Combinatorics, 2016 Fall, USTC

Week 13, November 29 & December 1

Ramsey's Theorem

Theorem 1. *Let n, k satisfying $\binom{n}{s}2^{1-\binom{s}{2}} < 1$. Then $R(s,s) > n$.*

Proof. We need to find a 2-edge-coloring of K_n such that it has NO monochromatic clique K_s.

Consider a random 2-edge-coloring of K_n: each edge is colored by blue or red, each with probability $\frac{1}{2}$, independent of other edges.

Let A be the event that the so-defined K_n has a monochromatic K_s. For $X \in \binom{[n]}{s}$, let A_X be the event that X is a monochromatic K_s. $\longleftrightarrow A = \cup_{X\in\binom{[n]}{s}} A_X$

$$P(A) = P\left(\cup_{X\in\binom{[n]}{s}} A_X\right) \leq \sum_{X\in\binom{[n]}{s}} P(A_X) = \binom{n}{s}2^{1-\binom{s}{2}} < 1,$$

Thus $P(A_C) > 0$, that is the probability that K_n has NO monochromatic K_s is positive. So there must exist a 2-edge-coloring of K_n such that it has NO monochromatic clique K_s.

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Corollary 2. $R(k,k) \geq \frac{1}{e\sqrt{2}}k2^{\frac{k}{2}}$.

Proof. Let $n = \frac{1}{e\sqrt{2}}k2^{\frac{k}{2}}\left(\frac{e}{2}\right)^{1/k}$. Recall that $\binom{n}{k} < \frac{n^k}{k!}$ and $k! \geq e\left(\frac{k}{e}\right)^k$, thus we have that

$$\binom{n}{k}2^{1-\binom{k}{2}} < \frac{n^k}{e\left(\frac{k}{e}\right)^k}2^{1-\binom{k}{2}} = \left(\frac{en}{k}\right)^k \cdot \left(\frac{2}{e}\right) \cdot 2^{-\binom{k}{2}};$$

by substituting $n = \frac{1}{e\sqrt{2}}k2^{\frac{k}{2}}\left(\frac{e}{2}\right)^{1/k}$, it becomes that $\binom{n}{k}2^{1-\binom{k}{2}} < 1$. By Theorem, we get

$$R(k,k) > n = \frac{1}{e\sqrt{2}}k2^{\frac{k}{2}}\left(\frac{e}{2}\right)^{1/k} \geq \frac{1}{e\sqrt{2}}k2^{\frac{k}{2}}.$$

∎

Corollary 3.

$$\frac{1}{2} \leqslant \lim_{s\longrightarrow\infty} \frac{\log_2 R(s,s)}{s} \leqslant 2$$

Proof. This can be derived from $R(s,s) \leqslant \binom{2s-2}{s-1}$ ∎

The Probabilistic Methods in Combinatorics

We remark on the following two ideas in the proof.

(i). Imagine we need to find some combinatorial object satisfying certain property, call them "good" object. We consider a random object. If the probability that the random object is "good" is positive, then there must exist "good" objects.

(ii). To compute the probability of being "good", we often compute the probability of being "bad" and aim to prove this probability is strictly less than 1.

Definition 4. A *probability space* is (Ω, P), where Ω is a finite set and $P : 2^\Omega \to [0,1]$ is a function assigning a number in the interval $[0,1]$ to every subset of Ω such that

(i) $P(\emptyset) = 0$,

(ii) $P(\Omega) = 1$, and

(iii) $P(A \cup B) = P(A) + P(B)$ for disjoint sets $A, B \subset \Omega$.

- Any subset A of Ω is called an *event*, and $P(A) = \sum_{\omega \in \Omega} P(\{\omega\})$.
- A random variable is a function $X : \Omega \to R$
- Expectation: $E[X] := \sum_{\omega \in \Omega} P(\{\omega\}) \cdot X(\omega)$.
- Two events A, B in the probability space (Ω, P) are *independent* if

$$P(A \cap B) = P(A)P(B).$$

- **The linearity of expectations.** For any two random variables X and Y on Ω, we have

$$E[X + Y] = E[X] + E[Y]$$

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Definition 5. Let $\mathcal{F}$ be a family of sets. We say $\mathcal{F}$ is a k-family, if every set in $\mathcal{F}$ is of size k.

Definition 6. Let $X = \cup_{A \in \mathcal{F}} A$, we say $\mathcal{F}$ is *2-colorable* if there exists a function $f : X \to \{\text{blue, red}\}$ such that every set $A \in \mathcal{F}$ is not monochromatic (i.e. A has at least one blue element of X and at least one red element of X).

Remark. When $k = 2$, 2-family $\mathcal{F}$ can be viewed as a graph G. Then $\mathcal{F}$ is 2-colorable iff G is bipartite.

