Let A be a property we are intersested in.

Let $Pr(A) = Pr(G(n, \frac{1}{2})$ satisfies property $A) = \dfrac{\#\text{ graphs } G \in \mathcal{F}_n \text{ satisfying property } A}{|\mathcal{F}_n|}$.

a function about n.

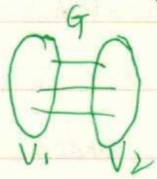
Def: We say random graph $G(n, \frac{1}{2})$ almost surely satisfies property A, if $\lim_{n\to+\infty} Pr(A) = 1$. Similarly, $G(n, \frac{1}{2})$ almost surely not satisfy property A, if $\lim_{n\to+\infty} Pr(A) = 0$.

Now we consider property A = bipartitness of graphs.

Thm: Random graph $G(n, \frac{1}{2})$ almost surely is NOT bipartite.

Proof: $Pr(A) = \dfrac{\#\text{ bipartite graphs on } [n]}{|\mathcal{F}_n|}$

A graph G is bipartite iff $\exists$ subset $U \subseteq V(G)$ s.t. all edges of G are between U and its complement U^c.



For subset $U \subseteq [n]$, Let A_U be the event that all edges of random graph $G := G(n, \frac{1}{2})$ are between U and U^c.

$$\Rightarrow Pr(A_U) = \frac{\#\text{ graphs } G \in \mathcal{F}_n \text{ satisfying } A_U}{2^{\binom{n}{2}}} = \frac{2^{|U|(n-|U|)}}{2^{\binom{n}{2}}}$$

U与U^c内边也一定不存在
U与U^c间边也任意可能.

$$\le \frac{2^{\frac{n^2}{4}}}{2^{\frac{n^2-n}{2}}} = 2^{-\frac{n^2}{4}+\frac{n}{2}}.$$

Let A be the event that $G(n, \frac{1}{2})$ is bipartite.

$$\Rightarrow A = \bigcup_{U\subseteq[n]} A_U.$$

A_U与A_{U^c}相同, 这不要紧.

$\Rightarrow$ By Union bound, $P_r(A) = P_r(\bigcup_{U\subseteq[n]} A_U) \le \sum_{U\subseteq[n]} P_r(A_U) \le 2^n \cdot 2^{-\frac{n^2}{4}+\frac{n}{2}} = 2^{-\frac{n^2}{4}+\frac{3}{2}n}$.

So $\lim_{n\to\infty} P(A) \le \lim_{n\to\infty} 2^{-\frac{n^2}{4}+\frac{3}{2}n} = 0$,

$\Rightarrow$ Random graph $G(n, \frac{1}{2})$ almost surely is NOT bipartite. #

Independent events : A property of Tournaments .

Def: k events $A_1, A_2, \cdots, A_k$ are independent, if $\forall I \subseteq [k]$, $P(\bigcap_{i\in I} A_i) = \prod_{i\in I} P(A_i)$.

竞赛图.

Def: A Tournaments of n vertices is a directed graph obtained from the complete graph K_n by assigning a direction to each edge.

We say a vertex i beats vertex j, if there exists an arc $i \to j$.

Def: We say a Tournament T has property S_k : for any subsets A of size k, there exists a vertex beats all vertices of A.

Question: For $\forall k \geq 2$, $\exists$ a T with property S_k ? ✓

Thm: $\forall k \geq 2$, if $\binom{n}{k}(1-\frac{1}{2^k})^{n-k} < 1$, then there exists a tournament on n vertices satisfying S_k.

Proof: We try to show the existence of such tournament by considering random tournament on $[n]$, where for any pair $i<j$, the arc $i\to j$ occurs with prob. $\frac{1}{2}$, independent of other choices.

Let B be the event that random tournament T doesn't have S_k.

$\Rightarrow B = \bigcup_{A\in\binom{[n]}{k}}$ (NO vertex can beats all vertices of A)

For $A \in \binom{[n]}{k}$, let B_A be the event that NO vertex in $[n]\backslash A$ can beat all vertices of A. so $B = \bigcup_{A\in\binom{[n]}{k}} B_A$.

