• graph G with NO $K_{3,3}$. $e(G) \le O(n^{\frac{5}{3}})$.

$$\sum \# \, v\!\left\langle \begin{smallmatrix} u_1 \\ u_2 \\ u_3 \end{smallmatrix}\right. = \sum_{v \in V(G)} \binom{d(v)}{3} = \sum_{\{u_1 u_2 u_3\}} \# \, v\!\left\langle \begin{smallmatrix} u_1 \\ u_2 \\ u_3 \end{smallmatrix}\right. \le \sum_{\{u_1 u_2 u_3\}} 2 \le 2\binom{n}{3}.$$

Counting Spanning Trees.

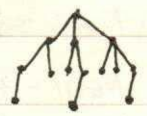
Def: A graph G is connected, if for any 2 vertices u and v, G has a path from u to v; otherwised, we say G is disconnected.

eg: K_n ✓, G =  disconnected.
component.

Def: A component of a graph G is a maximal connected subgraph of G.
极大连通子图.

Rmk: G is disconnected iff G has ≥ 2 components.

Def: A graph T is called a tree if it is connected and has **NO** cycle.

A vertex in a tree with degree one is a leaf. 

Fact 1: Any tree (with $n \geq 2$ vertices) has at least one leaf.

Pf: Suppose to the contrary that any vertex has degree ≥ 2.

$a_1 \; a_2 \; a_3 \; a_4 \cdots a_{k-1} \; a_k$

k有限，一定会有圈.

As the tree has finite vertices, this process must terminate. When terminating, we find a cycle, a contradiction! #

Euler's formula: If $T=(V,E)$ is a tree, then $|V|=|E|+1$.

Pf: By induction, when $|V|=1$, clearly hold.

Assuming this holds for all trees with less than $|V|$.

Consider $T=(V,E)$. By Fact 1, T has a leaf v. i.e. $d(v)=1$.

Let $T'=T-\{v\}$. Then T' is also a tree: $\begin{cases} \text{NO cycle} \\ \text{connected.} \end{cases}$

$\because (V',E')$

So by induction, $|V'|=|E'|+1$. but $|V|=|V'|+1$, $|E|=|E'|+1$.

v is a leaf.

$\Rightarrow |V|=|E|+1$. #

Fact 2: Any tree T (with ≥ 2 vertices) has at least 2 leaves.

Pf: Suppose T has exactly one leaf v.

$2|E| = \sum_{x\in V} d(x) \geq 1 + 2(|V|-1) = 2|V|-1 = 2(|E|+1)-1 = 2|E|+1$.

a contradiction!

#

Tree Characterization Theorem: Let $T=(V,E)$ be a graph.

The following are equivalent:

(i). T is a tree. (i.e. connected and NO cycle).

(ii). T is connected, but deleting any edge will result in a disconnected graph.

(iii). T has NO cycle, but adding any new edge f will creat a cycle in $T+\{f\}$.

Rmk: (ii) tells us that a tree is a "minimal" connected graph.

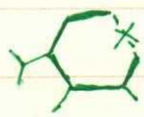
(iii) ... ... a "maximal" graph with NO cycle.

Proof: (i) $\Rightarrow$ (ii).

Suppose $\exists$ an edge $e \overset{xy}{=}$ s.t. $T-\{e\}$ is still connected.

Then $T-\{e\}$ has a path P from x to y. But $P\cup\{xy\}$ forms a cycle in T, a contradition!

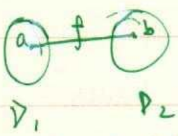
(ii) $\Rightarrow$ (i). Suppose T has a cycle C. Now if we delete any edge e in C, $T-\{e\}$ is still connected. a contradition!



圈中删一条边不影响连通性.

(i) $\Rightarrow$ (iii). Adding any new edge $f=uv$ to get $T+\{f\}$.
Since T is connected, T has a path Q from u to v.
Now $Q \cup \{f\}$ gives a cycle in $T+\{f\}$.

(iii) $\Rightarrow$ (i). Suppose T is disconnected, so T has ≥ 2 components say D_1, D_2.



Pick $a \in D_1$, $b \in D_2$. If we add a new edge ab to T. the $T+\{ab\}$ will have NO cycle. a contradition to (iii).
加条f并不能产生圈. So T must be connected. *

Def: Given a graph $G=(V,E)$, a graph $H=(V',E')$ is a spanning subgraph if H is a subgraph of G and $V'=V$.

Def: Given a connected graph G with n numbered vertices, say $v_1, v_2, \cdots v_n$.
Let $ST(G)$ = Number of spanning trees in G.
连通图中至少有1个 spanning tree.（删边，删到没有圈）.

