Combinatorics, 2016 Fall, USTC

Week 14, December 6 & 8

The Probabilistic Methods in Combinatorics

Theorem 1. *Let $G = (V, E)$ be a graph on n vertices and with minimum degree $\delta > 1$. Then G contains a dominating set of at most $\frac{1+\ln(1+\delta)}{1+\delta} \cdot n$ vertices.*

Proof. For $p \in (0,1)$(will determine the value of p later). We pick each vertex in $V(G)$ with probability p uniformly at random. Let X be the random set of vertices picked. Let Y be the random set of vertices $y \in V \setminus X$, which has no neighbors in X. That is, $y \in Y$ if and only if y is not picked and all neighbors of y are not picked. So

$$P(y \in Y) = (1-p)^{1+d(y)} \leqslant (1-p)^{1+\delta} \leqslant e^{-p(1+\delta)}$$

Then,

$$E[|Y|] = E[\sum_{y \in V} 1_{\{y \in Y\}}] = \sum_{y \in V} P(y \in Y) \leqslant n \cdot e^{-p(1+\delta)}$$

Also, $E[|X|] = np$.

Claim: $X \cup Y$ is a dominating set of G. Why? (exercise)

Since

$$E[|X \cup Y|] = E[|X|] + E[|Y|] \leqslant n(p + e^{-p(1+\delta)})$$

Check when $p = \frac{\ln(1+\delta)}{1+\delta}$, $p + e^{-p(1+\delta)}$ is minimized. So we fix $p = \frac{\ln(1+\delta)}{1+\delta}$ to get $E[|X \cup Y|] \leqslant \frac{1+\ln(1+\delta)}{1+\delta} \cdot n$. ∎

Definition 2. For $G = (V, E)$, an independent set (or a stable set) $I \subseteq V$ is a subset of vertices which has NO edges in it.

Let $\alpha(G) = max|I|$ over all independent set $I \subseteq V$.

Theorem 3. *For any graph G, $\alpha(G) \geqslant \sum_{v \in V} \frac{1}{d(v)+1}$ where $d(v)$ devotes the degree of v in G.*

Proof. Let $V(G) = [n]$. For $i \in [n]$, let N_i be the neighborhood of i in G. Let $S_n = \{$ permutations $\pi : [n] \to [n]\}$.

For given $\pi \in S_n$, we say a vertex $i \in [n]$ is $\pi-$dominating, if $\pi(i) < \pi(j)$ in π for $\forall j \in N_i$, Let $M_\pi = \{$ all $\pi-$dominating vertices $\}$.

Claim: $\forall \pi \in S_n$, M_π is an independent set.

Pf of Claim: Suppose not, then $\exists i, j \in M(\pi)$ with $ij \in E(G)$. Let $\pi(i) < \pi(j) \Rightarrow j \notin M(\pi)$, a contradiction.

Pick an $\pi \in S_n$ uniformly at random, compute $E[|M_\pi|]$?

Note $|M_\pi| = \sum_{i \in [n]} 1_{\{i \text{ is } \pi-dominating\}}$. So $E|M_\pi| = \sum_{i \in [n]} P(i \text{ is } \pi - dominating)$

Recall: i is $\pi-$ dominating iff $\pi(i)$ is the minimum over $\{i\} \cup N_i$. Since π is random, every vertex in $\{i\} \cup N_i$ has the equal probability to achieve the minimum in π, which is $\frac{1}{1+d(i)}$. Thus

$$E[|M_\pi|] = \sum_{i \in [n]} P(i \text{ is } \pi - dominating) = \sum_{i \in V} \frac{1}{1+d(i)}$$

Pf: Exercise

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Corollary 4 (Turán's Thm exact form)**.** *If an n-vertex graph G is K_{r+1}-free, then $e(G) \leqslant e(Tr(n)) \approx \frac{r-1}{2r} n^2$*

Definition 5. Turán's graph $Tr(n)$ is a graph on n vertices s.t. $V(G) = V_1 \cup V_2 \cup ... \cup V_r$ and $||V_i| - |V_j|| \leqslant 1$ where $ab \in E(G)$ iff $a \in V_i$ and $b \in V_j$ for some $i \neq j$

Theorem 6. *(Turán's Thm approximate form) If G is $K_{r+1}-$free, then* $e(G) \leqslant \frac{r-1}{2r}n^2$

Pf1: By the Corollary.

Pf2: Consider the vertices of G as $[n]$ and for $\forall i \in [n]$. Assign a weight p_i to it such that

$$\sum_{i\in[n]} p_i = 1 \quad \& \quad p_i \geqslant 0 \tag{1}$$

Find the max of $f(p) = \sum_{ij\in E(G)} p_i p_j$ over all weight functions $p : [n] \to [0,1]$ satisfying (1).