For $u \notin A$, let $B_{u,A}$ be the event that u can't beats all vertices of A. $\Rightarrow$ $B_A = \bigcap_{u\notin A} B_{u,A}$.

Fact 1: $P(B_{u,A}) = 1-(\frac{1}{2})^k$.

Fact 2: $P(B_A) = P(\bigcap_{u\notin A} B_{u,A}) = \prod_{u\notin A} P(B_{u,A}) = (1-(\frac{1}{2})^k)^{n-k}$. since all events $B_{u,A}$

for all $u \notin A$ are independent, because only the arcs between u and A will effect $B_{u,A}$ and these are disjoint for each u.

By Union bound,

$$\Pr(B) = \Pr\left(\bigcup_{A \in \binom{[n]}{k}} B_A\right) \le \sum_{A \in \binom{[n]}{k}} \Pr(B_A) = \binom{n}{k}\left(1-\left(\tfrac{1}{2}\right)^k\right)^{n-k} \underset{\text{by condition.}}{<} 1.$$

so $\Pr(\text{Random Tournament satisfies } S_k) = \Pr(B^c) = 1-\Pr(B) > 0$.

$\Rightarrow$ There exists a tournament on n vertices satisfying property S_k. #.

Corollary: $\forall k \ge 2$, $\exists f(k)$ s.t. for $\forall n \ge f(k)$ (least), $\exists$ a tournament on n vertices satisfying property S_k.

$k=3$. as $\binom{91}{3}\left(\frac{7}{8}\right)^{88} < 1$, $\Rightarrow$ $f(3) \le 91$.

Def: For $(\Omega, \Pr)$, a mapping $X : \Omega \to \mathbb{R}$ is a random variable.

eg: We toss a fair coin n times, Let X denote the numbers of heads.

$\Omega = \{H, T\}^n$. $\Pr \leftarrow$ uniformly distributed.

$X: \Omega \to \mathbb{R}$ by $X(w) = \#$ H's in w.

$$\mathbb{E}(X) = \sum_{w\in\Omega} \Pr(\{w\}) \cdot X(w) = \sum_{w \in \Omega} \frac{X(w)}{|\Omega|} = \sum_{w\in\Omega} \frac{X(w)}{2^n} = \sum_{i=0}^{n} \sum_{w\in\Omega \text{ with } i \text{ heads}} \frac{i = X(w)}{2^n}$$

$$= \sum_{i=0}^{n} \frac{\binom{n}{i} i}{2^n} = \frac{n 2^{n-1}}{2^n} = \frac{n}{2}. \quad \#.$$

The linearity of expectation: $\forall X, Y$ r.v.'s, $\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y)$.

Def: For event $A \subseteq \Omega$, the indicator random variable $\mathbb{1}_A(w) = \begin{cases} 1, & w \in A \\ 0, & \text{o.w.} \end{cases}$

so, $\mathbb{E}(\mathbb{1}_A) = \Pr(A)$.

Let A_i be the events that i^{th} toss gives a head $\Pr(A_i) = \frac{1}{2}$.

$\Rightarrow X = \sum_{i=1}^{n} \mathbb{1}_{A_i} \Rightarrow \mathbb{E}(X) = \mathbb{E}(\sum \mathbb{1}_{A_i}) = \sum_{i=1}^{n} \mathbb{E}(\mathbb{1}_{A_i}) = \sum_{i=1}^{n} \Pr(A_i) = \frac{n}{2}$.

1 2 3 4 5 6 7 8 9 10 11 12 The month
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Fact: For any random variable X, $\Pr(X \geq \mathbb{E}(X)) > 0$ and $\Pr(X \leq \mathbb{E}(X)) > 0$. #

$\Pr(X \geq \mathbb{E}(X)) > 0$ iff $\exists \omega \in \Omega$ s.t. $X(\omega) \geq \mathbb{E}(X)$.

Thm: Let G be a graph on $2n$ vertices and with m edges. Then its vertex set can be partitioned into disjoint sets A and B with $|A| = |B| = n$, such that more than $\frac{m}{2}$ edges are between A and B. Max-Cut.