Cayley's Formula: For $\forall n \geq 2$, $ST(K_n) = n^{n-2}$.

Proof 1: We first count the number of spanning trees with given degree sequence say $d_1, d_2, \cdots d_n$. where $\sum_{i=1}^{n} d_i = 2(n-1)$.

Lemma: Let $d_1, \cdots d_n$ be positive integers with $\sum d_i = 2(n-1)$.
Then the number of spanning trees on vertex set $\{v_1, v_2, \cdots v_n\}$ and satisfying $d(v_i) = d_i$ is equal to $\frac{(n-2)!}{(d_1-1)!(d_2-1)!\cdots(d_n-1)!}$

Pf of Lemma: By induction. Base Case: $n=2$, $d_1=d_2=1$. ✓

We assume that this holds for any sequence ~~for~~ of $n-1$ integers with sum $2(n-2)$. Let $\mathcal{F} = \{$spanning tree T on $\{v_1 \cdots v_n\}$ with $d(v_i)=d_i\}$.

Note that $\frac{\sum d_i}{n} = \frac{2(n-1)}{n} < 2$, so there exists some i s.t. $d_i=1$.

W.L.O.G. Let $d_n = 1$. so v_n is a leaf.

Let $\mathcal{F}_i = \{T - \{v_n\}$: the unique neighbor of v_n is v_i in $T \in \mathcal{F}\}$, $i=1,2,\cdots,n-1$.

so $|\mathcal{F}| = \sum_{i=1}^{n-1} |\mathcal{F}_i|$

$\mathcal{F}_i = \{$spanning tree T on $\{v_1, \cdots v_{n-1}\}$ with degrees $d_1, \cdots d_i - 1, \cdots d_{n-1}\}$.

By induction for each $\mathcal{F}_i$:

$$|\mathcal{F}_i| = \frac{(n-3)!}{(d_1-1)! \cdots (d_i-2)! \cdots (d_{n-1}-1)!}$$

$$\Rightarrow |\mathcal{F}| = \sum_{i=1}^{n-1} |\mathcal{F}_i| = \sum_{i=1}^{n-1} \frac{(n-3)!\,(d_i-1)}{(d_1-1)! \cdots (d_{n-1}-1)!} = \frac{(n-3)!}{(d_1-1)! \cdots (d_{n-1}-1)!} \sum_{i=1}^{n-1}(d_i-1)$$

$$= \frac{(n-2)!}{(d_1-1)! \cdots (d_n-1)!} \quad *$$

$\sum_{i=1}^{n-1}(d_i-1) = 2(n-2)-(n-1) = n-2$.
$(d_n-1)! = 1$

Binomial Thm: $(x+y)^n = \sum_{\substack{i+j=n \\ i,j \geq 0}} \frac{n!}{i!\,j!} x^i y^j$.

An extension: $(x_1 + \cdots + x_k)^n = \sum_{\substack{i_1+\cdots+i_k=n \\ i_j \geq 0}} \frac{n!}{i_1! \cdots i_k!} x_1^{i_1} \cdots x_k^{i_k}$

When $x_1 = \cdots = x_n = 1$, $k^n = \sum_{\substack{i_1+\cdots+i_k=n \\ i_j \geq 0}} \frac{n!}{i_1! \cdots i_k!} \quad \cdots (*)$

Back to Proof 1: $ST(K_n) = \sum_{\sum d_i = 2(n-1)} \#$ spanning trees with degree $d_1, \cdots d_n$

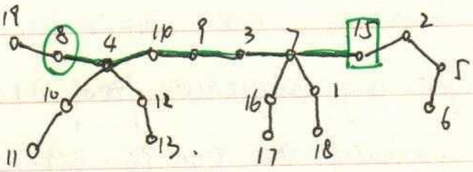
$$\overset{\text{by Lemma}}{=} \sum_{\sum d_i = 2(n-1)} \frac{(n-2)!}{(d_1-1)! \cdots (d_n-1)!}$$

$$= \sum_{\sum (d_i-1) = n-2} \frac{(n-2)!}{(d_1-1)! \cdots (d_n-1)!} \overset{(*)}{=} n^{n-2}.$$

$\Rightarrow \quad ST(K_n) = n^{n-2}$.

#

Proof 2 :



两个特殊点可以相同.

Given a spanning tree, choose 2 special vertices.
(One with a circle and the other with a square).

We call such a subject (the spanning tree with 2 special vertices) as a vertebrate
Let $\mathcal{V} = \{$all vertebrate (脊椎动物) on n number vertices say $1,2,\cdots,n\}$

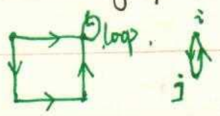
- $|\mathcal{V}| = ST(K_n) \cdot n^2$.

