Combinatorics, 2016 Fall, USTC
Homework 6

- The due is on Tuesday, Nov. 1, at beginning of the class.
- Solve all problems.

1. Let L be a set of n distinct lines in the plane and P a set of n distinct points in the plane. Prove that the number of pairs (p, ℓ), where $p \in P$, $\ell \in L$, and p lies on ℓ, is bounded from above by $O(n^{3/2})$.

2. Let $t_n = ST(K_n)$. Prove the following recurrent formula

$$(n-1)t_n = \sum_{k=1}^{n-1} k(n-k)\binom{n-1}{k-1} t_k t_{n-k}.$$

3. Given a graph G on n vertices, let Q be its Laplace matrix, and let Q^* be the matrix whose element at position (i, j) equals $(-1)^{i+j} \det Q_{ij}$. Prove the following:

(1). $\det Q = 0$.

(2). If G is connected, then Q has rank $n-1$.

(3). If G is disconnected, then the rank of Q is at most $n-2$.

(4). If G is connected and $\vec{x} \in R^n$ is an arbitrary vector, then $Q\vec{x} = \vec{0}$ if and only if $\vec{x}$ is a multiple of the vector $\vec{1} = (1, 1, ..., 1)^T$.

(5). Show that QQ^* is a zero matrix. Using (4), infer that Q^* is an identical matrix.

4. For all values of positive integers n, determine the number of spanning trees on given n vertices in which all vertices have degree 1, 2, or 3.

5. For each $1 \leq k \leq n-1$, let N_k denote the number of spanning trees of K_n in which the vertex n has degree k. (Here we assume that $V(K_n) = \{1, 2, ..., n\}$.)

(a) Prove that $(n-1-k)N_k = k(n-1)N_{k+1}$.

(b) Prove by induction on k that $N_k = \binom{n-2}{k-1}(n-1)^{n-1-k}$.

(c) Use the above equality to derive the Cayley's formula.

6. Let D be a digraph defined on the vertex set $[n]$ such that every vertex has out-degree one. Prove that each component of such a digraph is a directed cycle, possibly with some trees hanging at the vertices of the cycle, with edges directed towards the cycle. Then use this to show that the mapping $\varphi : \mathcal{D} \to \mathcal{V}$ defined in the class indeed is a bijection.

7. Calculate $ST(K_{n,m})$, i.e., the number of spanning trees of the complete bipartite graph $K_{n,m}$.