Combinatorics, 2016 Fall, USTC
Homework 2

- The due is on Tuesday, September 27, at beginning of the class.
- Solve any all problems.

1. How many functions $f : [n] \to [n]$ are there such that for all $i < j$, we have $f(i) \le f(j)$?

2. Let $\pi(n)$ denote the number of primes not exceeding the number n.

(a) Show that the product of all primes p with $m < p \le 2m$ is at most $\binom{2m}{m}$.

(b) For a prime p, show that if p^k divides $\binom{2m}{m}$, then $p^k \le 2m$.

(c) Using (a) and (b), prove that there exist absolute constants $c_1, c_2 > 0$ such that

$$c_1 \cdot \frac{n}{\log n} \le \pi(n) \le c_2 \cdot \frac{n}{\log n}.$$

3. An *equivalence* of $[n]$ is just a partition of $[n]$ into k non-ordered non-empty subsets for some k. Denote the n^{th} *Bell number* B_n to be the total number of equivalence of $[n]$. Prove that

$$B_n = \frac{1}{e} \sum_{i=0}^{\infty} \frac{i^n}{i!}.$$

4. (a). Given a natural number N, determine the probability that two number $m, n \in [N]$ chosen independently at random are relatively prime.

(b). Prove that the limit of the probability in (a) for $N \to \infty$ equals the infinite product $\prod_p (1 - 1/p^2)$, where p runs over all primes.

5. Show that the number $(6 + \sqrt{37})^{999}$ has at least 999 zero following the decimal point.

6. Let $f_n(x) = x(x-1)...(x-n+1)$. Prove that

$$f_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} f_k(x) f_{n-k}(y).$$

7. Let A_n be the number of ways of going up n stairs, if we may take one or two steps at a time. Find the generating function $f(x)$ of $\{A_n\}_{n\ge 0}$.