Combinatorics, 2016 Fall, USTC
Homework 5

- The due is on Tuesday, October 25, at beginning of the class.
- Solve all problems.

1. Let G be a graph such that for any two non-adjacent vertices u, v, it holds that $N_G(u) = N_G(v)$. Prove that G must be a *complete multipartite graph* (i.e., $V(G)$ can be partitioned into $V_1, V_2, ..., V_k$ for some $k \geq 2$ such that for any $u \in V_i, v \in V_j$ where $i \neq j$, we have $uv \in E(G)$).

2. Let $n > 0$ be an even integer. Let $\mathcal{F} \subset 2^{[n]}$ be a family of subsets of $[n]$ such that $\mathcal{F}$ contains no four distinct sets A, B, C, D satisfying $A \subset B \subset C \subset D$. Show that $|\mathcal{F}| \leq 3\binom{n}{n/2}$.

3. Recall that in the second proof of Sperner's Theorem we define an equivalence $\sim$ on the family $2^{[n]}$, by letting $M \sim M'$ hold if and only if both M and M' have the same partial pairing of their sequences.

Prove the claim that each equivalence class indeed is a symmetric chain.

4. Show that the set families $\binom{X}{\lfloor n/2 \rfloor}$ and $\binom{X}{\lceil n/2 \rceil}$ are the only independent systems on an n-element set X with the largest possible number of sets.

5. Let X be an n-element set and let $S_1, S_2, ..., S_n$ be subsets of X such that $|S_i \cap S_j| \leq 1$ for any $i \neq j$. Prove that there exists some subset S_i with $|S_i| \leq C\sqrt{n}$, for absolute constant C (independent of the choice of n).

6. Show that if a graph G on n vertices does not contain $K_{s,t}$ as a subgraph, then it has at most $C \cdot n^{2-1/s}$ edges for some absolute constant C only depending on t and s.

7. A set S is *sum-free* if no elements x, y, z such that $x + y = z$. Let S be a sum-free subset of $[n]$, where n is even. Prove that $|S| \leq n/2$.
Extra points. Characterize all sum-free subsets S of $[n]$ with $|S| = n/2$.