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Ramsey's Thm (on graphs)

Def: A r-edge-coloring of K_n is a function $f: E(K_n) \to [r]$, which assigns a color $\in [r]$ to each edge of K_n.

If $r=2$, usually we assume that the colors are "red" and "blue".

Def: A k-clique (or K_k) is monochromatic if all its edges are colored by the same color.

Thus, <u>A party of six</u> $\Leftrightarrow$ any 2-edge-coloring of K_6 has a monochromatic triangle.

Thm (Ramsey): Let k, l be integers at least 2. There exists an integer N s.t. any 2-edge-coloring of K_N has a blue K_k or a red K_l.

The Pigeonhole Principle: Let $A_1, \cdots, A_k$ be disjoint sets, which form a partition of X, with $|X| = 1 + \sum_{i=1}^{k}(a_i - 1)$. Then there exists some i s.t. $|A_i| \geq a_i$.

Proof: We will show that N can be picked as $N = \binom{k+l-2}{k-1}$.

<u>By Induction on the sum $(k+l)$</u>

The base case ($k=l=2$) is trivial.

In fact, when $k=2$ or $l=2$, it is easy to see that N can be picked as $\binom{k+l-2}{k-1}$.

We need to show: any 2-edge-coloring of $K_{\binom{k+l-2}{k-1}}$ has a blue K_k or a red K_l.

Write: $N_1 = \binom{(k-1)+l-2}{k-1-1}$, $N_2 = \binom{k+(l-1)-2}{k-1}$.

$(k-1, l)$ $(k, l-1)$

so that $N_1 + N_2 = \binom{k+l-3}{k-2} + \binom{k+l-3}{k-1} = \binom{k+l-2}{k-1} = N$.

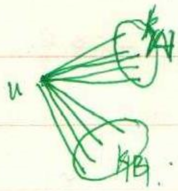
Fix a vertex $u \in V(K_N)$ which can be arbitrary.

Then we portion $V-\{u\}$ into a disjoint sets A and B by:

- $A = \{v \in V-\{u\} : uv \text{ is blue}\}$.
- $B = \{v \in V-\{u\} : uv \text{ is red}\}$.

$\Rightarrow |A| + |B| = N-1 = N_1 + N_2 - 1$. By P-P, $|A| \geq N_1$ or $|B| \geq N_2$.

- Case 1: $|A| \geq N_1$. The vertices of A will contain a complete graph K_{N_1},



whose edges are colored by blue or red.

By induction for the pair $(k-1, l)$, the set A contains a blue K_{k-1} or a red K_l.

If A has a red K_l, then we are done.

Otherwise, A has a blue K_{k-1}. Since all edges between u and the vertices of A are blue, this blue K_{k-1} plus u give us a blue K_k. This proves case 1.

- Case 2: $|B| \geq N_2$. Similarly as case 1.

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Def: For $\forall k, l \geq 2$, the Ramsey Number $R(k,l)$:

Denote the smallest integer N s.t. any-2-edge-coloring of K_N contains a blue K_k or a red K_l.

(i). $R(k,l) \leq M \Leftrightarrow$ any 2-edge-coloring of K_M has a blue K_k or a red K_l.

(ii). $R(k,l) > P \Leftrightarrow \exists$ a 2-edge-coloring of K_P NOT containing blue K_k nor red K_l.

Fact: $R(3,3) = 6$.

Pf: · A part of six tells us $R(3,3) \leq 6$.

· $R(3,3) > 5$  没有蓝或红 in K_3. #

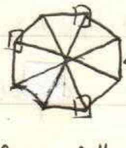
Corollary: $R(k,l) \leq \binom{k+l-2}{k-1}$

Fact 2: $R(k,l) = R(l,k)$. Fact 3: $R(2,l) = l$, $R(k,2) = k$.

Fact 4: $R(3,4) = 9 < 10$.

Pf: · $R(3,4) \le \binom{3+4-2}{3-1} = 10$.

· $R(3,4) > 8 \iff \exists$ a 2-edge-coloring of K_8 with NO blue K_3 nor red K_4.



← blue edge. The max independent set is of size 3.

$\Rightarrow$ NO red K_4.

· $R(3,4) \le 9$. follows by the Next Thm2: $R(3,4) \le R(2,4) + R(3,3) - 1 = 9$.

(Note that from the proof of Ramsey's Thm:

☆ $R(k,l) \le R(k-1,l) + R(k,l-1)$. Exercise.)

