n-letters 选择 in 有序、无序.

Fact: For subsets $I_1, \dots, I_k \subseteq \{0,1,2,\dots\}$, Let $f_j(x) = \sum_{m \in I_j} x^m$. and $f(x) = \prod_{j=1}^{k} f_j$

Then $[x^n] f = \sum_{\substack{i_1+\dots+i_k=n \\ i_j \in I_j}} \left(\prod_{j=1}^{k} [x^{i_j}] f_j \right) = \sum_{\substack{i_1+\dots+i_k=n \\ i_j \in I_j}} 1$ = # of solutions to $i_1+\dots+i_k=n$. $i_j \in I_j$

$\sum a_n x^n = \prod_{j=1}^{k} f_j(x)$.

Pro. (1). Let S_n be the number of selections (无序选择) of n letters choses from an unlimited supply of a's, b's, c's, where number of a's and b's are even.

Sol: $\Leftrightarrow S_n = \#\{x_1 + x_2 + x_3 = n.\ x_1, x_2 \in \{0,2,4,\dots\},\ x_3 \in \{0,1,2,\dots\}\}$.

$$\sum S_n x^n = \Big(\sum_{i_1 \in 2\mathbb{N}} x^{i_1}\Big)\Big(\sum_{i_2 \in 2\mathbb{N}} x^{i_2}\Big)\Big(\sum_{i_3 \in \mathbb{N}} x^{i_3}\Big) = \left(\frac{1}{1-x^2}\right)^2 \left(\frac{1}{1-x}\right)$$

(2). How about words? (有序选择).

Number of words of n letters formed by x a's. y b's and z c's.

$\boxed{x}\ \boxed{y} \cdots \boxed{z}$.

There are $\binom{n}{x}$ ways for the a's, $\binom{n-x}{y}$ ways for the b's. the remaining are c's.

$$\binom{n}{x}\binom{n-x}{y} = \frac{n!}{x!\,y!\,z!}$$

solution of Pro(2) is $T_n = \sum_{\substack{x+y+z=n \\ x,y \in 2\mathbb{N} \\ z \in \mathbb{N}}} \frac{n!}{x!\,y!\,z!}$.

(**Def** the exponential G.F. of $\{a_n\}_{n \geq 0}$ defined as $f(x) = \sum_{n=0}^{+\infty} \frac{a_n}{n!} x^n$.)

$f(x) = \sum_{n=0}^{\infty} \frac{T_n}{n!} x^n = f_1(x) f_2(x) f_3(x)$ where $f_1(x) = f_2(x) = \sum_{i=0}^{+\infty} \frac{x^{2n}}{(2n)!}$. $f_3(x) = \sum \frac{x^n}{n!}$

$$[x^n] f = \sum_{\substack{i+j+k=n \\ i,j \text{ even}}} [x^i] f_1\, [x^j] f_2\, [x^k] f_3 = \sum \frac{1}{i!} \frac{1}{j!} \frac{1}{k!}$$

$$= \left(\sum \frac{n!}{i!\,j!\,k!}\right) \frac{1}{n!} = \frac{T_n}{n!}$$

$$\begin{cases} e^x = \sum \frac{x^i}{i!} \\ e^{-x} = \sum \frac{(-1)^i x^i}{i!} \end{cases} \Rightarrow \frac{e^x + e^{-x}}{2} = \sum_{i \in \{0,2,4,\dots\}} \frac{x^i}{i!} = f_1(x) f_2(x).$$

$$\Rightarrow f(x) = f_1 f_2 f_3(x) = \left(\frac{e^x + e^{-x}}{2}\right)^2 e^x = \frac{1}{4}\left(e^{3x} + 2e^x + e^{-x}\right) = \sum_{i=0}^{\infty} \frac{3^i + 2^i + (-1)^i}{4} \frac{x^i}{i!}$$

$$\Rightarrow \boxed{T_n = \frac{3^n + 2^n + (-1)^n}{4}}.$$

- Ordinary G.F. can be used to find the number of selectures (not involve ordering)
- Exporential G.F. can be used to find the number of arrangements or some combiratorial objects involving ordering.

$\Rightarrow$ • Given $I_1, I_2, \cdots I_k \subseteq \mathbb{N}$. Let $f_j(x) = \sum_{i \in I_j} x^i$, Let $a_n = \sum_{\substack{i_1+\cdots+i_k=n \\ i_j \in I_j}} 1$.

$\prod_{j=1}^{k} f_j(x)$ is the G.F. of $\{a_n\}$.

• Let $g_j(x) = \sum_{i \in I_j} \frac{x^i}{i!}$, $b_n = \sum_{\substack{i_1+\cdots+i_k=n \\ i_j \in I_j}} \frac{n!}{i_1! \cdots i_k!}$, Then $\prod_{j=1}^{k} g_j(x)$ is the exporential G.F. of $\{b_n\}$.

eg1: Find the number a_n of ways to send n students to numbered classroom (R_1, R_2, R_3) such that each room has at least 1 students.

