- We discuss a generalization of Fisher's Inequality.

Def: $L \subseteq \{0,1,2,\cdots n\}$. We say $\mathcal{F} \subseteq 2^{[n]}$ is L-intersecting, if $|A \cap B| \in L$ for $\forall A \neq B \in \mathcal{F}$.

Rmk: Fisher's Ineq. just says that $|L|=1$.

Thm1 (Frankl – Wilson, 1981)

If $\mathcal{F}$ is an L-intersecting family in $2^{[n]}$, then $|\mathcal{F}| \leq \sum_{k=0}^{|L|} \binom{n}{k}$.

Rmk: This is best possible, considering all subsets of $[n]$ of size at most l for $L=\{0,1,\cdots l-1\} \Rightarrow |\mathcal{F}| = \sum_{k=0}^{l} \binom{n}{k}$.

12.14 Proof: Let $\mathcal{F} = \{A_1, A_2, \cdots A_m\}$ where $|A_1| \leq |A_2| \leq \cdots \leq |A_m|$

Lemma: $\exists f_1 \cdots f_m$, $\exists a_1 \cdots a_m$ $\begin{cases} f_i(a_i) \neq 0 \\ f_i(a_j) = 0, \ \forall j<i \end{cases}$

$\Rightarrow f_1 \cdots f_m$ linearly indep...

For each $A_i \in \mathcal{F}$, define polynomial $f_i(\vec{x})$ on $\mathbb{R}^n$, by

$$f_i(\vec{x}) = \prod_{l \in L \text{ with } l<|A_i|} (\vec{x} \cdot \vec{\mathbb{1}}_{A_i} - l).$$

so $f_i(\vec{x})$ is a polynomial with n variables and with degree $\leq |L|$.

Note: $f_i(\vec{\mathbb{1}}_{A_i}) = \prod_{l \in L, l<|A_i|} (|A_i| - l) > 0, \ \forall i$

For $1 \leq j < i \leq m$, $f_i(\vec{\mathbb{1}}_{A_j}) = \prod_{l \in L, l<|A_i|} (|A_i \cap A_j| - l) = 0$, since $\mathcal{F}$ is L-intersecting, $\exists l \in L$, s.t. $|A_i \cap A_j| = l \leq |A_i|$ where $|A_j| \leq |A_i| \Rightarrow |A_i \cap A_j| < |A_i|$.

By Lemma, we see $f_1(\vec{x}) \cdots f_m(\vec{x})$ are linearly independent.

Next, we consider the dimension of the space containing such polynomials.

$\vec{x} \in \{0,1\}^n$. $f_i(\vec{x}) = x_1^* x_2^* x_3 + x_5 x_6^* x_7 + \cdots$ 可以将 x_j^* 看作 x_j.

Observation: All vectors we consider are 0/1 vectors. Thus, we can define $\tilde{f}_i(\vec{x})$ from $f_i(\vec{x})$ by replacing x_i^k with x_i.

$\Rightarrow \tilde{f}_i(\vec{\mathbb{1}}_{A_j}) = f_i(\vec{\mathbb{1}}_{A_j}) = \begin{cases} 0, & j<i \\ >0, & j=i \end{cases}$.

$\Rightarrow \tilde{f}_1(\vec{x}), \cdots \tilde{f}_m(\vec{x})$ are linearly independent.

We see that each $\tilde{f}_i(\vec{x})$ is a linear combination of the monomials $\prod_{i\in I} x_i$, where $I \subseteq [n]$ and $|I| \le |L|$. And the number of such monomials is $\sum_{k=0}^{|L|}\binom{n}{k}$, which is the dimension of the space containing $\tilde{f}_1, \cdots \tilde{f}_m$.

#

$\Rightarrow |\mathcal{F}| = m \le \sum_{k=0}^{|L|}\binom{n}{k}$.

Thm2: Let p be a prime and $L \subseteq \mathbb{Z}_p = \{0, 1, \cdots p-1\}$.
Let $\mathcal{F} \subseteq 2^{[n]}$ be s.t. $\begin{cases} (1). & |A| \notin L \pmod p \\ (2). & |A\cap B| \in L \pmod p, \quad \forall A \ne B \in \mathcal{F}. \end{cases}$

Then $|\mathcal{F}| \le \sum_{k=0}^{|L|}\binom{n}{k}$.

$\mathcal{F}$ is L-intersecting, 对Thm1 in L, 可能可以找到一个 p, $L \to L \pmod p$, 即缩小 $|L|$, 使上界更好 (by Thm2).

Proof: All operations are mod p.

~~Define~~ a polynomial $f_i(\vec{x})$ over $\mathbb{Z}_p^n$ ~~for~~ each $A_i \in \mathcal{F} = \{A_1, \cdots A_m\}$.

as $$f_i(\vec{x}) = \prod_{l\in|L|} (\vec{x}\cdot \vec{I}_{A_i} - l)$$ $\mathbb{Z}_p$ 中不能排序, 不能有 $l<|A_i|$, 故有条件(1)

Then we have: $\begin{cases} f_i(\vec{I}_{A_i}) = \prod_{l\in L}(|A_i| - l) \ne 0 & \text{where } |A_i| \notin L. \\ f_i(\vec{I}_{A_j}) = \prod_{l\in L}(|A_i\cap A_j| - l) = 0. & \forall j \ne i. \end{cases}$

$\Rightarrow f_1, f_2 \cdots f_m$ are linearly independent over $\mathbb{Z}_p^n$.