Definition 7. For $\forall k$, let $m(k) := min|\mathcal{F}|$ over all k-family $\mathcal{F}$ which are NOT 2-colorable.

(1) $m(k) \leqslant t \Leftrightarrow \exists k$-family $\mathcal{F}$ which is not 2-colorable but $|\mathcal{F}| = t$.

(2) $m(k) > t \Leftrightarrow \forall k$-family $\mathcal{F}$ with $|\mathcal{F}| = t$ is 2-colorable.

Fact:$m(2) = 3$

Theorem 8. *$m(k) \geqslant 2^{k-1} \Leftrightarrow$ every k-family $\mathcal{F}$ with $|\mathcal{F}| = 2^{k-1} - 1$ is 2-colorable.*

Proof. We need to find a function $f : X \to \{\text{blue, red}\}$ where $X = \cup_{A\in\mathcal{F}} A$ s.t. $\forall A \in \mathcal{F}$ has a blue element and a red element. We sat such f is good. Otherwise call it bad. We then consider a random function φ on X, that is each $x \in X$ is colored by blue or red with probability $\frac{1}{2}$, independent of other choice. Let B be the event that φ is bad, i.e. there exists some $A \in \mathcal{F}$ which is monochromatic.

For $A \in \mathcal{F}$, let B_A be the event that A is monochromatic. So $B = \cup_{A\in\mathcal{F}} B_A$.

It is easy to see for $\forall A \in \mathcal{F}$

$$P(B_A) = 2(\frac{1}{2})^{1-k} = 2^{1-k}$$

Thus,

$$p(B) \leqslant \sum_{A\in\mathcal{F}} P(B_A) = |\mathcal{F}| 2^{1-k} < 1$$

So $P(\varphi \textit{ is good}) = P(B^C) > 0$.

Since

$$P(\varphi \textit{ is good}) = \frac{\#\ good\ functions}{all\ functions} > 0$$

$\Rightarrow$ there must exist good functions. ∎

Definition 9. The *random graph* $G(n,p)$ for $0 \leqslant p \leqslant 1$ is a graph with vertex set $\{1,2,...,n\}$, where each of potential $\binom{n}{2}$ edges appears with probability p, independent of other edges.

Let A be the property of graphs we are interested in.

$$\begin{aligned} Let \quad P_r(A) &= P_r(G(n,\frac{1}{2}) \quad has \quad property \quad A) \\ &= \frac{\#graphs \quad in \quad \mathcal{G}_n \quad satisfying \quad property \quad A}{2^{\binom{n}{2}}} \end{aligned} \tag{1}$$

which is a function of n.

Definition 10. We say random graph $G(n,\frac{1}{2})$ almost surely satisfies property A, if $lim_{n\to+\infty}P_r(A)=1$. If $lim_{n\to+\infty}P_r(A)=0$, then $G(n,\frac{1}{2})$ almost surely not satisfy property A.

Consider property A=bipartiteness.

Theorem 11. *Random Graph $G(n,\frac{1}{2})$ almost surely is NOT bipartite.*

Proof. Let A=the event that $G(n,\frac{1}{2})$ is bipartite. For $U \in 2^{[n]}$, let A_U be the event that all edges of G are between U and $[n]\backslash U$.

$$\Longrightarrow A = \bigcup_{U\in[n]} A_U$$

What is $P_r(A_U)$?

By definition,

$$\begin{aligned} P_r(A_U) &= \frac{\#bipartite \quad graph \quad G \subset (U,[n]\backslash U)}{2^{\binom{n}{2}}} \\ &= \frac{2^{|U|(n-|U|)}}{2^{\binom{n}{2}}} \leq \frac{2^{\frac{n^2}{4}}}{2^{\frac{n(n-1)}{2}}} = 2^{-\frac{n^2}{4}+\frac{n}{2}} \end{aligned} \tag{2}$$

So $P_r(A) \leq \sum_{U \subset [n]} P_r(A_U) \leq 2^n \cdot 2^{-\frac{n^2}{4}+\frac{n}{2}} = 2^{-\frac{n^2}{4}+\frac{3n}{2}}$.

So $lim_{n \to +\infty} P_r(A) = 0$. ∎

Independent Events

Definition 12. k events $A_1, A_2, ..., A_k$ are independent if $\forall I \subset [n]$, $P_r(\bigcap_{i \in I} A_i) = \prod_{i \in I} P_r(A_i)$.

Definition 13. A Tournament of n vertices is a directed graph obtained from the clique K_n by assigning a direction to each edge of K_n. We say a vertex i beats vertex j if there exists: $i \longrightarrow j$.

Definition 14. A tournament T has property S_k: for any subset A of size k, there are exists a vertex beats all vertices of A.