Proof: We choose $A \in \binom{V(G)}{n}$ uniformly at random, and let $B := V(G) \setminus A$.
$\Rightarrow |A| = |B| = n$.

Let $X := X(A,B)$ be the number of edges (a,b) between A and B where $a \in A$, $b \in B$. We want to compute $\mathbb{E}(X)$.

For each edge $uv \in E(G)$, let A_{uv} be the event that $uv \in (A,B)$. $\Leftrightarrow |\{u,v\} \cap A| = 1$.

Then $X = \sum_{uv \in E(G)} \mathbb{1}_{A_{uv}}$. So $\mathbb{E}(X) = \mathbb{E}(\sum \mathbb{1}_{A_{uv}}) = \sum \mathbb{E}(\mathbb{1}_{A_{uv}}) = \sum_{uv \in E(G)} \Pr(A_{uv})$.

where $\Pr(A_{uv}) = \dfrac{\#\, A \in \binom{V(G)}{n} \text{ with } |\{u,v\} \cap A| = 1}{\text{all subsets } A \in \binom{V(G)}{n}} = \dfrac{2 \cdot \binom{2n-2}{n-1}}{\binom{2n}{n}} = \dfrac{n}{2n-1}$

$\Rightarrow \mathbb{E}(X) = \sum_{uv \in E(G)} \Pr(A_{uv}) = \dfrac{mn}{2n-1} > \dfrac{m}{2}$.

By Fact, there exists a bipartite subgraph (A,B) with $|A| = |B| = n$, s.t. $X(A,B) \geq \mathbb{E}(X) > \frac{m}{2}$. #

若 $|A| = \frac{n}{3}$, $|B| = \frac{2n}{3}$. 同样处理.

Sum-free sets: A set B s.t. $\forall x,y \in B \Rightarrow x+y \notin B$.

Our hw exercise tells us that the maximum sum-free subset in $[2n]$ is of size n. e.g. $\{n+1, n+2, \cdots 2n\}$.

Thm: For any set A of non-zero integers, there is a sum-free subset $B \subseteq A$ with $|B| \geq \frac{1}{3}|A|$.

Proof: • We proceed by inducing this problem to a problem in the finite field $\mathbb{Z}_p = \{0, 1, \cdots p-1\}$ for some prime p.

• We will choose prime p large enough s.t. $p > |a|$ for $\forall a \in A$.

Observe that in $\mathbb{Z}_p$, there is a sum-free subset $S := \{\lceil \frac{p}{3} \rceil, \lceil \frac{p}{3} \rceil + 1, \cdots, \lfloor \frac{2p}{3} \rfloor\}$

$\Rightarrow |S| \geq \frac{p-1}{3}$. and S is sum-free mod p!

Pick a random element $x \in \mathbb{Z}_p^* = \mathbb{Z}_p \setminus \{0\}$, and let

$$A_x := \{a \in A : (ax \bmod p) \in S\}.$$

$$|A_x| = \sum_{a \in A} \mathbb{1}_{\{(ax \bmod p) \in S\}}$$

Claim: A_x is sum-free!

(Pf: For $a, b \in A_x$, then $(ax \bmod p) \in S$, $(bx \bmod p) \in S$,

$\Rightarrow (a+b)x \bmod p \notin S \Rightarrow a+b \notin A_x$.

Thus, A_x is sum-free!)

Next we compute $\mathbb{E}(|A_x|)$.

observe that for fixed $a \in \mathbb{Z}_p^*$, $\{(ax \bmod p) : x \in \mathbb{Z}_p^*\} = \mathbb{Z}_p^*$.

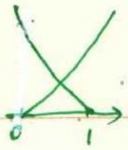
So running over $x \in \mathbb{Z}_p^*$, there are exactly $|S|$ many $x \in \mathbb{Z}_p^*$ satisfying $(ax \bmod p) \in S$.

for fixed a, $\Pr[(ax \bmod p) \in S] = \dfrac{\#\, x \in \mathbb{Z}_p^* \text{ with } (ax \bmod p) \in S}{|\mathbb{Z}_p^*|}$

$$= \frac{|S|}{|\mathbb{Z}_p^*|} \geq \frac{p-1}{3(p-1)} = \frac{1}{3}.$$

$\Rightarrow \mathbb{E}(|A_x|) = \sum_{a \in A} \Pr((ax \bmod p) \in S) \geq \frac{|A|}{3}$. $0 \notin A$.

