Combinatorics, 2016 Fall, USTC

Week 10, November 8

Partially Ordered Sets

Let X be a finite set.

Definition 1. R is called a "relation" on the set X, if $R \subseteq X \times X$ where $X \times X = \{(x_1, x_2) : x_1, x_2 \in X\}$. Denote the Cartesion product if $(x, y) \in R$, then xRy.

Definition 2. A partially ordered set (poset for short) is an ordered pair (X, R), where X is a finite set and R is a relation on X such that the following holds:

(1) R is reflective: xRy for $\forall x \in X$

(2) R is antisymmetric: if xRy and yRx, then $x = y$

(3) R is transitive: if xRy and yRz, then xRz

Examples. Consider the poset $(2^{[n]}, \subseteq)$, where "$\subseteq$" denotes the inclusion relationship.

We often use "$\preccurlyeq$" to replace the use of "R". So $(X, R) \Rightarrow (X, \preccurlyeq)$.

If $x \preccurlyeq y$ but $x \neq y$, then $x \prec y$, and we say x is predecessor of y.

Definition 3. Let $(X, \preccurlyeq)$ be a poset, we say element x is an immediate predecessor of y, if

(1) $x \prec y$

(2) NO $t \in X$ s.t. $x \prec t \prec y$

If x sa an immediate predecessor of y, then we write $x \lhd y$.

Fact: For $x, y \in (X, \preccurlyeq)$, $x \prec y$ if and only if there exists $x_1, x_2, ..., x_k \in X$ s.t. $x \lhd x_1 \lhd x_2 \lhd ... \lhd x_k \lhd y$(Note that $k = 0$ i.e. $x \lhd y$).

Proof. ($\Leftarrow$) trivial

($\Rightarrow$) For $x \prec y$, let $M_{xy} = \{t \in X : x \prec t \prec y\}$. We prove by induction on $|M_{xy}|$. Because $|M_{xy}| = 0 \Rightarrow x \lhd y$. So suppose it holds for $x \prec y$ with $|M_{xy}| < n$. Consider $x \prec y$ with $|M_{xy}| = n \geqslant 1$. Pick any $t \in M_{xy}$, consider M_{xt}, M_{ty}. Clearly $M_{xt} \subsetneq M_{xy}$ and $M_{ty} \subsetneq M_{xy}$(Because of transitivity). By induction on M_{xt}, M_{ty}, there exists $x_1, x_2, ..., x_k \in X$ and $y_1, y_2, ..., y_l \in X$ s.t. $x \lhd x_1 \lhd x_2 \lhd ... \lhd x_k \lhd t$ and $t \lhd y_1 \lhd y_2 \lhd ... \lhd y_l \lhd y$, $\Rightarrow x \lhd x_1 \lhd x_2 \lhd x_k \lhd t \lhd y_1 \lhd ... \lhd y_l \lhd y$. We are done. ∎

One property that posets have is that we can express them in diagrams

Definition 4. The Hassa diagrams of a poset $(X, \preccurlyeq)$ is a drawning in the plane such that

(1) Each element of X is drawn as a nod in the plane

(2) Each pair x, y with $x \lhd y$ is connected by a line segment

(3) If $x \lhd y$, then the nod x must appear lower in the plane then the nod y

The fact that $x \prec y$ iff $x \lhd x_1 \lhd x_2 \lhd ... \lhd x_k \lhd y$ now can be restated as follows: $x \prec y$ if and only if we can find a path in the Hassa diagram from nod x to nod y, strictly from bottom to top.

Definition 5. Let $(X, \preccurlyeq)$ and $(X^{'}, \preccurlyeq^{'})$ be two posets. A mapping $f : X \to X^{'}$ is called an embedding of $(X, \preccurlyeq)$ and $(X^{'}, \preccurlyeq^{'})$ if

(1) f is injective

(2) $f(x) \preccurlyeq f(y)$ iff $x \preccurlyeq y$.

Theorem 6. *For every poset* $(X, \preccurlyeq)$ *there exists an embedding into the poset* $\mathscr{B}_X = (2^X, \subseteq)$

Proof. Consider the mapping $f : X \to 2^X$ by $f(x) = \{y \in X : \ y \preccurlyeq x\}$. Let us verify that such f is an embedding of $(X, \preccurlyeq)$ into $\mathscr{B}_X$.

Firstly, f is injective

Suppose $f(x) = f(y)$ for $x, y \in X \ \Rightarrow x \in f(y) \ \Rightarrow x \preccurlyeq y$, similarly $y \preccurlyeq x$. Thus $x = y$.

Secondly, $f(x) \subseteq f(y)$ iff $x \preccurlyeq y$.

If $x \preccurlyeq y$, then $\forall t \in f(x)$ has $t \preccurlyeq x \preccurlyeq y \ \Rightarrow t \in f(y) \ \Rightarrow f(x) = f(y)$.

If $f(x) \subseteq f(y)$, then $x \subseteq f(x) \subseteq f(y) \ \Rightarrow x \preccurlyeq y$ ∎

Definition 7. Let $P = (X, \preccurlyeq)$ be a poset.

(1) For distinct $x, y \Rightarrow X$, if $x \prec y$ or $y \prec x$, then we say x, y are comparable. Otherwise x, y are incomparable

(2) The set $A \subseteq X$ is an antichain of P, if any two elements of A are incomparable. Let $\alpha(P)$ be the maximum size of an antichain in P

(3) The set $A \subset X$ is a chain of P, if any two elements of A are comparable. Let $\omega(P)$ be the maximum size of a chain of P

Consider the Hassa diagram, $\omega(P)$ means the max length of a path (from bottom to top) in this diagram. So $\omega(P)$ is also called the height of P. And $\alpha(P)$ is called the width of P.

Definition 8. An element $x \in X$ is minimal in $P = (X, \preccurlyeq)$, if x has NO predecessor in P.

Fact: The set of minimal elements of $P = (X, \preccurlyeq)$ forms an antichain of P.

Theorem 9. *For $\forall$ poset $P = (X, \preccurlyeq)$, $\alpha(P) \cdot \omega(P) = |X|$*

Proof. We will inductively define a sequence of poset P_i and set M_i for $1 \leqslant i \leqslant l$ s.t. M_i is the set of minimal elements of $P_i = (X_i, \preccurlyeq)$ and $X_i = X - \bigcup_{j=1}^{i-1} M_j$ as following. First, set $P_1 = P = (X, \preccurlyeq), X_1 = X$ and $M_1 = \varnothing$. Assume posets $P_i = (X_i, \preccurlyeq)$ and M_{i-1} are defined for all $1 \leqslant i \leqslant k$. Let $M_i = \{$ all minimal elements of $P_i\}$ and let $X_{i+1} = X - M_1 \bigcup ... \bigcup M_i$. Then let P_{i+1} be the subposet of P restricted on X_{i+1}. We keep doing this until $X_{l+1} = \varnothing$. By Fact 2, each M_i for $1 \leqslant i \leqslant l$ is an antichain of P_i and thus it is also an antichain of P. So $|M_i| \leqslant \alpha(P)$ ∎