Similarly, we replace each $f_i(\vec{x})$ by $\tilde{f}_i(\vec{x})$, where each factor x_i^k is replaced by x_i. As we only consider 0/1 ~~vectors~~, this does not affect the above properties. so $\tilde{f}_1, \cdots \tilde{f}_m$ remains linearly indep. which we are generated by monomials $\prod_{i\in I} x_i$ with $I \subset [n]$, $|I| \le |L|$.
so $|\mathcal{F}| \le m \le \text{dimension} \le \sum_{k=0}^{|L|}\binom{n}{k}$.

#.

Application:

We have construct a graph on $n=\binom{k}{3}$ vertices, which does not contain any clique or indep. set of size $>k$.

$\Rightarrow R(k+1,k+1) > \binom{k}{3} = \Omega(k^3)$.

We will improve the number of vertices to $n = k^{\Omega\left(\frac{\log k}{\log\log k}\right)}$ super-polynomial.

Thm3 (Frankl-Wilson): For any prime p, there is a graph G on $n=\binom{p^3}{p^2-1}$ vertices, such that the size k of maximum clique or independent set in G is at most $\sum_{i=0}^{p-1}\binom{p^3}{i}$.

Proof: $G=(V,E)$ is defined as follows:

- $V=\binom{[p^3]}{p^2-1}$ and for $A,B\in V$, $(A,B)\in E$ iff $|A\cap B| \not\equiv p-1 \pmod p$

(1). Consider a maximum clique, say with vertices $A_1,\cdots A_k$.

- $|A_i\cap A_j| \not\equiv p-1 \pmod p$ $\forall i\neq j$
- $|A_i| = p^2-1 \equiv p-1 \pmod p$.

we can use Thm2 by letting $L=\{0,1,\cdots p-2\}\subseteq \mathbb{Z}_p$. so $k\le \sum_{i=0}^{p-1}\binom{p^3}{i}$.

(2). Consider a maximum stable set, say $B_1,\cdots B_k$.

- $|B_i\cap B_j| \equiv p-1 \pmod p$ $\forall i\neq j$.
- $|B_i| = p^2-1$

so $|B_i\cap B_j|\in L=\{p-1, 2p-1,\cdots p(p-1)-1\}$, $|L|=p-1$. and $B_1\cdots B_k$ are L-intersecting.

By Thm1, $k\le \sum_{i=0}^{p-1}\binom{p^3}{i}$.

#

$n=\binom{p^3}{p^2-1} \simeq p^{\Theta(p^2)}$

$k=\sum_{i=0}^{p-1}\binom{p^3}{i} \simeq p\binom{p^3}{p} \simeq \left(\frac{p^3}{p}\right)^p \simeq p^{\Theta(p)}$

$\Rightarrow \log k \simeq p\log p \Rightarrow \log(\log k) \simeq \log p + \log\log p \simeq \log p$.

$\Rightarrow p \simeq \frac{\log k}{\log p} \simeq \frac{\log k}{\log\log k}$.

$\Rightarrow n \simeq p^{\Theta(p^2)} = (p^p)^p \simeq k^p \simeq k^{\frac{\log k}{\log\log k}}$.

$\stackrel{\text{Thm3}}{\Rightarrow} R(k,k) \geqslant k^{\Omega\left(\frac{\log k}{\log\log k}\right)}$.

Def: Given a set $S \subseteq \mathbb{R}^n$, (bounded) the diameter of S is denoted as $\mathrm{Diam}(S) = \sup\{\underset{\downarrow}{d(x,y)} : x, y \in S\}$.

Euclidean distance between x and y in $\mathbb{R}^n$.

Borsuk's Conjecture: Can every bounded $S \subseteq \mathbb{R}^d$ be partitioned into $d+1$ sets of strictly smaller diameter?

Known: S = sphere, S = a smooth convex body, $d \le 3$.

但后这个猜想并不对. In 1993, Kahn-Kalai disproved this conjecture

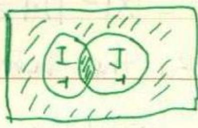
Lemma: For prime p, there exists a set of $\frac{1}{2}\binom{4p}{2p}$ vectors in $\mathcal{F} \subseteq \{-1,1\}^{4p}$, such that every subset of size $2\binom{4p}{p-1}$ vectors contains an orthogonal (正交的) pair of vectors.

Pf: Let $Q = \{I \in \binom{4p}{2p} : 1 \in I\}$.

For $\forall I \in Q$, define a vector $\vec{v}^I \in \{-1,1\}^{4p}$ by $\vec{v}^I_i = \begin{cases} 1, & i \in I \\ -1, & i \notin I \end{cases}$.

Let $\mathcal{F} = \{\vec{v}^I : I \in Q\}$.

Compute: $\vec{v}^I \cdot \vec{v}^J = |I\cap J| - |I \cap J^c| - |I^c \cap J| + |I^c \cap J^c|$
$= 4p - 2|I \Delta J|$



so $\vec{v}^I \perp \vec{v}^J$ iff $|I\Delta J| = 2p = |I| + |J| - 2|I\cap J| = 4p - 2|I\cap J|$ iff $|I \cap J| = p$.