Sol: $a_n = \sum_{\substack{i+j+k=n \\ i,j,k \geq 1}} \binom{n}{i}\binom{n-i}{j} = \sum_{\substack{i+j+k=n \\ i,j,k \geq 1}} \frac{n!}{i!j!k!}$ → use the exporential G.F.

$\Rightarrow \left(\sum_{i \geq 1} \frac{x^i}{i!}\right)^3$ is the exporential G.F. of $\{a_n\}$.

$\Rightarrow \sum_{n \geq 3} \frac{a_n}{n!} x^n = (e^x - 1)^3 = e^{3x} - 3e^{2x} + 3e^x - 1 = \sum_{i=0}^{+\infty} (3^i - 3 \cdot 2^i + 3) \frac{x^i}{i!} - 1$

$\Rightarrow a_n = 3^n - 3 \cdot 2^n + 3$.

eg2: Let $f_i(x) = \sum_{n \geq 0} \frac{a_n^{(i)}}{n!} x^n$ for $i = 1, 2, \cdots k$ and $f(x) = \prod_{j=1}^{k} f_j(x)$.

Then $f(x) = \sum_{n \geq 0} \frac{A_n}{n!} x^n$ iff $A_n = \sum_{i_1+\cdots+i_k=n} \left(\frac{n!}{i_1! \cdots i_k!} \prod_{j=1}^{k} a_{i_j}^{(j)} \right)$.

Pf: Since $f(x) = \prod f_j$, $[x^n] f = \sum_{i_1+\cdots+i_k=n} \prod_{j=1}^{k} [x^{i_j}] f_j$. #

eg3: Let a_n be the number of arrangements of type A for n people and let b_n . . — — — — — — — B — — — .

Define an arrangement of n people of a new type, say type C, as follows: dividing the given group of n people into 2 groups (the 1st and 2nd), and arranging the 1st group by an arrangement of type A and the 2nd group by an arrangement of type B.

C: n分成2组
1st → A
2nd → B.

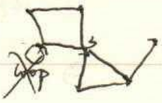
Let C_n be the number of arrangements of n people of type C. $A(x)$, $B(x)$, $C(x)$ be the responding exponential G.F. for $\{a_n\}$, $\{b_n\}$, $\{C_n\}$. Find $C(x)$?

Sol: $C_n = \sum_{i+j=n} \frac{n!}{i!\,j!} a_i b_j$. $A(x) = \sum \frac{a_n}{n!} x^n$. $B(x) = \sum \frac{b_n}{n!} x^n$.

$\Rightarrow$ $C(x) = \sum \frac{C_n}{n!} x^n = A(x) B(x)$. #.

Basic of Graphs.

Def: A (undivided) graph $G = (V, E)$, consists of a finite set V of vertices (V is called the vertex set) and a set E of edges (E is called the edge set), such that $E \subseteq V \times V = \{(u,v) : u, v \in V\}$ (unordered pair).



In our class, we often consider simple graph which contains **No** loops or multiple edges.

- If (i,j) is an edge of G, then we say vertices i and j are adjacent in graph G, write this as $i \underset{G}{\sim} j$.
- $E(G)$ denotes the edge set of G, $V(G)$ denotes the vertex set of G.

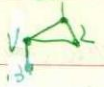
- If vertex i is an endpoint of an edge $e \in E(G)$, then we say e is incident to vertex i.
- $e(G) = |E(G)|$ = the number of edges in G.

度 • The degree of a vextex v in G denoted by $d_G(v)$ or $d(v)$, is the number of edges in G incident to v.

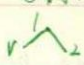
邻居点 • The neighborhood of vertex v in G, denoted by $N_G(v)$ or $N(v)$, is the set of vertices adjacent to v.
$\Rightarrow |N(v)| = d(v)$.

- A graph $G' = (V', E')$ is a subgraph of $G = (V, E)$. if $V' \subset V$ and $E \subset E' \cap (V' \times V')$

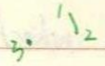
图



子图.

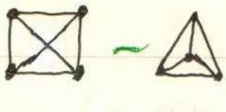


诱导子图.

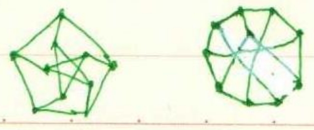


(induced subgraph: 也集完全[illegible]边集决定)

- A graph with n vertices is a complete (完全图) graph or clique, denoted by K_n, if all pairs of vertices are adjacent. $\Rightarrow e(K_n) = \binom{n}{2}$.
- A graph with n vertices is called an independent set (空图), denoted as I_n, if it contains **NO** edges.
- Given a graph $G = (V, E)$, its complement of G is a graph $(G^c)\ \bar{G} = (V, E)$, such that $\bar{E} = V \times V - E$. $\Rightarrow e(G) + e(\bar{G}) = \binom{n}{2}$



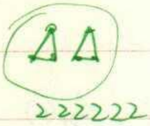
Def: Two graphs $G = (V, E)$. and $G' = (V', E')$ are isomorphic (同构) if there exists a one-to-one mapping $f: V \to V'$ such $i \underset{G}{\sim} j$ iff $f(i) \underset{G'}{\sim} f(j)$.



peterson graph.

