Combinatorics, 2016 Fall, USTC

Week 11, November 15 and 17

Poset

Poset: P=(X,$\preceq$)

Definition 1. An element x$\in$ X is minimal, if x has no predecessor.

Definition 2. $\alpha(P)$ =max size of anti-chain in P. $w(P)$=max size of a chain in P.

Fact: The set of all minimal element in P forms an anti-chain of P.

Then $\forall$ poset P=(X,$\preceq$), we have

$$\alpha(P) \cdot w(P) \geq |X|$$

Proof. We inductively define a sequence of posets $P_i = (x_i, \preceq)$ and a sequence of sets $M_i \subset P_i$, such that each M_i is the seet of minimal elements of P_i, and $X_i = X - \sum_{j=0}^{i-1} M_j$, where $M_0 = \phi$.

Now suppose we obtain $P_1, P_1, ..., P_l$ and $M_i \subset P_i$ for $1 \leq i \leq l$.

By the Fact, each M_i is an anti-chain of P_i; since P_i is the restricted subposet of P on X_i, M_i is also an anti-chain of P. So

$$|M_i| \leq \alpha(P).$$

It suffices to find a chain $x_1 < x_2 < ... < x_l$ in P, such that $x_i \in P_i = (X_i, \preceq)$.

If this holds,

$$X = M_1 \bigcup M_2 \bigcup ... \bigcup M_l$$

$$\Longrightarrow |X| = \sum_{i=1}^{l} |M_i| \leq \alpha(P) \cdot l \leq \alpha(P)w(P).$$

∎

We claim something stronger holds:

$\forall x \in M_{i+1}$ and $\forall i < l$, $\exists y \in M$, such that $y < x$.

proof: By definition of M_i.

Ramsey Theroem

The order from disorder!

Definition 3. (Erdös-Szekeres Theroem) Consider a squence $X = (x_1, x_2, ..., x_n)$ of real number of length n. A subsequence $(x_{i_1}, x_{i_2}, ..., x_{i_m})$ of X, where $i_1 < i_2 < ... < i_m$ is monotone, if either $x_{i_1} \leq x_{i_2} \leq ... \leq x_{i_m}$ or $(x_{i_1} \geq x_{i_2} \geq ... \geq x_{i_m}$.

For example, $(10, 9, 7, 4, 5, 1, 2, 3) \longrightarrow (10, 9, 7, 5, 1)$

Theorem 4. *(Erdös-Szekeres) For any sequence $(x_1, x_2, ..., x_{n^2+1})$ of length $n^2 + 1$, there exists one monotone subsequence of length n+1.*

Proof. Let $X = [n^2 + 1]$. We define a poset $P = (X, \preceq)$ as following:

$i \preceq j$ if and only if $i \leq j$ and $x_i \leq x_j$.

It is easy to verify that P indeed is a poset(refecsive antisymmetric & transitive)

By the previous result that $\alpha(P) \cdot w(P) \geq |X| = n^2 + 1$, we have 2 cases to consider:

case 1: $\alpha(P) \geq n + 1$.

So P has an anti-chian of size n+1, say $\{i_1, i_2, .., i_{n+1}\}$. We may assume $i_1 < i_2 < ... < i_{n+1}$. But each (i_j, i_k) is incomparable in P. Thus, assuming $i_j < i_k$, we see that $x_{i_j} > x_{i_k}$.

$$\Longrightarrow x_{i_1} > x_{i_2} > ... > x_{i_{n+1}}$$

. This is a decreasing subsequence of $(x_1, x_2, ..., x_{n^2+1})$.

case 2: $w(P) \geq n + 1$.

So P has a chain, say $x_{i_1} \preceq x_{i_2} \preceq ... \preceq x_{i_{n+1}}$. By definition, we have $i_1 < i_2 < ... < i_{n+1}$ and $x_{i_1} \leq x_{i_2} \leq ... \leq x_{i_{n+1}}$. So we have an increasing subsequence of length n+1.

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Rmk: In fact, the proof shows that we can have a strickly increasing subsequence or a decreasing subsequence.

Exercise: Find examples to show that E-S Thm is best possible.