Question: For $\forall k \geq 2$, does it exist a T with property S_k? Yes!

Theorem 15. *For $\forall k \geq 2$, if $\binom{n}{k}(1 - \frac{1}{2^k})^{n-k} < 1$, then there exists a tournament T on n vertices satisfying property S_k.*

Proof. We show this by considering a Random Tournament an [n]. For any i<j, the $i \longrightarrow j$ occurs with probability $\frac{1}{2}$, independent of other choices. Let B be the event that T doesn't satisfy S_k. For $A \in \binom{[n]}{k}$, let B_A be the event that all vertices in $[n] \backslash A$ can not beat every vertex of A.

$$\Longrightarrow B = \bigcup_{A \in \binom{[n]}{k}} B_A$$

For $x \in [n] \backslash A$, let $B_{A,x}$ be the event that x can not beat every vertex of A.

$$\Longrightarrow B_A = \bigcap_{x \in [n] \backslash A} B_{A,x}$$

Clearly, $P_r(B_{A,x}) = 1 - (\frac{1}{2})^k$.

Note that only the arcs between x and A will effect the event $B_{A,x}$, and these arcs for distinct vertices x's are disjoint. Thus, all events $B'_{A,x}s$ for all $x \in [n] \backslash A$ are independent.

$$\implies P_r(B_A) = P_r(\bigcap_{x \notin A} B_{A,x}) = \prod_{x \notin A} P_r(B_{A,x}) = (1 - (\frac{1}{2})^k)^{n-k}$$

By union bound,

$$P_r(B) \leq \sum_{A \in \binom{[n]}{k}} P_r(B_A) \leq \binom{n}{k}(1 - (\frac{1}{2})^k)^{n-k} < 1.$$

Thus, $P_r(B^c) > 0$, i.e. there exists a tournament on [n] satisfying property S_k.

∎

Corollary: $\forall k \geq 2$, there exists a minimal f(k) and a tournament on f(k) vertices satisfying property S_k.

- k=3, as $\binom{91}{3}(\frac{7}{8})^{88} < 1$, $\Rightarrow f(3) \leq 91$.

The Linearity of Expectation

- $\forall X, Y, E[X + Y] = E[X] + E[Y]$
- $P_r(X \geq E[X]) > 0$
- $P_r(X \leq E[X]) > 0$

Definition 16. Set A is a sum-free: if $\forall x, y \in A, x + y \notin A$.

Recall: The maximum sum-free set in [2n] is of size n. $A = \{n+1, n+2, ..., 2n\}$ or $A = \{\text{odd integers}\}$

Theorem 17. *For any set A of non-zero integers, there is a sum-free set* $B \subset A$ *with* $|B| \geq \frac{|A|}{3}$.

Proof. We will choose prime p large enough s.t.$p > |a|$ for $\forall a \in A$.

Consider $Z_p = \{0, 1, ..., p-1\}$. There is a sum-free set under $Z_p(mod p)$:

$$S = \{\lceil \frac{p}{3} \rceil, \lceil \frac{p}{3} \rceil + 1, ..., \lceil \frac{2p}{3} \rceil\}$$

We proceed by reducing the original problem to Z_p. For $x \in Z_p^* = Z_p \backslash \{0\}$, let $A_x = \{a \in A : (ax \mod p) \in S\}$.

Claim: $\forall x \in Z_p^*$, A_x is a sum-free subset of A.

Proof. For $a, b \in A_x$, $(ax \mod p) \in S$, $(bx \mod p) \in S$,

$\implies (ax + bx \mod p) \notin S$(as S is sum-free mod p)

∎

Then, we want to find some $x \in Z_p^*$, s.t. $|A_x| \geq \frac{|A|}{3}$. Choose $x \in Z_p^*$ uniformly at random. We compute $E[|A_x|]$.

Note that $|A_k| = \sum_{a \in A} 1_{\{(ax \mod p) \in S\}}$, so

$$E[|A_x|] = \sum_{a \in A} E[1_{\{(ax \mod p) \in S\}}] = \sum_{a \in A} P_r((ax \mod p) \in S)$$

Observe that for fixed $a \in A$, running over all $x \in Z_p^*$, then (ax modp) will also run over all Z_p^*.

$$\implies P_r((ax \mod p) \in S) = \frac{|S|}{p-1} \geq \frac{1}{3}$$

$$So \quad E[|A_x|] = \sum_{a \in A} P_r((ax \mod p) \in S) \geq \frac{|A|}{3}$$

Then, there must exist some $x \in Z_p^*$, s.t. $|A_x| \geq E[|A_x|] \geq \frac{|A|}{3}$, where A_x is sum-free.

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Definition 18. A dominator set of a graph G is a subset $A \subset V(G)$ s.t. every $u \in V \backslash A$ has a neighbor in A.