Claim: If $ij \notin E(G)$ and $p_i, p_j > 0$, then we can let $p_i \to 0,\ p_j \to p_i + p_j$ or $p_i \to p_i + p_j,\ p_j \to 0$ to increase the value of $f(p)$.

Pf of Claim: Let $S_i = \sum_{k\in N_i} p_k$ and $S_j = \sum_{k\in N_j} p_k$. Let $S_i \geqslant S_j$, then after assigning the new weight p^* satisfies

$$f(p^*) = f(p) - (p_i S_i + p_j S_j) + (p_i + p_j)S_i = f(p) + (S_i - S_j)p_j \geqslant f(p)$$

Now we keep applying this claim when stop, we arrive at some $(p_1, p_2, ..., p_n)$ s.t. the vertices i with $p_i > 0$ form a clique K_s is G. Since G is $K_{r+1}-$free, $\Rightarrow s \leqslant r$. So

$$f(p) = \frac{1}{2}[(\sum_{i\in V(K_s)} p_i)^2 - \sum_{i\in V(K_s)} p_i^2] = \frac{1}{2}[1 - \sum_{i\in V(K_s)} p_i^2]$$

As $\sum_{i\in V(K_s)} p_i^2 \geqslant \dfrac{1}{s}$

$$\Rightarrow f(p) \leqslant \frac{1}{2}(1-\frac{1}{s}) \leqslant \frac{1}{2}(1-\frac{1}{r}) = \frac{r-1}{2r}$$

$$\Rightarrow \frac{e(G)}{n^2} \leqslant max\ f(p) \leqslant \frac{r-1}{2r}$$

$$\Rightarrow e(G) \leqslant \frac{r-1}{2r} \cdot n^2$$

The Deleting Method

Previously, we often define an appropriate probability space and then show the random structure with desired property occurs with positive probability.

Today, we extend this and consider situation where random structure does not always have the desired property, and may have some very few "blemishes". After deleting all blemishes, we will obtain the wanted structure.

Recall: (Turán Thm) For any G, $\alpha(G) \geq \sum_{v\in V} \frac{1}{1+d(v)}$.

Corollary 7. *$\forall G$ with m edges and n vertices, $\Longrightarrow \alpha(G) \geq \frac{n^2}{2m+n}$. If $m = \frac{nd}{2}$, where d=average degree, then $\alpha(G) \geq \frac{n}{1+d}$.*

Next, we'll see a short argument, which shows the half-way of the previous result.

Theorem 8. *Let G be a graph on n vertices and with average degree d. Then $\alpha(G) \geq \frac{n}{2d}$.*

Proof. Let $S \subset V(G)$ be a random subset, where for $\forall v \in V$, $P_r(v \in S) = p$ and value of p will be determined later.

Let $X = |S|$ and $Y = e(S)$, $\Longrightarrow E[X] = np$ & $E[Y] = mp^2 = p^2 \cdot \frac{nd}{2}$

$$\Longrightarrow E[X - Y] = np - p^2 \cdot \frac{nd}{2} = n(p - \frac{d}{2}p^2)$$

By choosing $p = \frac{1}{d}$, we have $E[X - Y] = \frac{n}{2d}$. So there is a particular set S such that $|S| - e(S) \geq E[X - Y] = \frac{n}{2d}$. Now we delete one vertex for each edge of S. This leaves a subset $S^* \subset S$. Since all edges of S are destroyed, S^* must be an independent set of $size \geq |S| - e(S) \geq \frac{n}{2d}$ ∎

Recall: If $\binom{n}{k}2^{1-\binom{k}{2}} < 1$, then $R(k, k) > n$. $\Longrightarrow R(k, k) > \frac{1}{e\sqrt{2}}k2^{\frac{k}{2}}$.

Theorem 9. *For* $\forall n$, $R(k, k) > n - \binom{n}{k}2^{1-\binom{k}{2}}$.

Proof. Consider a random 2-edge-coloring of K_n, where each edge is colored by red or blue with probability $\frac{1}{2}$, independent of other choices. For $A \in \binom{[n]}{k}$, let X_A be the indicator random variable of the event that A is monochromatic.

Let $X = \sum_{A \in \binom{[n]}{k}} X_A$ be the number of monochromatic k-subsets.

$$E[X] = \sum_{A \in \binom{[n]}{k}} E[X_A] = \binom{n}{k}2^{1-\binom{k}{2}}$$

Then there exist a 2-edge-coloring of K_n s.t. $X \leq E[X] = \binom{n}{k}2^{1-\binom{k}{2}}$. Fix such a 2-edge-coloring, remove one vertex from each monochromatic k-subset. This will delete at most $X \leq \binom{n}{k}2^{1-\binom{k}{2}}$ vertices and destroy all monochromatic $K_k's$. So it remains at least $n - \binom{n}{k}2^{1-\binom{k}{2}}$ vertices, which has NO monochromatic K_k.

$$\Longrightarrow R(k,k) > n - \binom{n}{k} 2^{1-\binom{k}{2}}$$

∎

Find $max_n\{n - \binom{n}{k}2^{1-\binom{k}{2}}\}$, $\Longrightarrow R(k,k) > \frac{1}{e}(1+o(1))k2^{\frac{k}{2}}$.