The Pigeonhole Principle

Let X be a set with at least $1+\sum_{i=1}^{k}(n_i-1)$ elements and let $X_1, X_2, ..., X_k$ be disjoint sets forming a partition of X. Then, there exists i, s.t. $|X| \geq n_i$.

(1)Two equal degrees.

Theorem 5. *Any graph has two vertices of the same degree.*

Proof. Let G be a graph with n vertices. The degrees are from 0 to n-1. So the only exceptional case will be that there is exactly one vertex of degree i for $\forall i \in \{0, 1, ..., n-1\}$. But it is impossible to have a vertex with degree 0 and a vertex with degree n-1 at the same time. ∎

Exercise: For $\forall n$, find an n-vertex graph G, which has exactly two veertices with the same degree.

(2)Subsets without divisors.

Question: How large a subset $S \subset [2n]$ can be such that for $\forall i, j \in S$, we have $i \nmid j$ & $j \nmid i$?

Obviously, we can take $S = \{n+1, n+2, ..., 2n\}$ with $|S| = n$.

Theorem 6. *For any $S \subset [2n]$ with $|S| \geq n+1$, there exists $i, j \in S$ such that $i|j$.*

Proof. For each odd 2k-1, define $S_{2k-1} = \{2^i \cdot (2k-1) \in S, forsomei\}$ for some k=1,2,...,n.

Clearly, $S = \bigcup_{k=1}^{n} S_{2k-1}$ can be partitioned into n subsets. But $|S| \geq n+1$, by P-P, $\exists k \in [n]$, s.t. $|S_{2k-1}| \geq 2$. ∎

(3)Rational approximation.

Theorem 7. *Given n>0, for any $x \in R$,there is a rational number p/q with $1 \leq q \leq n$ such that $|x - \frac{p}{q}| < \frac{1}{nq}$.*

Proof. Consider x>0, let $\{x\} = x - \lfloor x \rfloor$ be the fractional part of x. Consider $\{ix\}$, for i=1,2,...,n+1, where are n+1 real numbers in [0,1). Partition [0,1) into n subintervals $[0, \frac{1}{n}), [\frac{1}{n}, \frac{2}{n}), ..., [\frac{n-1}{n}, 1)$. By P-P, there are 2 numbers say $\{ix\}, \{jx\}$(let $\{jx\} > \{ix\}$) belonging to the same subinterval.

$\Longrightarrow \{(j-i)x\} = \{jx\} - \{ix\} \in [0, \frac{1}{n})$.

Let q=j-i, then $qx = p + \epsilon$, where $\epsilon = \{qx\} \in [0, \frac{1}{n})$ and $p \in Z$.

$\Longrightarrow x = \frac{p}{q} + \frac{\epsilon}{q}$, where $|\frac{\epsilon}{q}| < \frac{1}{nq}$. ∎

Erdős-Szekeres Theroem

Theorem 8. *For any sequence of mn+1 real numbers* $\{a_0, a_1, ..., a_{mn+1}\}$*, there is an increasing subsequence of length m+1 or a decreasing subsequence of length n+1.*

Proof. **(the second proof)**

For each $i \in \{0, 1, ..., mn\}$, let t_i be the maximum length of an increasing subsequence starting at a_i. If ∃i, s.t.$t_i \geq m+1$,then we are done. So we may assume $t_i \in \{1, 2, ..., m\}$ for $\forall i \in \{0, 1, ..., mn\}$. By P-P, there exists some $s \in \{1, 2, ..., m\}$ such that there are at least n+1 many $t_i's$ satisfying that $t_i = s$. Let these indexes i's be $i_1 < i_2 < ... < i_{n+1}$.

Claim: $a_{i_1} \geq a_{i_2} \geq ... \geq a_{i_{n+1}}$.

Proof. Otherwise, there ∃j,s.t. $a_{i_j} < a_{i_{j+1}}$. Then we would extend the maximal increasing subsequence starting at $a_{i_{j+1}}$, by adding a_{i_j}, to get an increasing subsequence starting at a_{i_j} of length s+1. Therefore, this contradicts $t_{i_j} = s$. ∎

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Ramsey's Theorem

Fact:(A party of six) Suppose a party has 6 participants. Participants may know each other or not. Then there must be 3 participants who know each other or don't know each other.