Note that $|I|, |J| = 2p$. $\Rightarrow$ $1 \le \underset{1\in I\cap J}{|I\cap J|} \le 2p-1$. 令$1 \in I\cap J$, 保证了$|I\cap J| \neq 0$.

$\Rightarrow$ ✪: $\vec{v}^I \perp \vec{v}^J$ iff $|I\cap J| \equiv 0 \pmod p$

claim: For any subset $G \subseteq \mathcal{F}$ without orthogonal pairs, the $|G| \le \sum_{k=0}^{p-1}\binom{4p}{k}$

(Pf: $\forall A \in G$, $|A| = 2p \equiv 0 \pmod p$.

$\forall A, B \in G$, $|A\cap B| \not\equiv 0 \pmod p$.

Taking $L = \{1, 2, \cdots p-1\} \subseteq \mathbb{Z}_p$.

By Thm 2, $|G| \le \sum_{k=0}^{p-1}\binom{4p}{k}$. #)

$\Rightarrow$ max subset without orthogonal pairs $\le \sum_{k=0}^{p-1}\binom{4p}{k} \underset{\uparrow}{<} 2\binom{4p}{p-1}$

Claim2 (Exercise) #

Thm 4: For d sufficiently large, there exists a bounded $S \subseteq \mathbb{R}^d$ (in fact a finite set) s.t. any partition of S into $\le 1.1^{\sqrt{d}}$ contains a part of the same diameter.

As $1.1^{\sqrt{d}} >> d+1$, this disproves Borsuk's conj.

Proof: Let $\mathcal{F}_1$ be from the Lemma.

(Def: A tensor product of vector $\vec{v} \in \mathbb{R}^n$ is
$\vec{w} = \vec{v} \otimes \vec{v} \in \mathbb{R}^{n^2}$ by $w_{ij} = v_i \cdot v_j$, $\forall 1 \le i, j \le n$.)

Let $X = \{\vec{v} \otimes \vec{v} : \vec{v} \in \mathcal{F}_1\} \subseteq \{-1, 1\}^{n^2} \subseteq \mathbb{R}^{n^2}$, $n = 4p$.

- $\vec{w} \in \{-1, 1\}^{2n}$. $\Rightarrow \|\vec{w}\|^2 = n^2 \Rightarrow \|\vec{w}\| = n$.
- $\vec{w}, \vec{w}' \in X$, say $\vec{w} = \vec{v} \otimes \vec{v}$, $\vec{w}' = \vec{v}' \otimes \vec{v}'$
  $\therefore \vec{w} \cdot \vec{w}' = \sum_{i,j} w_{ij} \cdot w'_{ij} = \sum_{1 \le i,j \le n} (v_i v_j) \cdot (v_i' v_j') = (\vec{v} \cdot \vec{v}')^2$
- $\vec{w} \perp \vec{w}'$ iff $\vec{v} \perp \vec{v}'$
- $\|\vec{w} - \vec{w}'\|^2 = \|\vec{w}\|^2 + \|\vec{w}'\|^2 - 2\vec{w} \cdot \vec{w}' = 2n^2 - 2(\vec{v} \cdot \vec{v}')^2 \le 2n^2$.

$$\Rightarrow \begin{cases} \mathrm{Diam}(X) = \sqrt{2}n \\ |X| = |\mathcal{F}_1| = \frac{1}{2}\binom{4p}{2p} \end{cases}$$

By the Lemma, any subset of $2\binom{4p}{p-1}$ vectors in $\mathcal{F}_1$, contains an orthogonal pair. Thus, any subset of $2\binom{4p}{p-1}$ vectors in X contains a pair $\vec{w}, \vec{w}'$ of max distance $= \sqrt{2}n$.

Therefore, if we want to decrease the diameter, we must partition X into sets of size less than $2\binom{4p}{p-1}$, and so the number of parts is

$$\ge \frac{|X|}{2\binom{4p}{p-1}} = \frac{\frac{1}{2}\binom{4p}{2p}}{2\binom{4p}{p-1}} = \frac{1}{4} \frac{(3p+1) \cdots (2p+1)}{2p(2p-1) \cdots p} \ge \frac{1}{4}\left(\frac{3}{2}\right)^{p+1} \ge c\left(\frac{3}{2}\right)^{\frac{\sqrt{d}}{4}} \ge 1.1^{\sqrt{d}}$$

$X \subseteq \mathbb{R}^d = \mathbb{R}^{n^2}$, $d = n^2 = (4p)^2 = 16p^2$.

#

12.16

Recall: Sperner's Thm: For any antichain $\mathcal{F} \subseteq 2^{[n]}$ (i.e. $\forall A, B \in \mathcal{F}$, $A \not\subseteq B$, $B \not\subseteq A$)
Then we have $|\mathcal{F}| \le \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

In fact, we proved a stronger result:

LYM-Inequality: For any antichain $\mathcal{F} \subseteq 2^{[n]}$, $\sum_{A \in \mathcal{F}} \frac{1}{\binom{n}{|A|}} \le 1$.