$\Rightarrow$ There exists $x \in \mathbb{Z}_p^*$ s.t. $A_x = \{a \in A : (ax \bmod p) \in S\} \subseteq A$, which is sum-free and satisfies $|A| \geq \mathbb{E}|A_x| \geq \frac{|A|}{3}$. #



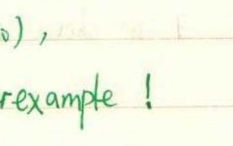
Problem: For $\forall k \geq 2$, let $f(x) = x^k + a_1 x^{k-1} + \dots + a_{k-1} x$ and
$g(x) = (1-x)^k + b_1(1-x)^{k-1} + \dots + b_{k-1}(1-x)$, where $a_1, a_2, \dots a_{k-1}, b_1, \dots b_{k-1} \geq 0$.
Let x_0 be the unique real in $[0,1]$, s.t. $f(x_0) = g(x_0)$,
Prove: $f(x_0) + g(x_0) \leq f(\frac{1}{2}) + g(\frac{1}{2})$ OR find a counterexample!
(TRUE for $k=2$)

Expectation.

Def: A dominating set of a graph $G=(V,E)$ is a set $U \subseteq V$ s.t. every $v \in V \setminus U$ has at least one neighbor in U.

Thm: Let $G=(V,E)$ be a graph on n vertices with minimum degree $\delta > 1$. Then, G has a dominating set of at most $\frac{1+\ln(1+\delta)}{1+\delta} n$ vertices.

Proof: Fix $p \in [0,1]$, We pick each vertex in V with probability p randomly and independently.



以概率 p 放到 A 中.

坏点放到 A 中. $A \to$ dominating

$Pr(u \text{ is bad}) = Pr(u \in V \setminus A \ \&$ all its neighbors are in $V \setminus A)$
$\leq (1-p)^{1+\delta}$.

Let X be the (random) set of vertices picked.

Let Y be the (random) set of vertices $u \in V \setminus X$, which has no neighbors in X. That is, $u \in Y$ iff u is not picked and all neighbors of u are not picked.

$$Pr(u \in Y) = (1-p)^{1+d(u)} \leq (1-p)^{1+\delta} \text{ (as } d(u) \geq \delta).$$

- $\mathbb{E}|X| = np$
- $\mathbb{E}|Y| = \mathbb{E} \sum_{u \in V} \mathbb{1}_{(u \in Y)} = \sum_{u \in V} P(u \in Y) \leq (1-p)^{1+\delta} n$

Observe that $X \cup Y$ is a dominating set.

So, $\mathbb{E}|X \cup Y| = \mathbb{E}(|X| + |Y|) \leq n(p + (1-p)^{1+\delta}) \leq n(p + e^{-p(1+\delta)})$ as $1-p \leq e^{-p}$

Check that $p + e^{-p(1+\delta)}$ is minimized at $p = \frac{\ln(1+\delta)}{1+\delta}$.

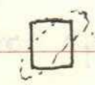
min at $\tilde{p} = 1 - (1+\delta)^{-\frac{1}{\delta}}$, δ 较大的时候两个界差不太多.

So we will fix our probability p to be $\frac{\ln(1+\delta)}{1+\delta}$.

Then $\mathbb{E}|X \cup Y| \leq n\left[\frac{\ln(1+\delta)}{1+\delta} + \frac{1}{1+\delta}\right] = \frac{1+\ln(1+\delta)}{1+\delta} n$.