$\#\{$mappings $f: [n] \to [n]\} = n^n$. Our goal is to find a bijection between $\mathcal{V}$ and the family $\mathcal{F} = \{$mapping $f: [n] \to [n]\}$.

Lemma 2: Such a bijection exists. $\Rightarrow |\mathcal{V}| = ST(K_n)\, n^2 = n^n \Rightarrow ST(K_n) = n^{n-2}$

Pf: Let $\mathcal{D} = \{$ all digraphs on $\{1,2,\cdots,n\}$ s.t. each vertex has exactly one out-neighbor $\}$.

digraph:

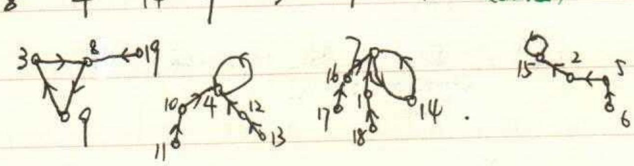


Let the unique path in W from $\bigcirc \to \square$ be the "chord" of W.

1^{st} row $\to$	3	4	7	8	9	14	15 (按大小)
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
2^{nd} row $\to$	8	4	14	9	3	7	15 (路径)

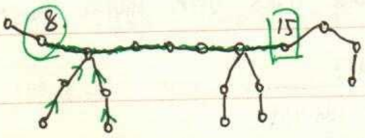
$\varphi: \mathcal{V} \to \mathcal{D}$

$\varphi(W) =$



每个点有一个出度.
定义了一个 $f: [n] \to [n]$.

$W =$



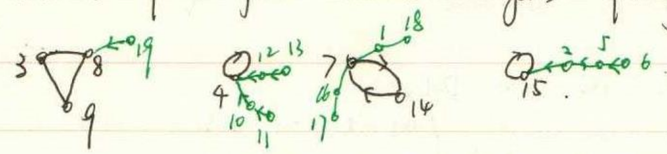
所有延伸出来的点都指向弧上点.

We then define a digraph P as follows:

- The vertex set consists of vertices of the chord.
- The edges are from the vertex in 1^{st}-row to the one below it.

So each vertex in P has exactly one edge going out and exactly one edge going in.

$\Rightarrow$ P consists of disjoint directed cycles (possibly containing loops and 2 cycles).

P: 

Next we extend P to all vertices $[n]$ by following:

(i) We can back to W and remove all edges of the chord.

(ii) Direct the remaining of the components s.t. they point to the vertices of the chord contained in that component.

(iii) The edges in (ii), together with the edges of P, define a new graph $\varphi(W)$ on $[n]$ in $\mathcal{D}$.

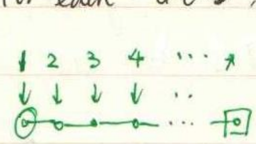
Let us define a mapping $\varphi: \mathcal{V} \to \mathcal{D}$. by $W \mapsto \varphi(W)$ as above.

We want to show: step 1: $\exists$ a bijection between $\mathcal{V}$ and $\mathcal{D}$.

step 2: $\exists$ a bijection between $\mathcal{D}$ and $\mathcal{F}$.

For step 2, it is easy, as for each digraph $G \in \mathcal{D}$ there is a unique $f \in \mathcal{F}$, such that if $i \to j$ is an edge of G, then $f(i) = j$. $G \in \mathcal{D} \leftrightarrow f \in \mathcal{F}$.

For Step 1, 2 things left: ①. Need to define $\varphi^{-1}: \mathcal{D} \to \mathcal{V}$ s.t. $\varphi^{-1} \circ \varphi = Id$.

(How to define φ^{-1}? For each $G \in \mathcal{D}$, the vertices of G belonging to direct cycles will from the chord: 

And the remaining vertices give rise to other edges of W.)

② $\forall\, G \in \mathcal{D}$. $\exists\, W \in \mathcal{V}$. s.t. $\varphi(W) = G$ (hw).

Combining ① & ②, now we see $\exists$ a bijection between $\mathcal{V}$ and $\mathcal{D}$.

$\Rightarrow$ by step 1 & 2, $\exists$ a bijection between $\mathcal{V}$ and $\mathcal{F}$. #

Proof 3: (Linear Algebra).

Def: For a graph G on n vertices define the Laplace matrix $Q = (q_{ij})_{n\times n}$ of G as follows:

- $q_{ii} = d_G(i)$
- $q_{ij} = \begin{cases} -1, & \text{if } ij \in E(G) \\ 0, & \text{otherwise} \end{cases}$ for $i \neq j$.

$$\begin{pmatrix} d(1) & 0 & -1 & 0 & -1 & \cdots \\ & & & \nearrow & & \end{pmatrix} \to 0$$

Rmk: the sum of all rows is $\vec{0}$. $\det Q = 0$.

eg1: Laplace matrix of K_n is $A \triangleq \begin{pmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & -1 \cdots & -1 \\ \vdots & & \ddots & \vdots \\ -1 & \cdots & \cdots & n-1 \end{pmatrix}_{n\times n}$.