Today, we study an even stronger result, namely, the Bollobás's Thm (extremal set theory)

Bollobás's Thm: Let $A_1, A_2, \cdots A_m$ and $B_1, B_2, \cdots B_m$ be two sequences of sets such that:

- $A_i \cap B_j \neq \phi$, $\forall i \neq j$ → 这个条件需要，否则 $\begin{matrix}\phi & \{1\} \\ \{1\} & \phi\end{matrix}$ 就不满足结论.
- $A_i \cap B_i = \phi$, $\forall i$. 这个条件 $\Rightarrow$ $A_1 \cdots A_m$ distinct, $B_1 \cdots B_m$ distinct

Then $\sum_{i=1}^{m} \frac{1}{\binom{a_i+b_i}{a_i}} \le 1$, where $a_i = |A_i|$ and $b_i = |B_i|$

Exercise: Prove that Bollobás's Thm can imply LYM-Inequality. ($B_i = A_i^c$)

Proof: Let $X = \bigcup_{i=1}^{m} (A_i \cup B_i)$, we prove by induction of $|X| = n$.
when $n = 1$, clearly. $\begin{matrix}\{1\} \\ \phi\end{matrix}$
So we assume it holds for $|X| \le n-1$. For each $x \in X$, define

$$I_x = \{1 \le i \le m : x \notin A_i\}$$

Define $\mathcal{F}_x = \{A_i : i \in I_x\} \cup \{B_i - \{x\} : i \in I_x\}$. Note that any set of $\mathcal{F}_x$ doesn't contain x, so $\mathcal{F}_x$ has less than n elements.
Hence we apply induction hypothesis for each $\mathcal{F}_x$. (check $\mathcal{F}_x$ satisfies the condition)

$$\sum_{i \in I_x} \frac{1}{\binom{|A_i| + |B_i - \{x\}|}{|A_i|}} \le 1 \quad \cdots ①$$

We summing up the above inequalities for all $x \in X$

$$\sum_{x \in X} \sum_{i \in I_x} \frac{1}{\binom{|A_i| + |B_i - \{x\}|}{|A_i|}} \le n \quad \cdots ②$$

For each i, it contributes either $\underset{\uparrow \atop i \notin I_x}{0}$, or $\underbrace{\frac{1}{\binom{a_i+b_i}{a_i}} \text{ or } \frac{1}{\binom{a_i+b_i-1}{a_i}}}_{i \in I_x}$ to each x.

The term $\frac{1}{\binom{a_i+b_i}{a_i}}$ corresponds to points $x \notin A_i \cup B_i$. thus this term appears exactly $(n-a_i-b_i)$ times.

While, the term $\frac{1}{\binom{a_i+b_i-1}{a_i}}$ corresponds to points $x \notin A_i$ & $x \in B_i$, thus this term appears exactly b_i times.

$$② \Leftrightarrow \sum_{i=1}^{m} \left[(n-a_i-b_i)\frac{1}{\binom{a_i+b_i}{a_i}} + b_i \frac{1}{\binom{a_i+b_i-1}{a_i}} \right] \le n.$$

Since $\frac{\binom{k-1}{l}}{\binom{k}{l}} = \frac{k-l}{k}$, we get $\frac{1}{\binom{a_i+b_i-1}{a_i}} = \frac{1}{\binom{a_i+b_i}{a_i}} \frac{a_i+b_i}{b_i}$,

Plugging in, $\sum_{i=1}^{m} \left[(n-a_i-b_i)\frac{1}{\binom{a_i+b_i}{a_i}} + \frac{a_i+b_i}{\binom{a_i+b_i}{a_i}} \right] \le n$

$$\Leftrightarrow \sum_{i=1}^{m} n \cdot \frac{1}{\binom{a_i+b_i}{a_i}} \le n$$

$$\Leftrightarrow \sum_{i=1}^{m} \frac{1}{\binom{a_i+b_i}{a_i}} \le n. \qquad \#$$

Def: Let $\mathbb{F}$ be a field, a set $A \subseteq \mathbb{F}^n$ is in general position, if any n vectors in A are lineraly independent over $\mathbb{F}$.

e.g. for $a \in \mathbb{F}$, define $\vec{m}(a) = (1, a, a^2, \dots a^{n-1}) \in \mathbb{F}^n$. (moment curve)

Then $\{\vec{m}(a) : a \in \mathbb{F}\}$ is in general position. $|A|$ 可以十分大.

Next, we use the so-called "general position" argument to prove a version of Bollobás's Thm, which is weaker than the previous one. But, on the other hand, the condition can be generalized to

$$A_i \cap B_j \ne \phi \quad \text{for } \forall\, i<j.$$

Bollobás's Thm (the skew version)

Let $A_1, \dots A_m$ be sets of size r and $B_1, \dots B_m$ be sets of size s, such that:

- $A_i \cap B_j \ne \phi$, $\forall\, i<j$
- $A_i \cap B_i = \phi$, $\forall\, i$

Then, $m \le \binom{r+s}{s}$.

Proof (By Lovász): Let $X = \bigcup_i (A_i \cup B_i)$

Take a set $V \subseteq \mathbb{R}^{r+1}$ of vectors $\vec{v} = (v_0, v_1, \cdots v_r)$ such that

- V is in general position
- $|V| = |X|$.

