Combinatorics, 2016 Fall, USTC
Homework 10

- The due is on **Tuesday**, Dec. 6, at beginning of the class.
- Solve all problems.

1. We toss a fair coin n times and let X be the number of heads appearing. Compute the expectation of X.

2. Let $k \geq 4$ be an integer and $\mathcal{F} \subset \binom{X}{k}$ be a k-family. Prove that if

$$|\mathcal{F}| < \frac{4^{k-1}}{3^k},$$

then there is a coloring of elements of X with 4 colors such that in any set of family $\mathcal{F}$ all 4 colors are represented.

3. Prove that if there is a real $p \in [0,1]$ such that

$$\binom{n}{k} \cdot p^{\binom{k}{2}} + \binom{n}{t} \cdot (1-p)^{\binom{t}{2}} < 1,$$

then Ramsey number $R(k,t) > n$. Using this, show that $R(4,t) \geq c \cdot \left(\frac{t}{\ln t}\right)^{3/2}$ for some constant $c > 0$.

4. Prove that there exists an absolute constant $c > 0$ with the following property. Let A be an n by n matrix with pairwise distinct entries. Then there is a permutation of the rows of A so that no column in the permuted matrix contains an increasing subsequence of length at least $c\sqrt{n}$.

5. Prove that for all integers n and $p \in [0,1]$,

$$R(k,l) > n - \binom{n}{k} p^{\binom{k}{2}} - \binom{n}{l} (1-p)^{\binom{l}{2}}$$

and then show

$$R(4,k) \geq c \cdot \left(\frac{k}{\ln k}\right)^2$$

for some constant $c > 0$.

6. For any $\mathcal{F} \subseteq \binom{[n]}{3}$ with $m := |\mathcal{F}| \geq n/3$, prove that there exists a subset $A \subseteq [n]$ with

$$|A| \geq \frac{2n^{3/2}}{3\sqrt{3m}}$$

such that none of the sets in $\mathcal{F}$ is contained in A.

7. Recall the definition of $m(k)$. Given the following result: if there exists $p \in [0, 1]$ with $t(1-p)^k + t^2 p < 1$, then $m(k) > 2^{k-1}t$. Prove that

$$m(k) \geq c \cdot 2^k \left(\frac{k}{\ln k}\right)^{1/2}$$

for some constant $c > 0$.