Thus, $\exists$ a choice of $X \cup Y$ with $|X \cup Y| \leq \mathbb{E}|X \cup Y| \leq \frac{1+\ln(1+\delta)}{1+\delta} n$ #

Problem: For a graph G with minimum degree $\delta \geq 100$, find a set $U \subseteq V(G)$ with $U \leq C \frac{\ln\delta}{\delta} \cdot n$ for some constant $C > 0$ such that for $\forall v \in V \setminus U$, v has at least one neighbor in U and at least one neighbor in $V \setminus U$.

Def: For $G = (V, E)$, an independent set (/ stable set) $I \subseteq V$ is a subset which has NO edges insider. eg: 

Let $\alpha(G) = \max |I|$ over all independent sets I.

Thm: For any $G = (V, E)$, $\alpha(G) \geq \sum_{v \in V} \frac{1}{d(v)+1}$ where $d(v) =$ degree of v.

Proof: For any $i \in V := [n]$, Let N_i be the neighborhood of vertex i in G, Let S_n be the set of all permutations of $[n]$.

For $\forall \pi \in S_n$, we say vertex i is π-dominating if $\pi(i) < \pi(j)$ for $\forall j \in N_i$.

Let $M(\pi) = \{$all π-dominating vertices$\}$

Claim: $M(\pi)$ is an independent set! $\Rightarrow |M(\pi)| \leq \alpha(G)$.

(Pf: Suppose NOT, then $\exists\, i, j \in M(\pi)$ with $ij \in E(G)$.

Assume that $\pi(i) < \pi(j)$. Since $i \in N_j$, j is NOT π-dominating, i.e. $j \notin M(\pi)$, a contradiction! #)

- Pick $\pi \in S_n$ randomly and uniformly,
- Let $X = |M(\pi)|$ is a random variable.

$\Rightarrow \mathbb{E}(X) = \sum_{\lambda \in S_n} \frac{|M(\lambda)|}{|S_n|} \leq \alpha(G)$.

Compute $\mathbb{E}X$?

$X = \sum_{i \in [n]} \mathbb{I}_{\{i \text{ is } \lambda\text{-dominating}\}} \Rightarrow \mathbb{E}(X) = \sum_{i \in [n]} \Pr(i \text{ is } \lambda\text{-dominating})$

vertex i is λ-dominating iff $\lambda(i)$ is the minimum over $\{i\} \cup N_i$. But λ is random, each vertex in $\{i\} \cup N_i$ has the equal probability, which is $\frac{1}{1+d(i)}$, to achieve the minimum.

Thus, $\Pr(i \text{ is } \lambda\text{-dominating}) = \frac{1}{1+d(i)}$

$\Rightarrow \mathbb{E}X = \sum_i \Pr(i \text{ is } \lambda\text{-dominating}) = \sum_i \frac{1}{1+d(i)}$

$\Rightarrow \exists \lambda \in S_n$, s.t. $|M(\lambda)| \geq \mathbb{E}|X| = \sum_i \frac{1}{1+d(i)}$

$\Rightarrow \alpha(n) \geq \sum_{v \in V} \frac{1}{d(v)+1}$. #

Fact: $\left(\sum_{i \in [n]} \frac{1}{a_i}\right)\left(\sum_{i \in [n]} a_i\right) = n + \sum_{1 \leq i < j \leq n} \left(\frac{a_i}{a_j} + \frac{a_j}{a_i}\right) \geq n + \sum_{1 \leq i < j \leq n} 2 = n^2$.

Corollary: For graph G with n vertices and m edges, $\alpha(G) \geq \frac{n^2}{2m+n}$.

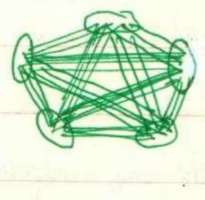
Proof: By Thm, $\alpha(G) \geq \sum \frac{1}{d(i)+1} \geq \frac{n^2}{\sum_i (d(i)+1)} = \frac{n^2}{2m+n}$. #

Turán's Thm: Let G be a graph on n vertices which has NO copy of K_{r+1}. Then $e(G) \leq \frac{r-1}{2r} n^2$, with equality if and only if $G = T_r(n)$.

Def: Turán graph $T_r(n)$ is a graph on n vertices.