- The degree sequence of graph G is a sequence of degrees of all vertices in an increasing order.

eg:



显然不同构.

- A path P is a sequence of vertices $v_1, v_2, \cdots v_k$ such that $(v_{i-1}, v_i) \in E(G)$. The length of path P, denoted by $|P|$ ($|P| = k-1$), is the number of edges it contains.
- A cycle C is a sequence of vertices $v_1, v_2, \cdots v_k$ such that $(v_{i-1}, v_i) \in E(G)$ of $2 \le i \le k$, and $(v_1, v_k) \in E(G)$.

  The length of cycle C denoted by $|C|$, is the number of edges in C.

Def: A planar (平面图) graph is a graph which can be draw on the plane. in such a way that its edges only intersect (也只在端点相交) at their endpoints.

In other words, it can be draw on the plane s.t. **NO** edges cross each other.

Exercise: Convince yourself that K_5 is NOT a planar graph.

Handshaking Lem: For any $G=(V,E)$, $\sum_{v \in V(G)} d(v) = 2e(G)$.

☆: Double-Counting.

↓

Pf: $F = \{(e,v) : \text{edge } e \text{ is incident to vertex } v\}$.

$$\sum_{v \in V(G)} (\# (e,v) \text{ using } v) = |F| = \sum_{e \in E(G)} (\# (e,v) \text{ using } e)$$

$$\sum_{v \in V(G)} d(v) = |F| = \sum_{e \in E(G)} 2 = 2e(G)$$

Therefore $\sum_{v \in V(G)} d(v) = 2e(G)$. #

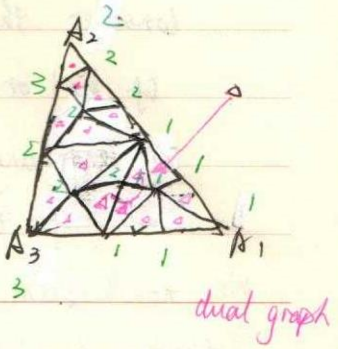
奇度点偶数个.

Corollary1: In any graph G, the number of vertices with odd degree is even.

Pf: Let $O = \{\text{vertex } v: d(v) \text{ is odd}\}$. $\mathcal{E} = \{\text{vertex } u: d(u) \text{ is even}\}$.

$$2e(G) = \sum_{v\in V} dv = \sum_{v\in O} d(v) + \sum_{u\in \mathcal{E}} d(u) \Rightarrow \sum_{v\in O} d(v) \text{ is even.} \Rightarrow |O| \text{ is even.} \quad \#$$

Corollary2: For any graph G, if there is a vertex of odd degree, than there are at least two vertices of odd degree.



Let's consider a triangle with the vertices $A_1A_2A_3$.
we devide it into small triangles arbitrarily such that

- No triangles has a vertex in an edge of other triangle.

We color the vertices using three colors say 1, 2, 3, according to the following rules:

- each A_i is colored by i.
- each vertex in the segment A_iA_j is colored by i or j.
- all interier vertices can be colored by any color.

Sperner's Lemma (A planar version):

In the situation as above, there is a small triangle where vertices are assigned by 3 different colors.

Pf: Define an auxiliary graph G as follows:

- Its vetices are the faces of our triangulation, i.e. the small triangles plus the outer face
- Two vertices of G, i.e. two faces of triangulation, are adjacent, if the two corresponding faces are neighboring faces and the

endpoint of their common edge are colored by 1 and 2.

call the outer face as vertex z in G.

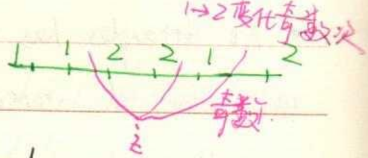
We consider the degree of v in G.

(1). If the face of v has NO colores 1 and 2 in its three vertices.
than $d_G(v)=0$.

(2). So assume the face of v has two vertices, which are of colores 1 and 2,
consider the color of the third vertices, say k.

If $k=1$ or 2. $d_G(v)=2$. If $k=3$. $d_G(v)=1$.

$\Rightarrow$ For any vertex $v \in V(G)-\{z\}$, $d(v)$ is odd iff $d(v)=1$,
iff the face of v has color 1, 2 and 3.