Markov's Inequality

Let $X \geq 0$ be a random variable and t>0, then $P(X \geq t) \leq \frac{E[X]}{t}$

Corollary 10. *Let $X_n \geq 0$ be integer value random variable for $n \in \mathbb{N}^+$ in (Ω_n, P_n). If $E[X_n] \to 0$ as $n \to +\infty$, then $P_r(X_n = 0) \to 1 (as \quad n \to +\infty)$ i.e. $X_n = 0$ almost surely occur.*

Theorem 11. *For a random graph G(n,p) for some fixed $p \in (0,1)$, then*

$$P_r(\alpha(x) \leq \lceil \frac{2lnn}{p} \rceil) \to 1 \quad as \quad n \to +\infty$$

Proof. Let $k = \lceil \frac{2lnn}{p} \rceil$. For any $S \in \binom{[n]}{k+1}$, let A_s be the event that S is an independent set. Let $X_n = \sum_{S \in \binom{[n]}{n+1}} 1_{A_S}$ be the number of independent set of size k+1. We want $P_r(X_n = 0) \to 1$ as $n \to +\infty$. Compute

$$\begin{aligned} E[X_n] &= \textstyle\sum_{S \in \binom{[n]}{k+1}} P_r(A_S) = \binom{n}{k+1}(1-p)^{\binom{k+1}{2}} \\ &\leq \frac{n^{k+1}}{(k+1)!} e^{-p\binom{k+1}{2}} \\ &= \frac{1}{(k+1)!}(ne^{-p \cdot \frac{k}{2}})^{k+1} \\ &\leq \frac{1}{(k+1)!} \to 0 \end{aligned}$$

By the corollary, $P_r(X_n = 0) \to 1$ as $n \to +\infty \Leftrightarrow P_r(\alpha(G) \leq \lceil \frac{2lnn}{p} \rceil) \to 1$

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Definition 12. For any G, the chromatic number $\chi(G)$ is the minimum integer k s.t. V(G) can be partitioned into k independent sets.

Fact 1: $\chi(K_n) = n$.

Fact 2: For any G on n-vertices, $\chi(G) \cdot \alpha(G) \geq n$.

Definition 13. The girth of G denoted by g(G) is the length of a shortest cycle in G.

Theorem 14 (Erdős)**.** *For any $k \in \mathbb{N}^+$, there exists a graph G with $\chi(G) \geq k \quad \& \quad g(G) \geq k$.*

Proof. Consider G=G(n,p) where p will be determined later.

Recall: Let $t = \lceil \frac{2lnn}{p} \rceil$, then $\alpha(G) \leq t$ almost surely.

Let $X = \#$of cycles of length less than k in G.

$$E[X] = \sum_{i=3}^{k-1} \frac{n(n-1)\cdots(n-i+1)}{2i} \cdot p^i$$

where $\frac{n(n-1)\cdots(n-i+1)}{2i}$ is the number of positive $C_i's$ in K_n.

$$\Rightarrow E[X] \leq \sum_{i=3}^{k-1} (np)^i = \frac{(np)^k - 1}{np - 1}$$

By Markov's inequality,

$$P_r(X > \frac{n}{2}) \leq \frac{E[X]}{n/2} \leq \frac{2[(np)^k - 1]}{n(np-1)}$$

Let $p = n^{-\frac{k-1}{k}}$,

$$\Rightarrow P_r(X > \frac{n}{2}) < \frac{2(n-1)}{n(n^{\frac{1}{k}} - 1)} \to 0 \quad as \quad n \to +\infty$$

$\Rightarrow \exists G$ on n vertices, $\alpha(G) \leq t$ and with $\leq \frac{n}{2}$ cycles of length less than k, where $t = \lceil \frac{2lnn}{p} \rceil \leq 3lnn \cdot n^{\frac{k-1}{k}}$.

By deleting one vertex from each cycle of length less than k, we have a graph $G^* \subset G$, with NO cycles of length less than k.

$$\begin{cases} |V(G^*)| \geq n - \frac{n}{2} = \frac{n}{2} \\ \alpha(G^*) \leq \alpha(G) \leq 3lnn \cdot n^{\frac{k-1}{k}} \end{cases}$$

$$\chi(G^*) \geq \frac{|V(G^*)|}{\alpha(G^*)} \geq \frac{n^{1/k}}{6lnn} >> k.$$

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