$V(G) = V_1 \cup V_2 \cup \cdots \cup V_r$, and $||V_i| - |V_j|| \leq 1$,

where $ab \in E(G)$, if and only if $a \in V_i$ and $b \in V_j$ for any $i \neq j$.

$T_r(n) =$  NO K_{r+1} !

Exercise: Use corollary to prove Turán's Thm!

11.25.

小结:

- For graph G, $\alpha(G) = \max_{\substack{\text{independent} \\ \text{set } I}} |I|$.

  ⇒ Thm: $\forall G$, $\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{d(v)+1}$.

  ⇒ Cor: $\alpha(G) \geq \frac{n^2}{2m+n}$. ⇒ prove Turán's Thm.

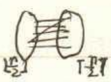
- Turán's Thm: G has NO copy of K_{r+1}. $e(G) \leq \frac{r-1}{2r} n^2$, "=" iff $G = T_r(n)$.

Def: Given a graph F, we say a graph G is F-free if G has NO copy of F.

Turán's Thm: For any K_{r+1}-free graph G on n vertices, $e(G) \leq e(T_r(n))$, with equality iff $G = T_r(n)$. $e(T_r(n)) \leq \frac{r-1}{2r} n^2$.

Proof 1: When $r=2$, Turán's Thm becomes Mentel's Thm:

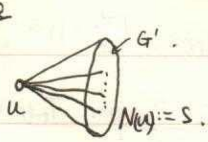
For any K_3-free graph G on n vertices satisfies $e(G) \leq \lfloor \frac{n^2}{4} \rfloor = e(T_2(n))$ with equality $G = T_2(n)$.



We will prove Turán's Thm by induction on r.

✓ Base case: $r=2$, that is Mentel's Thm.

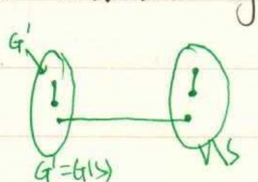
Let G be the K_{r+1}-free graph on n vertices with maximum number of edges. Pick a vertex u with maximum degree, Let $S := N(u)$.



Consider the subgraph G' of G on S.

Note that G' is K_r-free. (O.W. the K_r in G' plus u will give a K_{r+1} in G.)

Let $s := |S|$. Since G' is K_r-free on s vertices, by induction, $e(G') \leq e(T_{r-1}(s))$ ··· (1)

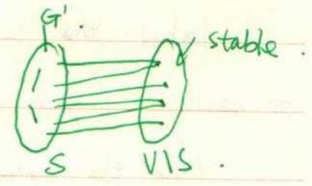


Let $e(S, V \setminus S)$ be the number of edges in E(G) with one endpoint in S and another endpoint in $V \setminus S$. Let $e(V \setminus S)$ be the number of edges contained in $V \setminus S$.

$\Rightarrow e(G) = e(G') + e(S, V\setminus S) + e(V\setminus S)$. ... (2).

Let H be obtained from G by deleting all edges in $V\setminus S$ and adding all missing edges across S and $V\setminus S$.

Claim: H is also K_{r+1}-free.



(pf: This is because G' is K_r-free, and $V\setminus S$ now is independent in H.)

$\Rightarrow e(H) = e(G) - e(V\setminus S) + \#$ missing edges across S and $V\setminus S$.

$= e(G) - e(V\setminus S) + |S|\cdot|V\setminus S| - e(S, V\setminus S)$. ... (3).

$\sum_{i\in V\setminus S} d(i) = 2e(V\setminus S) + e(S, V\setminus S)$

Since $N(u) = S$ is max, any $d_G(i) \leq |S|$,

$2e(V\setminus S) + e(S, V\setminus S) \leq |S|\cdot|V\setminus S|$

Thus, $e(H) = e(G) + |S|\cdot|V\setminus S| - e(V\setminus S) - e(S,V\setminus S) \geq e(G) + e(V\setminus S) \geq e(G)$.

But H is also K_{r+1}-free and $e(G)$ is maximum, $e(H) = e(G)$.