Identify the elements of X with vectors in V. $X \leftrightarrow V$

Hence, we will view A_i as a subset in V containing r vectors and B_j as a subset in V containing s vectors.

For each B_j, define $f_j(\vec{x}) = \prod_{v \in B_j} \langle \vec{v}, \vec{x} \rangle = \prod_{\vec{v} \in B_j} (v_0 x_0 + \cdots + v_r x_r)$.

For $x \in \mathbb{R}^{r+1}$, Note that $f_j(\vec{x}) = 0$ iff $\underline{\langle \vec{v}, \vec{x} \rangle = 0 \text{ for some } \vec{v} \in B_j}$ $\cdots$ ①.

Consider the subspace $\operatorname{span} A_i$, which is spanned by the r vector in A_i. Since $A_i \subseteq V \subseteq \mathbb{R}^{r+1}$ and V is in general position, we see that all r vectors in A_i are linearly independent and thus $\dim(\operatorname{span} A_i) = r$.

So, $(\operatorname{span} A_i)^{\perp}$ has dimension 1. Choose $\vec{a_i} \in (\operatorname{span} A_i)^{\perp}$ for $i = 1, \cdots m$

Then for each $\vec{v} \in V$, $\langle \vec{v}, \vec{a_i} \rangle = 0$ iff $\vec{v} \in \operatorname{span} A_i$. $\cdots$ ②

iff $\vec{v} \in A_i$

(O.W. $\vec{v} \notin A_i$, $\{\vec{v}\} \cup A_i$ has $r+1$ vectors in V, which must be linearly indep., contradicting to $\vec{v} \in \operatorname{span} A_i$).

Combining ① & ②, $f_j(\vec{a_i}) = \prod_{v \in B_j} \langle \vec{v}, \vec{a_i} \rangle = 0$ iff $A_i \cap B_j \neq \phi$.

$$\Rightarrow \begin{cases} f_j(\vec{a_i}) = 0, & \forall i < j \\ f_j(\vec{a_j}) \neq 0 \ (\text{since } A_j \cap B_j = \phi), & \forall j. \end{cases}$$

This shows that $\underline{f_1, \cdots f_m \text{ are linearly indep.}}$ $\quad (f_i(\vec{a_j})) = \begin{pmatrix} * & & 0 \\ & * & \\ * & & * \end{pmatrix}$

Next, we give an upper bound on the dimension of the space containing $f_1 \cdots f_m$.

<u>Recall</u>: $f_j(\vec{x}) = \prod_{\vec{v} \in B_j} (v_0 x_0 + \cdots + v_r x_r)$, it is homogeneous with degree $s = |B_j|$ and $r+1$ variables $(x_0, x_1, \cdots x_r)$. So this ploynomial space can be generated by all monomials of follows:

$$x_0^{i_0} x_1^{i_1} \cdots x_r^{i_r}, \text{ where } i_0 + i_1 + \cdots + i_r = s, \ i_j \geq 0. \rightarrow \binom{r+1+s-1}{r+1-1} = \binom{r+s}{r}$$

There are $\binom{r+s}{r}$ many solutions! So $m \leq$ the dimension $= \binom{r+s}{r}$. #

Subspace version: $V_1, \dots V_m$ be subspaces of dimension r &
$W_1, \dots W_m$ ... ... s.

$$\begin{cases} V_i \cap W_i = \phi \\ V_i \cap W_j \neq \phi, \ i \neq j \end{cases} \Rightarrow m \leq \binom{r+s}{r}.$$ #

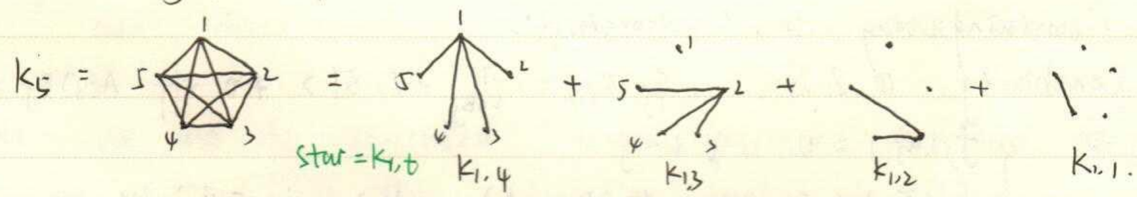
(补充).

12.21

Covering by complete bipartite subgraphs.

The following question was motivated by telephone communication problem.

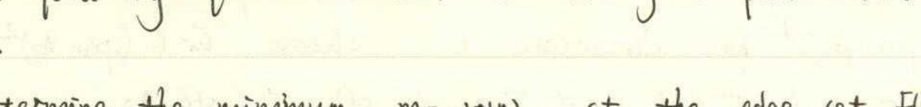
Q: Determine the minimum $m = m(n)$ s.t. the edge set $E(K_n)$ can be expressed as a disjoint union of edge sets of m complete bipartite subgraphs of K_n.



Fact: $m(n) \leq n-1$.

Pf: Because we can express $E(K_n)$ as a disjoint of $n-1$ stars. #

Rmk:



对 K_5 的分法不唯一.

We point out that there exist other partitions ~~of~~ $E(K_n)$, using $n-1$ complete bipartite subgraph, which is NO isomorphic to the star-decomposition.