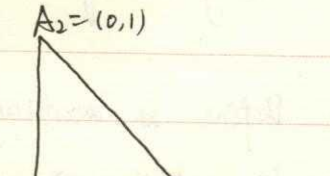
We now claim: z must be of odd degree.

why? The edges of G incident to v obviously only can go across A_1A_2. Let's write down the sequence of these colors in A_1A_2, from A_1 to A_2. So $d(z)=$ number of alternations between 1 and 2 in this sequence, which must be odd!

By corollary 2, there is a vertex $v \in V(G)-\{z\}$ with odd $d(v)$.

$\Rightarrow$ This face of v has colors 1, 2, 3 as observed. \#

Let Δ denote a triangle as follows: ⟶



Thm (Brouwer's fixed point Theorem in 2 dimension)

Every continuous function $f:\Delta \to \Delta$ has a fixed point x, namely $f(x)=x$.

Pf: ⟶

Pf: Define three auxiliary functions on Δ as follows:

for $a=(x,y)$. $\beta_1(a)=x$, $\beta_2(a)=y$, $\beta_3(a)=1-x-y$.

$\Rightarrow$ define $M_i=\{a\in\Delta : \beta_i(a)\geq\beta_i(f(a))\}$.

Fact 1: $\forall a\in\Delta$, $\exists i$ st $a\in M_i$.

Fact 2: If $p\in M_1\cap M_2\cap M_3$, then p is a fixed point.

Consider a sequence refinement of Δ:

$\Delta_1=\Delta$ Δ_2 Δ_3 ...

We will color Δ in such a way that:

① a vertex colored by i must belong to M_i.

② The assignment C satisfies the condition of sperner's lemma.

We show this indeed can be done:

- For A_i, say $i=1$, note that $A_1=(1,0)\in M_1$. so we can let $C(A_i)=M_i$ and this satisfies ①.
- Consider a vertex $a=(x,y)$ on the segment A_1A_2 i.e. $x+y=1$.
  If $a\notin M_1\cup M_2$, then $\begin{cases}\beta_1(a)<\beta_1(f(a))\\ \beta_2(a)<\beta_2(f(a))\end{cases}$ But $\beta_1+\beta_2\leq 1$. A contradiction!

  so $a\in M_1\cup M_2$.

We have proved: for any f, such $c:\Delta\to\{1,2,3\}$ exists.

Apply sperner's lemma to C on each Δ_j.

$\Rightarrow\exists$ a small triangle $A_{j1}A_{j2}A_{j3}$ in Δ_j satisfying $C(A_{jk})=k$ for $k=1,2,3$.

Consider $\{A_{11}, A_{21}, A_{31}, \cdots A_{j1} \cdots\}$.

Since $A_{jk}\in\Delta$, $\exists$ subsequence $\{A_{i_1 1}, A_{i_2 1}, \cdots A_{i_j 1} \cdots\}$ having $\lim_{j\to+\infty}A_{i_j 1}=P$ for some $p\in\Delta$.

Since the diameter of the small triangle $A_{j1}A_{j2}A_{j3}$ is going to 0 as $j \to +\infty$.
This shows that $\lim A_{i_j 2} = \lim A_{i_j 3} = p$.

$c(A_{i_j 1}) = 1 \Leftrightarrow A_{i_j 1} \in M_1 \Leftrightarrow \beta_1(A_{i_j 1}) \geq \beta_1(f(A_{i_j 1})) \Rightarrow \lim_{j \to \infty} \beta_1(f(A_{i_j 1})) \geq \lim_{j \to +\infty} \beta_1(f(A_{i_j 1}))$

$\beta_1(p) \geq \beta_1(f(p)) \Rightarrow p \in M_1$.

Similarly, $p \in M_2$. $p \in M_3$ i.e $p \in M_1 \cap M_2 \cap M_3$. $\Rightarrow p$ is a fixed point of f.

#

- Handshaking Lemma. • Sperner's Lemma • Brouwer's Fixed Point Thm.

Double - Counting.

Suppse that we give two finite sets A and B, and a subset $S \subseteq A \times B$. whenever $(a, b) \in S$, then we say a and b are incident. If N_a denotes the number of elements in B incident to $a \in A$, and N_b denotes the number of elements in A incident to $b \in B$.

$$\sum_{b \in B} N_b = |S| = \sum_{a \in A} N_a.$$

Define a matrix $X = (x_{ij})$, where $x_{ij} = \begin{cases} 1, & (i, j) \in S \\ 0, & (i, j) \notin S. \end{cases}$

eg:

A\B	1	2	3	4	5	6	⋯ j
1	1					1	
2	0					1	
3	0					1	
4	0					0	
5	0					0	
6	0					1	
⋮ i							

The (i, j)- coordinate this table is 1 if $i | j$, and 0 otherwise.
Let $T(j)$ be the number of divisors of j, that is the number of 1's in the j^{th} column.

