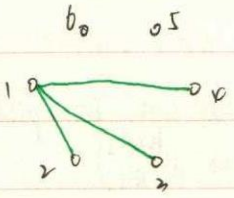
By P-P, there are at least $(n+1)$ x_i's with the same t_i call $x_{i_1}, \cdots x_{i_{n+1}}$, where $i_1 < i_2 < \cdots < i_{n+1}$. We claim: $x_{i_1} > x_{i_2} > \cdots > x_{i_{n+1}}$, which is strictly decreasing.

Why? If $x_{i_1} \le x_{i_2}$, we could extend the increasing subseq. starting at x_{i_2} by adding x_{i_1} to be the 1^{st} element, to get an increasing subseq of length $t_{i_2}+1$ starting at x_{i_1}. This shows $t_{i_1} \ge t_{i_2}+1$. But $t_{i_1} = t_{i_2}$, a contradiction! #

- <u>A part of six</u>.

Fact 4: Suppose a party has 6 participants. They may know each other or not. Then, there must be 3 person who know each other or don't know each other.

Pf: We construct a graph on 6 vertices. If i knows j, then connect ij O.W. not.
Now, we show: $\exists$ 3 vertices which are adjacent to each other, or aren't adjacent.



1认识3个以上，考虑2,3,4.
若2,3,4互不认识，$\Rightarrow$ #
若2认识3，$\Rightarrow$ #

#