Thm2: If both $R(k-1,l)$ and $R(k,l-1)$ are even, then $R(k,l) \le R(k-1,l) + R(k,l-1) - 1$.

Proof: Let $n = R(k-1,l) + R(k,l-1) - 1$. Consider any 2-edge-coloring of K_n.

For $\forall v$, $\begin{cases} B_v = \{y : yv \text{ is bule}\} \\ R_v = \{y : yv \text{ is red}\} \end{cases}$.

If $\exists$ some v, s.t. $|B_v| \ge R(k-1,l)$ or $|R_v| \ge R(k,l-1)$,

then we can repeat the previous argument to either get a blue K_k or a red K_l.

(没记) So we may assume: $\forall v$, $|B_v| \le R(k-1,l)-1$ & $|R_v| < R(k,l-1)-1$.

then $n-1 = |B_v| + |R_v| \le R(k-1,l) + R(k,l-1) - 2 = n-1$;

$\Rightarrow \forall v$, $|B_v| = R(k-1,l) - 1$ and $|R_v| = R(k,l-1) - 1$.

$\Rightarrow$ Consider the subgraph G induced by all blue edges.

Then, $\forall v$, $d(v) = |B_v| = R(k-1,l) - 1$ is odd,

On the other hand, G has n vertices, which is odd.

$2e(G) = \sum_{v \in G} d_G(v) =$ sum of odd many terms, which are odd = odd. This is a contradiction

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Fact 5: $R(4,4) = 18$, $\underbrace{R(3,5) = 14}$ (Exercise), $R(4,5) = 25$.

$43 \le R(5,5) \le 49$.

Diagonal Ramsey Number: $R(k,k)$.

Thm 3: Let n and k satisfy $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$. Then $R(k,k) > n$.

Proof: We look for a 2-edge-coloring of K_n s.t. it has NO monochromatic K_k.

Consider a random 2-edge-coloring of K_n, where each edge is colored by blue or red with probability $\frac{1}{2}$, independent of other edges.

For any $X \in \binom{[n]}{k}$. Let A_X be the event that X is a monochromatic K_k.

$\Rightarrow \Pr(A_X) = 2 \times (\frac{1}{2})^{\binom{k}{2}} = 2^{1-\binom{k}{2}}$.

$\Rightarrow \Pr(\bigcup_{X \in \binom{[n]}{k}} A_X)$ = probability of the event that $\exists$ a monochromatic K_k.

if $\Pr(\text{NO monochromatic } K_k) > 0$, then $R(k,k) > n$.

$\Pr(\text{NO monochromatic } K_k) = 1 - \Pr(\exists \text{ monochromatic } K_k)$

$= 1 - \Pr(\bigcup_X A_X) \geq 1 - \sum_X 2^{1-\binom{k}{2}} = 1 - \binom{n}{k} 2^{1-\binom{k}{2}} > 0$.

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Corollary: $R(k,k) > \frac{k}{e} \cdot 2^{\frac{k-1}{2}}$.

Pf: Let $n := \left(\frac{k}{e}\right) 2^{\frac{k-1}{2}} \left(\frac{e}{2}\right)^{\frac{1}{k}} > \frac{k}{e} \cdot 2^{\frac{k-1}{2}}$.

It suffices to show: $\binom{n}{k} 2^{1-\binom{k}{2}} < 1 \cdots (*) \Rightarrow R(k,k) > n > \frac{k}{e} 2^{\frac{k-1}{2}}$.

Verify $(*)$: $\binom{n}{k} < \frac{n^k}{k!} \le \frac{n^k}{e(\frac{k}{e})^k} = \frac{1}{e} \cdot \frac{e}{2} \cdot 2^{\frac{k-1}{2} \cdot k} = \frac{1}{2} 2^{\binom{k}{2}} = 2^{\binom{k}{2}-1}$.

$k! \geq e(\frac{k}{e})^k$ $\qquad \Rightarrow \binom{n}{k} 2^{1-\binom{k}{2}} < 1$.

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推广 Ramsey Number? 多色.

Def: For any integer $s_1, s_2, \cdots s_k \geq 2$, the Ramsey Number $R(s_1, \cdots s_k)$ is the least integer N s.t. any k-edge-coloring of K_N has a clique K_{s_i} of color i.

HW: $R(s_1, \cdots s_k) < +\infty$.

Application:

Schur's Thm: For any $k \geq 2$, there exists $N := N(k)$ s.t. for any k-coloring $c: [N] \to [k]$, we can find $x, y, z \in [N]$ s.t. $c(x) = c(y) = c(z)$ and $x + y = z$.

