Combinatorics, 2016 Fall, USTC
Homework 1

- The due is on Tuesday, Sep. 13.
- Solve all problems.

1. How many 9-digit numbers can be made of digits $1, 2, \ldots, 9$ so that no i is put in the i'th position, and the number of not a *palindrome*. (A palindrome is a word or a number that reads the same in either direction. For example, "rotator" and "12321" are palindromes, but "1231231" is not.)

2. Let n, r be be positive integers and $n \geq r$. Give a **combinatorial proof** of

$$\binom{2n}{2r} \equiv \binom{n}{r} (mod\ 2).$$

3. Find a closed formula for

$$\sum_{k=m}^{n} \binom{k}{m} \binom{n}{k}.$$

4. Suppose there are n objects of r different kinds, where the ith kind has k_i identical objects and $k_1 + k_2 + ... + k_r = n$. How many distinct arrangements of those n objects (listed in a row from left to right) in total are there?

5. Let n be a positive integer. Prove that the identity

$$x^n = \sum_{k=1}^{n} S(n, k)(x)_k$$

holds for every real number x, where $S(n, k)$ is the Stirling number of the second kind, and $(x)_k := x(x-1)...(x-k+1)$.

Hint: first prove the case when x is a positive integer by double-counting certain mappings.

6. Find a closed formula of

$$\sum_{k=0}^{\lfloor n/7 \rfloor} \binom{n}{7k}.$$

(Try to simplify this expression to a summation with at most four terms)

Hint: let $\epsilon = e^{2\pi i/7}$, where i is the imaginary unit, and then consider the value of $\sum_{j=0}^{6} \epsilon^{kj}$.

7. Let $n \geq m$. Prove

$$\sum_{k=0}^{m} \binom{m}{k} \binom{n+k}{m} = \sum_{k=0}^{m} \binom{m}{k} \binom{n}{k} 2^k.$$