$\Rightarrow \begin{cases} e(V\setminus S) = 0 \\ \forall i \in V\setminus S,\ d(i) = |S| \end{cases} \Rightarrow H = G$.

So, $e(G) = e(G') + |S|\cdot|V\setminus S| \overset{(1)}{\leq} e(T_{r-1}(s)) + s(n-s)$ is a function of variable s.

$s = \frac{r-1}{r} n \to \max$.

$\overset{(r-1)\mid s}{\approx} \frac{r-2}{2(r-1)} s^2 + s(n-s) = -\frac{r}{2(r-1)}\left(s^2 - \frac{2(r-1)}{r} sn\right)$

(HW) One can check: this function defined on integer-value s is maximized when $s = \lceil \frac{r-1}{r} n \rceil$ then $e(G) = e(T_r(n))$.

Therefore $e(G) \leq e(T_{r-1}(s)) + s(n-s) \leq e(T_r(n))$. And when $e(G) = e(T_r(n))$, $e(G') = e(T_{r-1}(n))$, again by induction, we get $G' = T_{r-1}(\lceil \frac{r-1}{r} n \rceil)$, this plus $V\setminus S$ gives that $G = T_r(n)$.

$T_r(n)$:

#

proof 2 更漂亮，然而并不能给出 "=" 成立时情况.

Proof 2: (Will show: $e(G) \le \frac{r-1}{2r} n^2$ for all K_{r+1}-free G).

Label the vertices of G by $1, 2, \cdots n$, and assign a value p_i to each i such that $\sum_{i \in [n]} p_i = 1$ where $p_i \geq 0$.

Find the maximum of $P = \sum_{ij \in E(G)} p_i p_j$ by adjusting p_i's.

Claim: If $ij \notin E(G)$ and $p_i, p_j > 0$, then we can change

$$① \begin{cases} p_i \to 0 \\ p_j \to p_i + p_j \end{cases} \quad \text{or} \quad ② \begin{cases} p_i \to p_i + p_j \\ p_j \to 0 \end{cases} \quad \text{to increase } \underset{P \to P'}{P} \text{ strictly.}$$

最后 $p_i>0$ 的点构成了一个完全图.

Pf of claim: Let $S_i = \sum_{k \in N(i)} p_k$ and $S_j = \sum_{k \in N(j)} p_k$, let $S_i \geq S_j$ (by symmetry).

Then after ②, the new value P' satisfies:

$$\begin{aligned} P &= * + p_i S_i + p_j S_j \quad , \quad ij \notin E(G) \\ \downarrow & \\ P' &= * + (p_i + p_j) S_i + 0 \end{aligned}$$

$$\Rightarrow P' - P = p_j (S_i - S_j) \geq 0 . \quad \blacksquare$$

Now we keep applying the above claim.

When stop, we arrive on vertex $(p_1, p_2, \cdots p_n)$ s.t. the vertex i with $p_i > 0$ form a clique say Q.

So, $P = \sum_{ij \in E(Q)} p_i p_j = \frac{1}{2} \left[\left(\sum_{i \in V(Q)} p_i \right)^2 - \sum_{i \in V(Q)} p_i^2 \right] = \frac{1}{2} \left[1 - \sum_{i \in V(Q)} p_i^2 \right]$

- By Cauchy-Schwartz, $\sum_{i \in V(Q)} p_i^2 \geq \frac{1}{|Q|} \left(\sum_{i \in V(Q)} p_i \right)^2 = \frac{1}{|Q|}$
- Note that G is K_{r+1}-free, so $|Q| \leq r$.

$\Rightarrow P \leq \frac{1}{2}\left(1 - \frac{1}{|Q|}\right) \leq \frac{1}{2}\left(1 - \frac{1}{r}\right) = \frac{r-1}{2r}$. the maximum $P \leq \frac{r-1}{2r}$.

But we can assign $\frac{1}{n}$ to each vertex, then

$$P_0 = \sum_{ij \in E(G)} p_i p_j = \frac{e(G)}{n^2} \leq \max P \leq \frac{r-1}{2r}.$$

$\Rightarrow e(G) \leq \frac{r-1}{2r} n^2$.

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