Proof: Let $N := R(\underbrace{3, 3, \cdots 3}_{k})$

Define a k-edge-coloring f on K_N by: $f(ij) := c(|i-j|)$.

By the definition of Ramsey Number $R(3,3,\cdots,3)$, f has a monochromatic triangle, say $i < j < l$,

$c(j-i) = c(l-i) = c(l-j)$.

Let $x := j - i$, $y := l - j$, $z := l - i$. $\Rightarrow \begin{cases} c(x) = c(y) = c(z) \\ \text{and } x + y = z \end{cases}$ #.

Schur used this to prove that the Fermat Last Thm holds in $\mathbb{Z}_p$ for large prime p.

Thm: For $\forall$ integer $m \geq 1$, there is a prime number $p(m)$ such that for any prime $p \geq p(m)$, $x^m + y^m \equiv z^m \pmod{p}$ has a nontrivial solution.

Proof: For prime p, define the multiplicative group $\mathbb{Z}_p^*$, there is a $g \in \mathbb{Z}_p^*$, s.t. $\forall\, x \in \mathbb{Z}_p^*$ can be written as $x = g^{im+j}$ for some i, j, $0 \leq j \leq m-1$

Define $c: \mathbb{Z}_p^* \to [m]$ by $c(x) = j$.

By Schur's Thm, $\exists N(m)$ s.t. as long as $p > N(m)$, $\exists x, y, z \in \mathbb{Z}_p^*$, satisfying $c(x) = c(y) = c(z)$ and $x + y = z$.

Write $x := g^{i_1 m + j}$, $y := g^{i_2 m + j}$, $z := g^{i_3 m + j}$.

Then $x + y = z \Rightarrow g^{i_1 m + j} + g^{i_2 m + j} \equiv g^{i_3 m + j} \pmod{p}$.

$\Rightarrow (g^{i_1})^m + (g^{i_2})^m \equiv (g^{i_3})^m \pmod{p}$

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The Probability Method in Combinatorics.

- A probability space is (Ω, P_r), where P_r is a normalized measure (i.e. $P_r(\Omega)=1$).

  $P_r : 2^{\Omega} \to [0,1]$: ① $P_r(\phi)=0$; ② $P_r(\Omega)=1$; ③ A, B disjoint $\subseteq \Omega$. $P_r(A \cup B) = P_r(A) + P_r(B)$.

In Combinatorics, it's mostly sufficient to work with finite Ω.

Any $A \subseteq \Omega$ is "event", $P_r(A) = \sum_{w \in A} P_r(\{w\})$

- A random varible is a function $X : \Omega \to \mathbb{R}$.
- Expectation : $\mathbb{E}[X] = \sum_{w \in \Omega} X(w) P_r(\{w\}) = \sum_a a P_r(X=a)$.
- Two events $A, B \subseteq \Omega$ are independent, if $P(A \cap B) = P(A) \cdot P(B)$.

  Two random varible X, Y are independent, if for $\forall a, b \in \mathbb{R}$,

  the events $\{X=a\}$, $\{Y=b\}$ are independent.

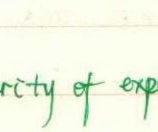
Fact 1: $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$, if X & Y independent.

Fact 2: $P_r(A_1 \cup \cdots \cup A_k) \le \sum_{i=1}^{k} P_r(A_i)$ union bound.

Fact 3: For $\forall$ random varible X, Y, $\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y)$ linearity of expectation

Union - Bound

Def: Let $\mathcal{F}$ be a family of subsets of size k, we say $\mathcal{F}$ is 2-colorable if there exists a function $f : \bigcup_{A \in \mathcal{F}} A \to \{red, blue\}$, such that every set $A \in \mathcal{F}$ is not monochromatic (i.e. every A has at least one element colored blue and one element colored red)



Rmk: where $k=2$, $\mathcal{F}$ can be viewed as a graph G.
with $V(G) = \bigcup_{A \in \mathcal{F}} A$ and $E(G) = \mathcal{F}$. In this case ($k=2$),
$\mathcal{F}$ is 2-colored iff G is bipartite.

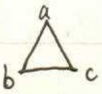
$A = \{$红, 蓝$\}$ 可视为一条边.

Def: $\forall k$, Let $\mathcal{F} = \{\text{set } A : |A| = k\}$, Let $m(k) := \min |\mathcal{F}|$ over all $\mathcal{F}$, which is NOT 2-colorable.

ii). $m(k) \le L$, iff there exists such a family $\mathcal{F}$ with L k-sets, which is NOT 2-colorable.

iii). $m(k) > L$, iff $\forall$ family $\mathcal{F}$ with $|\mathcal{F}| \le L$ can be 2-colorable.