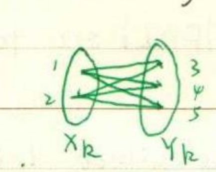
Fact 1: For any $n\times m$ matrixs M_1, M_2. $\operatorname{rank}(M_1+M_2) \le \operatorname{rank}(M_1) + \operatorname{rank}(M_2)$.

Def: The adjacency matrix of a graph H on n vertices is an $n\times n$ matrix $A=(a_{ij})_{n\times n}$ s.t. $a_{ij} = \begin{cases} 1, & ij \in E(H) \\ 0, & \text{o.w.} \end{cases}$, (so A is symmetric)

Thm (Graham–Pollak): $m(n) \geq n-1$.

Proof: Suppose the complete bipartite graphs $B_1, B_2, \cdots B_m$ disjointly cover all edges of K_n, i.e. $E(K_n) = E(B_1) \dot\cup \cdots \dot\cup E(B_m)$.

Let X_i and Y_i be the color classes of B_i, i.e. all edges of B_i go between X_i and Y_i. For each B_k, we define an $n\times n$ matrix $A_k = (a_{ij}^{(k)})_{n\times n}$ by $a_{ij}^{(k)} = \begin{cases} 1, & i\in X_k \,\&\, j\in Y_k \\ 0, & \text{o.w.} \end{cases}$

eg: 

$$\Rightarrow A_k = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \cdot & - & & 0 \\ 0 & - & - & - & 0 \end{pmatrix}$$

Claim: $\operatorname{rank}(A_k) = 1$, $\forall\, 1\le k\le m$.

(Pf: Because $B_k = (X_k, Y_k)$ is complete, the i^{th} rows (for $i\in X_k$) are identical. #.

Let $A = A_1 + A_2 + \cdots + A_m$. As each $ij \in E(K_n)$ belongs to exactly one of the graphs B_k, we have

$$a_{ij}^{(k)} = 1,\ a_{ji}^{(k)} = 0 \quad \text{or} \quad a_{ij}^{(k)} = 0,\ a_{ji}^{(k)} = 1.$$

$\Rightarrow A + A^T = J_n - I_n$, where $J_n = (1)_{n\times n}$. $= A(K_n)$.

Note that $\operatorname{rank} A \le \sum_{i=1}^{m} \operatorname{rank}(A_i) = m$.

It suffices to prove: $\operatorname{rank} A \geq n-1$. Suppose a contradiction that $\operatorname{rank} A \le n-2$. Let A' be an $(n+1)\times n$ matrix obtained from A by adding an extra row $(1,1,\cdots 1)$. $A' = \begin{pmatrix} A \\ 1,1,\cdots,1 \end{pmatrix}$

so $\operatorname{rank} A' \le n-1$.

Then there exists non-zero vector $\vec{x} \in \mathbb{R}^n$ s.t. $A'\vec{x} = \vec{0}$.

$\Leftrightarrow$ $A\vec{x} = 0$ & $\sum_{i=1}^{n} x_i = 0$.

Consider $\vec{x}^T (A + A^T)\vec{x} = \vec{x}^T (J_n - I_n)\vec{x}$

$\Leftrightarrow$ $0 = \vec{x}^T(A\vec{x}) + (A\vec{x})^T \vec{x} = \vec{x}^T(\underbrace{J_n \vec{x}}_{0}) - \vec{x}^T I_n \vec{x} = 0 - \sum_{i=1}^{n} x_i^2 < 0$. contradiction

Thus, $\operatorname{rank} A \geq n-1$.

$\Rightarrow$ $m(n) \geq n-1$. #

Finite Projective Plane (FPP)

Def: Let X be a finite set and $\mathcal{L} \subseteq 2^X$ be a family. The pair $(X, \mathcal{L})$ is called a finite projective plane (FPP for short), if it satisfies three axioms:

(P0) There exists a 4-set $F \subseteq X$ s.t. $|F \cap L| \leq 2$ for $\forall L \in \mathcal{L}$.

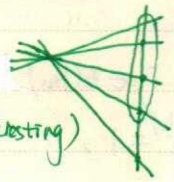
(P1) $\forall L_1, L_2 \in \mathcal{L}$ has $|L_1 \cap L_2| = 1$

(P2) $\forall x_1, x_2 \in X$, there exists exactly one subset $L \in \mathcal{L}$ with $\{x_1, x_2\} \subseteq L$.

We call elements of X as points and the sets in $\mathcal{L}$ as lines.

$\Rightarrow$ (P1): 两直线交于1点. (P2): 两点确定一条直线.

(P0)?



NOT satisfies (P0).

(NOT-interesting)

So we explains for (P0) – (P2).

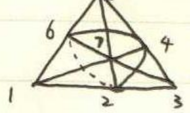
(P1) says: $\forall$ 2 lines intersect at exactly one point.

(In geometry, parellel lines do not!)

(P2) says: $\forall$ 2 points a, b determine a line, denoted by $\overline{ab}$.

(P0) is used to exclude some not-interesting cases.

e.g. The Fano plane (The smallest FPP)



$\{1\,2\,3\}$ $\{3\,4\,5\}$ $\{5\,6\,1\}$
$\{1\,4\,7\}$ $\{3\,6\,7\}$ $\{2\,5\,7\}$
$\{2\,4\,6\}$.