Consider the average $\overline{T(n)} = \frac{1}{n} \sum_{j=1}^{n} T(j)$.

Fact. $|\overline{T(n)} - H(n)| < 1$, when $H(n) = \sum_{i=1}^{n} \frac{1}{i}$ is the n^{th} Hamonic number.

pf: $\overline{T(n)} = \frac{1}{n} \sum_{i=1}^{n} \lfloor \frac{n}{i} \rfloor$.

...... #

[redacted] Sperner's Thm.

Def: Let $\mathcal{F} \leq 2^{[n]}$ be a family of subsets of $[n]$. [redacted]: we call $\mathcal{F}$ an independent system, if $\forall A, B \in \mathcal{F}$, we have $A \nsubseteq B$ & $B \nsubseteq A$. (In other words, there is no "containment" relationship between any 2 subsets in $\mathcal{F}$)

Def: (i). A chain of subsets of $[n]$ is a sequence of distinct subsets $A_1 \subset A_2 \subset \cdots \subset A_t$.

(ii). A maximal chain is a chain with the property that no other set can be inserted.

Fact 1. Any maximal chain must look like: $\mathcal{C}: \phi \subset \{x_1\} \subset \{x_1, x_2\} \subset \cdots \subset \underset{[n]}{\underset{\parallel}{\{x_1 \cdots x_n\}}}$.

Fact 2. There are $n!$ maximal chains in total.

why? Each maximal chain define a unique permutation $\pi: [n] \to [n]$ by $\pi(i) = x_i$. #.

Sperner's Thm: For any independent system $\mathcal{F}$ formed by subsets of $[n]$.

$$|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$ 所有的k元素的集合是 independent system. $\binom{n}{\lfloor \frac{n}{2} \rfloor}$最大,

Proof: First, this bound is tight, as $\binom{[n]}{\lfloor \frac{n}{2} \rfloor}$ and $\binom{[n]}{\lceil \frac{n}{2} \rceil}$ both are independent systems.

We use double-counting, by considering the number of pairs $(\mathcal{C}, A)$ such that (i). $\mathcal{C}$ is a maximal chain

(ii). $A \in \mathcal{C} \cap \mathcal{F}$

$\sum_{\mathcal{C}} N_{\mathcal{C}} = \#\text{pairs}(\mathcal{C}, A) = \sum_{A \in \mathcal{F}} N_A$.

✓ $N_{\mathcal{C}} = \#\text{subsets } A \in \mathcal{C} \cap \mathcal{F}. = |\mathcal{C} \cap \mathcal{F}| \leq 1$

Because any chain and any independent system can have at most 1 common subset, 有2个就有包含关系.

✓ $N_A = \#\text{ max. chains } \mathcal{C} \text{ st } A \in \mathcal{C}. = |A|!(n-|A|)!$

$\underbrace{\phi \subset \{\cdot\} \subset \{\cdot,\cdot\} \subset \cdots}_{k!} \subset \underset{k\text{-subset}}{\underset{\downarrow}{A}} \subset \underbrace{\cdots \subset [n]}_{(n-k)!}$

$$n! = \sum_{\mathcal{C}} 1 \geqslant \sum_{\mathcal{C}} N_{\mathcal{C}} = \#\text{pairs } (\mathcal{C}, A) = \sum_{A \in \mathcal{F}} N_A = \sum_{A\in \mathcal{F}} |A|!\,(n-|A|)!$$

$$\Rightarrow 1 \geqslant \sum_{A\in\mathcal{F}} \frac{|A|!\,(n-|A|)!}{n!} = \sum_{A\in\mathcal{F}} \frac{1}{\binom{n}{|A|}} \geqslant \sum_{A\in\mathcal{F}} \frac{1}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} = \frac{|\mathcal{F}|}{\binom{n}{\lfloor \frac{n}{2}\rfloor}}$$

$$\Rightarrow |\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$$ (等号可以取到. 称这个上界是"紧的"). ∗

Proof 2: **Def:** A chain is <u>symmetric</u>, if it consists of subsets of sizes $k, k+1, \dots, \lfloor \frac{n}{2} \rfloor, \dots, n-(k+1), n-k$. (consecutive sequence)

eg: $n=3$ $\{1\}\ \{13\}\ \{123\}$ is NOT symmetric.

$n=\phi$. ϕ, $\{123\}$ is NOT ~.

Def: A partition of $2^{[n]}$ into symmetric chains is a way of expressing $2^{[n]}$ as a disjoint union of symmetric chains.

Prop(∗). Note that each symmetric chain contains <u>exactly one</u> subset of size $\lfloor \frac{n}{2} \rfloor$, since there are $\binom{n}{\lfloor \frac{n}{2}\rfloor}$ subsets of size $\lfloor \frac{n}{2} \rfloor$, any partition of $2^{[n]}$ into symmetric chains has to consist of <u>exactly</u> $\binom{n}{\lfloor \frac{n}{2}\rfloor}$ symmetric chains.

