Combinatorics, 2016 Fall, USTC
Outlines in Week 1

2016.9.6

Combinatorics

1.Basic
2.Graph Theory
3.Extrenal Combinatorics

Some notations

- Write $[n] = \{1, 2, ..., n\}$
- For set X, $|X| = \#elements\ in\ X$
- $2^X = \{A : A \subseteq X\}$, So $2^{[n]} = \{all\ sets\ of\ [n]\}$
- Fact 1. $|2^X| = 2^{|X|}$
- $\binom{X}{k} = \{A : A \subseteq X, |A| = k\}$
- Fact 2. Let $|X| = n$, then
$$|\binom{X}{k}| = \frac{n!}{k!(n-k)!} = \binom{n}{k}$$
  binomial coefficient
- Remark. $\binom{n}{k}$ stands for the number of selections of size k out of n distinct objects. For $n < k$, let $\binom{n}{k} = 0$, $\binom{n}{0} = 1$

Properties on Binomial coefficient

- $\binom{n}{k} = \binom{n}{n-k}$ for $k \leqslant n$
- $\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$
  Combinatorial proof?
- Pascal triangle
- Fact 3. The number of integer solutions $(x_1, x_2, ..., x_n)$ to $x_1 + x_2 + ... + x_n = k$, where $x_i \in \{0, 1\}$, is $\binom{n}{k}$
- Fact 4. The number of integer solution $(x_1, x_2, ..., x_n)$ to $x_1 + x_2 + ... + x_n = k$, where $x_i > 0$, is $\binom{n+k-1}{n-1}$

Counting functions

- Let X,Y be sets such that $|X| = n,\ Y = [r]$

  Let $X^Y = \{all\ functions\ f : Y \to X\}$
- Claim 1. $|X^Y| = |X|^{|Y|} = n^r$
- Claim 2. There are $(n)_r$ injections $f : Y \to X$, where $n \geqslant r$

  $\Rightarrow \#of\ such\ strings = n(n-1)...(n-r+1) = (n)_r$
- **Definition.**(The Stirling number of the 2nd kind)

  Let S(r,n) be the number of partitions of [r] into n unordered non-empty subsets

  i.e. S(r,1)=1, S(3,2)=3, S(4,2)=7
- Exercise. S(r,2)=$\frac{1}{2}\sum_{i=1}^{r-1}\binom{r}{i} = \frac{1}{2}\binom{r}{2-2} = \binom{r-1}{2-1}$
- **Thm.** Let $r \geqslant n$, then # surjection $f : Y \to X = S(r,n) \cdot n!$

2016.9.8

Binomial Thm

- Consider a polynomial $f(x)$. Let $[x^k]f$ be the coefficient of the term x^k in $f(x)$.

  i.e. $f(x) = 3 + 2x - 10x^5 \Rightarrow [x^4]f = 0$
- Exercise 1. How many ways to form 16 cents, given 2 dimes, 3 nickels and 6 pennies?

  To see this, often multiplying out the parenthesis, each term x^{16} is formed by multiplying some x^{i_1} in 1st factor, some x^{i_2} in 2nd factor and x^{i_3} in 3rd factor with $i_1 + i_2 + i_3 = 16$, and each term x^{16} will contribute 1 to $[x^{16}]f$. This finishes the proof.
- Fact 1. For $j = 1, 2, ..., n$, let $f_j(x) = \sum\limits_{k \in I_j} x^k$ where I_j is a set of non-negative integers, and let $f(x) = \Pi_{j=1}^n f_j(x)$. Then, $[x^k]f$ equals the number of solutions $(i_1, i_2, ..., i_n)$ to $i_1 + i_2 + ... + i_n = k$ where $i_j \in I_j$
- Fact 2. Let $f_1, ..., f_n$ be polynomials and $f = f_1 f_1 ... f_n$. Then,

$$[x^k]f = \sum_{i_1+i_2+...+i_n=k, i_j \geqslant 0} ([x^{i_1}]f_1)([x^{i_2}]f_2)...([x^{i_n}]f_n)$$

- **Binomial Thm.** For $\forall$ positive integer n and $\forall$ real x,

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

- Exercise 2. $\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$
- Exercise 3. $\sum_{k=odd, 0 \leqslant k \leqslant n} \binom{n}{k} = \sum_{k=even, 0 \leqslant k \leqslant n} \binom{n}{k} = 2^{n-1}$
- Exercise 4. $\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$
- Exercise 5. (Vandermonde's Convolution Thm): $\forall n, m, k \geqslant 0,\ \binom{n+m}{k} = \sum_{j=0}^{k} \binom{n}{j}\binom{m}{k-j}$

Estimating

- **Thm.** For $\forall n \geqslant 1$, $e(\frac{n}{e})^n \leqslant n! \leqslant en(\frac{n}{e})^n$

  $\Rightarrow (n-1)! \leqslant e^{nlgn-n+1} = e(\frac{n}{e})^n$

  $\Rightarrow n! \leqslant ne(\frac{n}{e})^n$

  The prove of lower bound is simple, which is left as an exercise.
- Recall(Stirling formula): $n! \sim \sqrt{2\pi n}(\frac{n}{e})^n$ where $f(n) \sim g(n)$ means $\lim_{n \to +\infty} \frac{f(n)}{g(n)} = 1$
- Exercise. $n! \leqslant e\sqrt{n}(\frac{n}{e})^n$
- Fact. If n is even, $\binom{n}{0} < \binom{n}{1} < ... < \binom{n}{\frac{n}{2}} > \binom{n}{\frac{n}{2}+1} > ... > \binom{n}{n}$

  If n is odd, $\binom{n}{0} < \binom{n}{1} < ... < \binom{n}{\lfloor \frac{n}{2} \rfloor} = \binom{n}{\lceil \frac{n}{2} \rceil} > ... > \binom{n}{n}$
- **Corollary.** $\frac{2^n}{n+1} \leqslant \binom{n}{\lfloor \frac{n}{2} \rfloor} \leqslant 2^n$
- Remark. By Stirling formula $\binom{n}{\frac{n}{2}} \sim \sqrt{\frac{2}{\pi}} \cdot \frac{2^n}{\sqrt{n}}$
- **Thm.** For $\forall 1 \leqslant k \leqslant n$, $(\frac{n}{k})^k \leqslant \binom{n}{k} \leqslant (\frac{en}{k})^k$