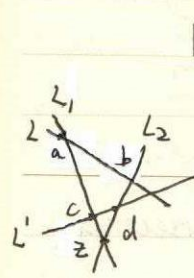
} lines.

※ 7 points & 7 lines, each lines with 3 points.

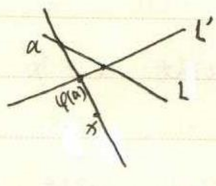
Prop 1: Let $(X, \mathcal{L})$ be a finite projective plane, Then for any 2 lines $L, L' \in \mathcal{L}$, $|L| = |L'|$.

Pf: Claim: $\exists\, x \in X$ with $x \notin L \cup L'$.

(Pf: Let $F \subseteq X$ be from (P0), Then $|F \cap L| \le 2$ & $|F \cap L'| \le 2$.
If $F \nsubseteq L \cup L'$, then we are done, so $F \subseteq L \cup L'$ with $|F \cap L| = |F \cap L'| = 2$, Let's say $F \cap L = \{a, b\}$, $F \cap L' = \{c, d\}$.
Let $L_1 = \overline{ac}$, $L_2 = \overline{bd}$. Let $z \in L_1 \cap L_2$ be the unique point.



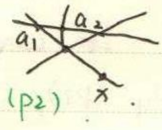
If $z \notin L$, $z \notin L'$, then again we are done,
so we assume $z \in L$. $\Rightarrow z \in L \cap L_1$. But $a \in L \cap L_1$.
By (P1), $z = a \in L_2$, $b, d \in L_2$. $\Rightarrow a, b, d \in L_2 \cap F$, a contradiction to (P0)!
$\Rightarrow \exists\, x \notin L \cup L'$. #.



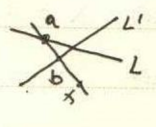
Now fix $x \notin L \cup L'$, we define a mapping $\varphi: L \to L'$ as follows: for $\forall\, a \in L$, let $\varphi(a) \in L'$ be the unique point in $L' \cap \overline{ax}$.

Next we show φ is bijective.

- φ is injective: If $\exists\, a_1, a_2 \in L$ s.t. $\varphi(a_1) = \varphi(a_2)$ then $a_1, a_2, \varphi(a_1), x$ are in the line $\overline{\varphi(a_1)x} = \overline{a_1x} = \overline{a_2x}$. (P2)



  But $\{a_1, a_2\} \subseteq \overline{\varphi(a_1)x} \cap L$, a contradiction to (P1).
- φ is surjective: For any $b \in L'$, let a be the unique point in $\overline{bx} \cap L$. a, b, x are in the line $\overline{ax} = \overline{bx}$.



  $\Rightarrow b \in \overline{ax} \cap L' \Rightarrow \varphi(a) = b$.

$\Rightarrow \varphi$ is bijective. $\Rightarrow |L| = |L'|$. #.

Def: Let $(X, \mathcal{L})$ be a finite projective plane. The order of $(X, \mathcal{L})$ is the number $|L|-1$, for $\forall L \in \mathcal{L}$.

Prop 2: Let $(X, \mathcal{L})$ be a FPP of order n, Then

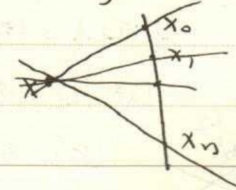
(i). exactly $n+1$ lines pass through each point $x \in X$

(ii). $|X| = n^2+n+1$

(iii). $|\mathcal{L}| = n^2+n+1$.

Proof: (i). Consider $\forall x \in X$, Let F be the 4-set satisfying (P0). $F=\{a,b,c,d\}$. Let $a, b, c \in F$ be distinct from x.

- Then, at least one of lines $\overline{ab}$, $\overline{ac}$ doesn't contain x. (otherwise a, b, c, x are in the same line, contradicting (P0).)
- There exists a line L with $x \notin L$. Let $L = \{x_0, x_1, \cdots x_n\}$, then $\overline{x_i x}$ define $n+1$ lines.



- On the other hand, any line containing x must intersecting L at some point, say x_i.

Therefore, there are exactly $n+1$ lines conta[in]ing x. #

(ii). Choose some line $L=\{x_0, x_1, \cdots x_n\} \in \mathcal{L}$ and a point $a \in X$ with $a \notin L$. Let $L_i = \overline{a x_i}$ for $i = 0, 1, \cdots n$.

By (P1), any two lines L_i, L_j interset at a single point, that is a. so $|L_0 \cup L_1 \cup \cdots \cup L_n| = n(n+1)+1 = n^2+n+1$.

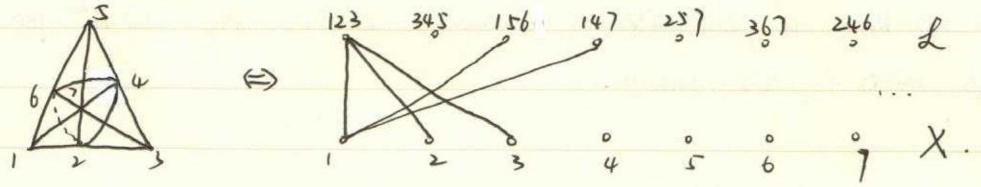
It remains to show that any $x \in X - \{a\}$ must belong to $L_0 \cup L_1 \cdots \cup L_n$. By (P1), line $\overline{ax}$ must intersect L at some point x_i, then $\overline{ax} = \overline{ax_i} = L_i \Rightarrow x \in L_i$.