Thm 2: The family $2^{[n]}$ has a partition into symmetric chains

(Because each chain has at most 1 subset in common with any independent system. In view of Prop(∗), Thm 2 implies that any independent system $\mathcal{F}$ can intersect with each symmetric chain in the partition by 1 set. thereby in view of property(∗), implying $|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2}\rfloor}$)

Proof of Thm 2: For each $A \in 2^{[n]}$, we assign a sequence "$a_1 a_2 \cdots a_n$" consisting of left and right parentheses "(" ")" by defining

$$a_i = \begin{cases} \text{"("} & \text{if } i \in A \\ \text{")"} & \text{if } i \notin A. \end{cases}$$

eg. $n=7$. $A=\{2,6\}$ $)\underline{()}))\underline{()}$

eg: $))))\underbrace{(\underline{()}\underline{()})})($ 最终会是 "$))\cdots)(\cdots(($"

We can define the so-called "partial pairing of parentheses." as follows:

①First, we pair up all pairs "()" of adjacent parentheses

② Then, we ignore these already paired parentheses.

③ Repe the above steps until nothing can be done.

Note that when this process stops, the remaing unmatched parentheses must look like "$\underset{\text{right}}{))\cdots)}\underset{\text{left.}}{((\cdots(}$". we say two subsets have "the same partial pairing", if the paired parentheses are the same (even in the same position).

eg: $n=11$. $A_1=\{5\ 6\ 8\}$ $))))\underbrace{(\underline{()}\underline{()})})$

$A_2=\{5\ 6\ 8\ 11\}$ $))))\underbrace{(\underline{()}\underline{()})}($

$A_3=\{4\ 5\ 6\ 8\ 11\}$ $)))(\underbrace{(\underline{()}\underline{()})}($

$\vdots$

$A_6=\{1\,2\,3\,4\,5\ 6\ 8\ 11\}$

$\{A_1, A_2, \cdots A_6\}$ is a symmetric chain.

We then define an equivalence "$\sim$" on $2^{[n]}$, by letting $A\sim A'$ iff A, A' have the same partial pairing.

Claim (HW): Each eqivalence class indeed formed a symmetric chain.

Now the proof is completed. ≠.

Littlewood - Offord Problem: Fix a vector $\vec{a}=(a_1, a_2, \cdots a_n)$ with each $|a_i|\geq 1$. Let $S=\{\vec{\varepsilon}=(\varepsilon_1, \cdots \varepsilon_n):\ \varepsilon_i=\pm 1.\ \forall i,\ \vec{a}\cdot\vec{\varepsilon}\in(-1,1)\}$. Then $|S|\leq\binom{n}{\lfloor\frac{n}{2}\rfloor}$

Proof : For $\forall\ \vec{\varepsilon} \in S$, define $A_{\vec{\varepsilon}} = \{ i \in [n] : \overset{|a_i|}{\overset{\shortparallel}{a_i \cdot \varepsilon_i}} > 0 \}$

Let $\mathcal{F}_1 = \{ A_{\vec{\varepsilon}} : \vec{\varepsilon} \in S \}$. Note that $|\mathcal{F}_1| = |S|$.

Claim : $\mathcal{F}_1$ is an independent system on $[n]$.

Suppose NOT, say $A_{\vec{\varepsilon}_1} \subsetneq A_{\vec{\varepsilon}_2} \in \mathcal{F}_1$

① $\vec{a} \cdot \vec{\varepsilon}_1 = \sum_{i \in A_{\vec{\varepsilon}_1}} a_i \varepsilon_i - \sum_{i \notin A_{\vec{\varepsilon}_1}} a_i \varepsilon_i = \sum_{i \in A_{\vec{\varepsilon}_1}} |a_i| - \sum_{i \notin A_{\vec{\varepsilon}_1}} |a_i|$.

$= 2 \sum_{i \in A_{\vec{\varepsilon}_1}} |a_i| - \left(\sum_{i=1}^{n} |a_i| \right)$.

② $\vec{a} \cdot \vec{\varepsilon}_2 = 2 \sum_{i \in A_{\vec{\varepsilon}_2}} |a_i| - \left(\sum_{i=1}^{n} |a_i| \right)$ Similarly.

② - ① $\vec{a} \cdot \vec{\varepsilon}_2 - \vec{a} \cdot \vec{\varepsilon}_1 = 2 \sum_{i \in A_{\vec{\varepsilon}_2}} |a_i| - 2 \sum_{i \in A_{\vec{\varepsilon}_1}} |a_i|$

$= 2 \sum_{i \in A_{\vec{\varepsilon}_2} \setminus A_{\vec{\varepsilon}_1}} |a_i| \cdot \geq 2$.

