Combinatorics, 2016 Fall, USTC
Homework 12

- The due is on Tuesday, Dec. 20, at beginning of the class.
- Solve all problems.

1. Given a graph G with $\chi(G) = n$ and a proper coloring $f : V(G) \to [n]$, prove that for any color $i \in [n]$, there exists a vertex of color i adjacent to a vertex of every other color.

2. Let $\mathcal{F} \subseteq 2^{[n]}$ be such that

- $|A|$ is even for all $A \in \mathcal{F}$, and
- $|A \cap B|$ is even for all distinct $A, B \in \mathcal{F}$.

Prove that $\mathcal{F} \leq 2^{n/2}$.

3. Let m red clubs $R_1, ..., R_m \subseteq [n]$ and m blue clubs $B_1, ..., B_m \subseteq [n]$ be such that

- $|R_i \cap B_i|$ is odd for every i, and
- $|R_i \cap B_j|$ is even for every $i \neq j$.

Show that $m \leq n$. And then answer the question if you can weaken the second condition to all $i < j$ (explain why).

4. (s-distance sets.) Let $s \geq 3$ be an integer. Let $a_1, a_2, ..., a_m$ be points in $\mathcal{R}^n$ and suppose that the pairwise distances between them take at most s values. Prove that $m \leq \binom{n+s+1}{s}$.

5. Construct a two-distance set of size $\binom{n+1}{2}$ in some $\mathbb{R}^n$.

6. Prove that among any $2^{k-1} + 1$ vectors in $\mathbb{F}_2^n$, there are k vectors which are linearly independent.