Fact: $m(2) \le 3$.

eg:  $\rightarrow \mathcal{F} = \left\{ \begin{matrix} \{ab\} \\ \{bc\} \\ \{ac\} \end{matrix} \right\}$ is NOT 2-colorable.

$m(2) > 2$

eg: $\mathcal{F} = \{\{1,2\}, \{3,4\}\}$ ✓, $\{\{1\,2\}, \{2\,3\}\}$ ✓ ..

$\Rightarrow m(2) = 3$.

Thm: $m(k) \geq 2^{k-1}$. ($\Leftrightarrow$ $\forall$ k-family $\mathcal{F}$ with $|\mathcal{F}| < 2^{k-1}$ can be 2-colorable.)

Proof: To show $\mathcal{F}$ is 2-colorable, we need to find a function $f : \underbrace{\bigcup_{A \in \mathcal{F}} A}_{= X} \rightarrow \{\text{blue}, \text{red}\}$

s.t. ($\forall A \in \mathcal{F}$ has a blue element and a red element) ⊛ "good."

We consider a random 2-coloring φ on X, that is, each $x \in X$ is colored blue or red with probability $\frac{1}{2}$, independent of other elements.

For $A \in \mathcal{F}$, Let M_A be the event that A is monochromatic. $Pr(\text{good}) > 0$. $\Leftrightarrow Pr(\text{not good}) < 1$.

$\Rightarrow P(M_A) = 2 \times (\frac{1}{2})^k = 2^{1-k}$.

$Pr(\underbrace{\varphi \text{ is "bad"}}) = Pr(\bigcup_{A \in \mathcal{F}} M_A) \le \sum_{A \in \mathcal{F}} P(M_A) = |\mathcal{F}| \cdot 2^{1-k} < 2^{k-1} \cdot 2^{1-k} = 1$.

i.e. NOT satisfying ⊛ $\Leftrightarrow \exists A$ s.t. A is monochromatic $\Leftrightarrow \exists A \in \mathcal{F}$ st. M_A occurs.

$\Rightarrow Pr(\underbrace{\varphi \text{ is "good"}}) = 1 - Pr(\varphi \text{ is "bad"}) > 0$.

i.e. φ satisfying ⊛.

Since $Pr(\varphi \text{ is "good"}) = \dfrac{\#\text{ good functions}}{\text{all functions}: X \rightarrow \{\text{red}, \text{blue}\}} > 0$.

we know #"good" functions is positive. #

We remark on the following two simple but useful ideas in proofs.

(i). Imagine we need to find some combinatorial object satisfying some property, call them "good" object. We consider a random object. If the probability that the random object is "good" is positive, then there must exist "good" object!

(ii). To compute the probability of being "good", we after compute the probability of being "bad" and aim to prove it is strictly less than 1.

<u>Classical Probability Space</u>: for $\forall A \subseteq \Omega$, $Pr(A) = \frac{|A|}{|\Omega|}$ and we say Pr is uniformly distributed.

e.g. Let S_n be the set of all permutations of $[n]$.
Define $Pr(A) = \frac{|A|}{|S_n|}$ for any $A \subseteq S_n$. We denote this probability space (S_n, Pr) simply by S_n.

Def: The <u>random graph</u> $G(n,p)$ ($0 \le p \le 1$) is a graph on vertex set $[n]$, where each of the potential $\binom{n}{2}$ edges appears with probability p, independents of other pairs.
Let $\mathcal{G}_n$ be the family of all graphs on $[n]$. $|\mathcal{G}_n| = 2^{\binom{n}{2}}$.

Let $p = \frac{1}{2}$. For any fixed $H \in \mathcal{G}_n$, $Pr(G(n,\frac{1}{2}) = H) = (\frac{1}{2})^{\binom{n}{2}} = \frac{1}{|\mathcal{G}_n|}$.
Thus the random graph $G(n,\frac{1}{2})$ can be viewed as a graph choosen from $\mathcal{G}_n$ uniformly at random.
In a larger sense, we can view $G(n,\frac{1}{2})$ as the probability space $(\mathcal{G}_n, Pr)$, where Pr is uniformly distributed.

uniformly distribution.

$G(n,\frac{1}{2}) \longleftrightarrow (\mathcal{F}_n, \mathbb{P}_r)$

probability space $\quad$ $= \{$all graphs with vertex set $[n]\}$