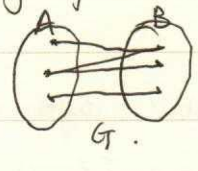
But $\vec{a}\vec{\varepsilon}_2, \vec{a} \cdot \vec{\varepsilon}_1 \in (-1, 1)$. $|\vec{a} \cdot \vec{\varepsilon}_2 - \vec{a} \cdot \vec{\varepsilon}_1| < 2$, a contradiction!

Thus $|S| = |\mathcal{F}_1| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$. #

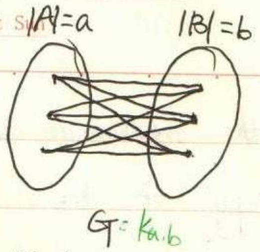
Double Counting

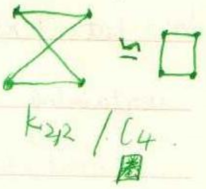
- Sperner's Thm : $\forall$ independent system $\mathcal{F}_1 \subseteq 2^{[n]}$, $|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$
  - pf1 : $\#(C, A)$.
  - pf2 : $2^{[n]} = \bigcup$ (symmetric chain).

Def: A graph G is bipartite (二部图), if its vertex set can be partitioned into two parts, say A and B, such that each edge joins one vertex in A and the one in B.



Def: Denote $K_{a,b}$ to be the complete bipartite graph (完全二部图)



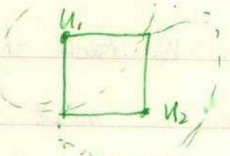


with two parts a and b.
That is a bipartite graph with edge set $\{(i,j): i\in A,\ j\in B\}$ where $|A|=a$, $|B|=b$.

Thm (Istvan Reiman) If a graph G with n vertices contains NO copy of C_4, then $e(G)\le\frac{n}{4}(1+\sqrt{4n-3})$. (“=”能不能取到？不知道).

(**Cauchy-Schwartz Ineq**: $\frac{\sum_{i=1}^{n}x_i^2}{n}\geqslant\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)^2$.)

Proof: Consider $S=\{(\{u_1,u_2\},v): \overset{u_1\ \ v\ \ u_2}{\overline{\quad\bullet\quad}}\text{ in } G\}$.



固定 u_1,u_2，至多有1个元素.
有多少个 $\{u_1,u_2\}$？由此给出 S 上界

C_4中u_1,u_2有公共的2个邻居

Since G has NO C_4, any two vertices u_1,u_2 can have at most 1 common neighbor v i.e. $\#\, u_1\overset{v}{\text{—}}u_2\le 1$. 若无$C_4$，则$u_1,u_2$至多一个公共邻居

Thus $|S|=\sum_{\{u_1,u_2\}}(\#\, \overset{u_1\ v\ u_2}{\text{—}})\le\sum_{\{u_1,u_2\}}1=\binom{n}{2}$.

On the other hand, $|S|=\sum_{v\in V(G)}(\#_v\angle)=\sum_{v\in V(G)}\binom{d(v)}{2}\le\binom{n}{2}=\frac{n^2-n}{2}$

$$\Rightarrow\ \frac{n^2-n}{2}\geqslant\sum_v\binom{d(v)}{2}=\sum_v\frac{d(v)^2-d(v)}{2}=\frac{n}{2}\Big(\sum_{v\in V(G)}\frac{d(v)^2}{n}\Big)-|E|.$$

$$\geqslant\frac{n}{2}\left(\frac{\sum_v d(v)}{n}\right)^2-|E|=\frac{2}{n}|E|^2-|E|.$$

$$\Rightarrow\ |E|^2-\frac{n}{2}|E|-\frac{n^2(n-1)}{4}\le 0.$$

$\frac{\frac{n}{2}\pm\sqrt{\frac{n^2}{4}+n^2(n-1)}}{2}=\frac{n}{4}(1\pm\sqrt{4n-3})$

$$\Rightarrow\ |E|\le\frac{n}{4}(1+\sqrt{4n-3})\sim\frac{1}{2}n^{\frac{3}{2}}$$

\# → → $\boxed{\text{Turán - Type Problem}}$.

Next, we consider graphs G with NO copy of $K_3=\Delta$

Question: How many edges G can have? $\frac{n^2}{4}$. ⟶ eg: 二部图不含 K_3. 完全二部图 → $\le\frac{n^2}{4}$.

Let $T(n)$ be the maximum number of edges in graphs with n vertices containing NO copy of K_3.

Mantal 's Thm: for any $n \geqslant 1$, $T(n) = \lfloor \frac{n^2}{4} \rfloor$

Proof: consider $K_{\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor}$, which has NO K_3 and $\lfloor \frac{n}{2} \rfloor \lfloor \frac{n}{2} \rfloor = \lfloor \frac{n^2}{4} \rfloor$ edges.