$\Rightarrow X = L_0 \cup L_1 \cup \cdots \cup L_n$.

$\Rightarrow |X| = n^2+n+1$. #

(iii). Exercise. #.

Def: The <u>incidence graph</u> of a FPP $(X, \mathcal{L})$ is a bipartite graph G with parts X and $\mathcal{L}$, where $x \in X$ is adjacent $L \in \mathcal{L}$ iff $x \in L$.



From this we can prove that $|X| = |\mathcal{L}|$. $\quad (e = |X| d(x_0) = |\mathcal{L}| d(L_0))$.

Def: The <u>dual</u> $(\mathcal{L}, \Lambda)$ of a FPP $(X, \mathcal{L})$ is obtained by taking the incidence graph G of $(X, \mathcal{L})$ and interpreting the points in $(X, \mathcal{L})$, as the lines in the new system and the lines in $(X, \mathcal{L})$ as the point in new system. swapping the roles of "points" and "lines".

Rmk: $\forall x \in X \to L_x = \{ L \in \mathcal{L} : x \in L \}$

$\Lambda = \{ L_x : x \in X \}$.

Prop 3. The dual $(\mathcal{L}, \Lambda)$ of a FPP $(X, \mathcal{L})$ is also a FPP.

Proof: We point out that for $i = 1, 2$.

(P_i) for $(X, \mathcal{L})$ gives rise to (P_{3-i}) for $(\mathcal{L}, \Lambda)$.

$$\begin{cases} (P1) \Rightarrow (P2)^*: \forall L_1, L_2 \in \mathcal{L}, \exists! \, L_x \in \Lambda, \text{ s.t. } \{L_1, L_2\} \subseteq L_x \iff x \in L_1 \cap L_2. \\ (P2) \Rightarrow (P1)^*: \forall L_{x_1}, L_{x_2} \in \Lambda, \exists! \, L \in \mathcal{L}, \text{ s.t. } L \in L_{x_1} \cap L_{x_2} \iff \exists! \, L \text{ go through } \{x_1, x_2\} \end{cases}$$

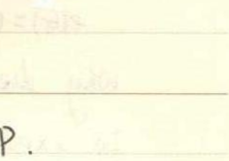
$(P0)^*$: It suffices to show that $(P0)$ holds for $(\mathcal{L}, \Lambda)$, that is

✪ we need to find 4 lines L_1, L_2, L_3, L_4, s.t. NO 3 of them have a common point!

$L_i, L_j, L_k \in L_x \iff x \in L_i \cap L_j \cap L_k$.

Let $F = \{a, b, c, d\}$ be the 4-set satisfying $(P0)$ for $(X, \mathcal{L})$.

Note that for such F, $|F \cap L| \le 2$, $\forall L \in \mathcal{L}$.



Since any 3 points of F don't lie on a line, we can define 4 distinct lines as follows:

$L_1 = \overline{ab}$, $L_2 = \overline{cd}$, $L_3 = \overline{ad}$, $L_4 = \overline{bc}$.

By Property of F, for any 3 lines of $\{L_1, L_2, L_3, L_4\}$ (by symmetry, say L_1, L_2, L_3), we see $L_1 \cap L_3 = \{a\}$, $L_2 \cap L_3 = \{d\} \Rightarrow L_1 \cap L_2 \cap L_3 = \emptyset$. This proves ✪.

#

Thm: A finite projective plane of order n exists whenever a field with n elements exists.

And we know a field with n elements exists iff $n = p^k$ for a prime p.

Q: $n = p_1 p_2$ exists? NO known.

<u>An Application of FPP</u>:

<u>Recall</u>: $\forall$ C_4-free graph G on m vertices has $e(G) \le \frac{m}{2}(1+\sqrt{4m-3})$ 

$ex(n, C_4) \le \frac{n}{2}(1+\sqrt{4n-3})$.

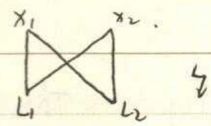
Thm: For infinitely many integers m, there exists a C_4-free graph on m vertices and with at least $0.35\, m^{\frac{3}{2}}$.

Proof: Take a FPP $(X, \mathcal{L})$ with order n and take its incidence graph G.

G has $m = |X| + |\mathcal{L}| = 2(n^2+n+1)$ vertices and $e(G) = \sum_{x\in X} d_G(x) = (n+1)(n^2+n+1)$.

$e(G) = (n+1)(n^2+n+1) \geqslant (n^2+n+1)^{\frac{3}{2}} = (\frac{m}{2})^{\frac{3}{2}} \geqslant 0.35\, m^{\frac{3}{2}}$.

why does such G has NO C_4?



If existing, then in the language of the FPP, it says there exist 2 points x_1, x_2 and 2 lines L_1, L_2 s.t. $x_i \in L_j$. $i,j=1,2$ contradiction to (P1) & (P2). #