Let Q_{ij} be an $(n-1)\times(n-1)$ matrix obtained from Q by deleting the i^{th} row and the j^{th} column. Thm: For $\forall$ multigraph G, $ST(G) = \det Q_{11}$.

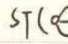
In particular, $ST(K_n) = \det A_{11} = \begin{vmatrix} n-1 & -1 & \cdots & -1 \\ \vdots & \ddots & & \vdots \\ -1 & \cdots & & n-1 \end{vmatrix}_{(n-1)\times(n-1)} = n^{n-2}$.

eg:  $d(1) = 3$.

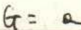
两个点间可以有多于1条边. (No loops).

Letting $\begin{cases} q_{ii} = d_G(i) \\ q_{ij} = -m \text{ where } m \text{ is } \#\text{ edges between } i \text{ and } j. \end{cases}$

Spanning tree: 

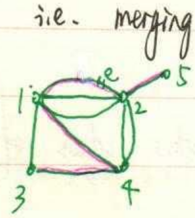
Proof of Thm: we count each spanning tree with different edges in $ST(G)$ (eg: $ST($$) = 6$)

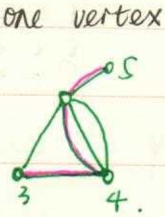
Prove by induction on number of edges of G:

- Base case: $G =$  ✓.
- Let e be any edge in G, say $e = 12$.

Define $G - e$ = the graph obtained from G by deleting e.

$G : e$ = the graph obtained from G by containg the edge e, i.e. merging 1 and 2 into one vertex.

eg2: $G =$ 

$\Rightarrow$ $G:e =$ 

claim: $ST(G) = ST(G-e) + ST(G:e)$. $\forall e$.

Pf of claim: we can divide the spanning trees of G into 2 classes.

- The 1^{st} class contains the spanning trees of G without containing e.
  (Note that these trees are exactly the spanning trees of $G-e$, which are of size $ST(G-e)$)
- The 2^{nd} class contains the spanning trees of G with e. We see that these trees are in one-to-one correspondence with the spanning trees of $G:e$.
  so that $\# 2^{nd}$ class $= ST(G:e)$.

Next, we analyze how the operations of $G-e$ and $G:e$ effect the Laplace matrix

- Let Q' be the laplace matrix of $G-e$.

G in eg2. $$Q = \begin{pmatrix} 5 & -3 & -1 & -1 & 0 \\ -3 & 6 & 0 & -2 & -1 \\ -1 & 0 & 2 & -1 & 0 \\ -1 & -2 & -1 & 4 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{pmatrix} Q_{11}. \qquad Q' = \begin{pmatrix} 4 & -2 & * \\ -2 & 5 & \\ & * & * \end{pmatrix} Q'_{11}$$

Q'_{11} is obtained from Q_{11} by substracting 1 from the element in the upper left corner of Q_{11}. i.e. $Q'_{11} = Q_{11} - \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & \cdots & 0 \end{pmatrix}_{(n-1)\times(n-1)}$.

- Let Q'' be the laplace matrix of $G:e$.

Here we relabel the vertices of $G:e$ as follows: (i). The new vertex get label 1
(ii) Other i get new label $i-1$.

$$Q'' = \begin{pmatrix} 5 & -1 & -3 & -1 \\ -1 & 2 & -1 & 0 \\ -3 & -1 & 4 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} Q''_{11} \quad \Rightarrow Q''_{11} = Q_{11,22}.$$

$$\Rightarrow Q_{11} = \begin{pmatrix} x_{11} & x_{12} & \cdots \\ * & \multicolumn{2}{c}{Q_{11,22}} \end{pmatrix}_{(n-1)\times(n-1)} \qquad \det Q_{11} = \det Q'_{11} + \det \begin{pmatrix} 1 & 0 \cdots 0 \\ * & Q_{11,22} \end{pmatrix}$$

$$Q'_{11} = \begin{pmatrix} x_{11}-1 & x_{12} & \cdots \\ * & \multicolumn{2}{c}{Q_{11,22}} \end{pmatrix}$$

$$= \det Q'_{11} + \det Q_{11,22} = \det Q'_{11} + \det Q''_{11}$$

$$\overset{\text{by induction}}{=} ST(G-e) + ST(G:e) \overset{\text{claim}}{=} ST(G)$$

$\Rightarrow ST(G) = Q_{11}$.

#