This shows $\underbrace{T(n) \geqslant \lfloor \frac{n^2}{4} \rfloor}$. Next, we show $T(n) \leqslant \frac{n^2}{4}$.

It suffices to show $T(n) \leqslant \frac{n^2}{4}$.

We will prove this by induction: (★) any n-vertex graph G with no K_3 has $e(G) \leqslant \frac{n^2}{4}$.

Base case: $n=1$, $T(1)=0$; $n=2$, $T(2)=1$.

We have the inductive hypothese: any graph H with less than $(n-1)$ vertices and with NO K_3 has at most $\frac{|V(H)|^2}{4}$ edges.

Let G be a n-vertex graph with NO K_3.

$e_0 = \overset{x}{\bullet}\!\!-\!\!-\!\!-\!\!\overset{y}{\bullet} \in E(G)$. Pick an edge $e_0 = xy \in E(G)$.

Let $E_x = \{$all edges incident to x, expect $e_0\}$.
$E_y = \{$ ‥ ‥ ‥ to y. ‥ ‥ $\}$.

$E_x \cap E_y = \varnothing$.

Since x and y have NO common neighbor (otherwise we have a K_3).

$\Rightarrow |E_x| + |E_y| \leqslant n-2$

Note that $G - \{x, y\}$ is a graph with $(n-2)$ vertices and with NO K_3.

So by induction, $e(G - \{x,y\}) \leqslant \frac{(n-2)^2}{4}$

$\Rightarrow e(G) = e(G - \{x,y\}) + |E_x| + |E_y| + 1 \leqslant \frac{(n-2)^2}{4} + (n-2) + 1 = \frac{n^2}{4}$.

$\Rightarrow \underbrace{T(n) \leqslant \frac{n^2}{4}}$ $\Rightarrow T(n) = \frac{n^2}{4}$ #.

Rmk: 当这样的 G 边数达到 $T(n)$ 时, G 什么样子? extremal graph 极图

$G = K_{\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor}$ (可类似归纳证明)

Thm: For ∀n, the unique n-vertex graph with NO K_3 and with maximum number of edges is $K_{\lfloor\frac{n}{2}\rfloor,\lfloor\frac{n}{2}\rfloor}$.

Pf: By induction. It is easy to check this when n=1,2.

Now we assume this holds for all integers less than n.

- consider a graph G with NO K_3 and with $\lfloor\frac{n^2}{4}\rfloor$ edges.

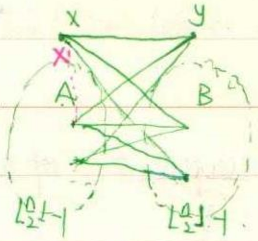
We need to show $G = K_{\lfloor\frac{n}{2}\rfloor,\lfloor\frac{n}{2}\rfloor}$.

$e_0 = \overset{x}{\bullet}\!\!-\!\!\overset{y}{\bullet} \in E(G)$. E_x, E_y, $\Rightarrow e(G-\{x,y\}) \le \lfloor\frac{(n-2)^2}{4}\rfloor$.

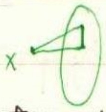
$\Rightarrow \lfloor\frac{n^2}{4}\rfloor = e(G) = e(G-\{x,y\}) + |E_x| + |E_y| + 1 \le \lfloor\frac{(n-2)^2}{4}\rfloor + (n-2)+1 = \frac{n^2}{4}$.

therefope, all inequalities must be equaties. ie. $|E_x|+|E_y| = n-2$, $e(G-\{x,y\}) = \lfloor\frac{(n-2)^2}{4}\rfloor$

By induction, $G-\{x,y\} = K_{\lfloor\frac{n-2}{2}\rfloor,\lfloor\frac{n-2}{2}\rfloor}$



Observation: $N_G(x)$, $N_G(y)$ are two independent set.



otherwise we find K_3.

Define the two parts in $G-\{x,y\}$ as A, B.

where $|A| = |B| = \lfloor\frac{n}{2}\rfloor - 1$. Then $\begin{cases} N_G(x)-\{y\} \subseteq A \text{ or } B \\ N_G(y)-\{x\} \subseteq A \text{ or } B \end{cases}$ … Fact 1.

Fact 2. $N_G(x) \cap N_G(y) = \phi$. Fact 3. $|N_G(x)| + |N_G(y)| = |E_x| + |E_y| + 2 = n$.

$\dots \Rightarrow \begin{cases} N_G(x)-\{y\} = A \\ N_G(y)-\{x\} = B \end{cases}$ or $\begin{cases} N_G(x)-\{y\} = B \\ N_G(y)-\{x\} = A \end{cases}$.

$\Rightarrow G = K_{\lfloor\frac{n}{2}\rfloor,\lfloor\frac{n}{2}\rfloor}$ #