Proof. We can construct a graph G on [6]. Each vertex i represents one participants: i and j are adjacent iff they know each other. Then we need to show that there are 3 vertices in G which form a triangle K_3 or an independent set I_3.

Consider vertex 1. From the point of view of 1. 1 is adjacent to ≥ 3 vertices or i not adjacent to ≥ 3 vertices. By symmetry, 1 is adjacent to 2,3,4. If one of pairs $\{2,3\}, \{2,4\}, \{3,4\}$ is adjacent, then we have a K_3. Otherwise, we have an $I_3 = \{2,3,4\}$. ∎

Definition 9. A r-edge-coloring of K_n is a function f:$E(K_n) \longrightarrow \{1, 2, ..., r\}$ which assigns one of the colors 1,2,...,r to each edge of K_n.

Definition 10. Suppose there is an r-edge-coloring of K_n. A clique in K_n is called monochromatic, if all its edges are colored by the same color.

Theorem 11. *(Ramsey's Thm(2-colors-version)) Let $k, l \geq 2$ be integers. There exists an integer N=N(k,l), s.t. any 2-edge-coloring of K_N (with colors red and blue) has a blue K_k or a red K_l.*

Proof. We will prove by induction on k+l that $N = \binom{k+l-2}{k-1}$ will suffice.

Base case: $k + l = 4 \Longleftrightarrow k = l = 2$. It is trivial.

Assume that it holds for $k'+l' \leq k+l-1$. Let $N_1 = \binom{k+l-3}{k-2}$, $N_2 = \binom{k+l-3}{k-1}$, and $N = \binom{k+l-2}{k-1}$.

Note that $N_1 + N_2 = N$.

Consider any 2-edge-coloring of K_N. Consider any vertex x. Let $A = \{y \in V(K_n) - \{x\} :$ edge xy is blue$\}$ and $B = \{y \in V(K_n) - \{x\} :$ edge xy is red$\}$. So $|A| + |B| = N - 1 = N_1 + N_2 - 1$. Thus, either $|A| \geq N_1$ or $|B| \geq N_2$.

Case 1: $|A| \geq N_1 = \binom{k+l-3}{k-2}$.

The vertices of A contains a $K_{\binom{(k-1)+l-2}{(k-1)-1}}$ where edges are blue or red. By induction on this $K_{\binom{(k-1)+l-2}{(k-1)-1}}$ for the pair $\{k-1, l\}$, so A has a blue K_{k-1} or a red K_l.We can add the vertex x to get a blue K_k. So we have done.

Case 2: $|B| \geq N_2 = \binom{k+l-3}{k-1}$.

Similarly. ∎

Definition 12. For $k, l \geq 2$, the Ramsey Number R(k,l) denotes the smallest integer N s.t. any 2-edge-coloring of K_N has a blue K_k or a red K_l.

Corollary 1: $R(k, l) \leq \binom{k+l-2}{k-1}$.

Let us try to understand this definition more:

- $R(k, l) \leq L \iff$ any 2-edge-coloring of K_L has a blue K_k or a red K_l.
- $R(k, l) \geq M \iff$ there exists a 2-edge-coloring of K_M which has no blue K_k nor red K_l.

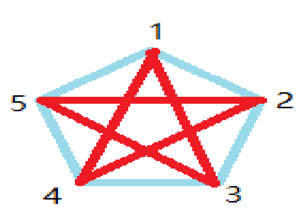
Corollary 2: (Exercise)$R(k, l) \leq R(k-1, l) + R(k, l-1)$.

Fact 1: $R(k, l) = R(l, k)$

Fact 2: $R(2, l) = l$ and $R(k, 2) = k$.

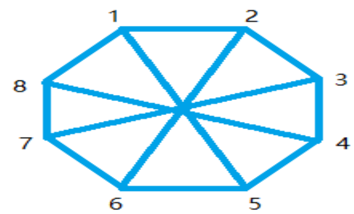
Fact 3: $R(3, 3) = 6$.

Why? A party of six tells us that $R(3, 3) \leq 6$; in the other hand, the following example tells us that $R(3, 3) > 5$.



Fact 4: $R(3,4) = 9$.

Consider the graph:



It has NO K_3 nor I_4. $\Longrightarrow R(3,4) > 8$. The fact $R(3,4) \leq 9$ will follow by a theorem which we prove next time.