Combinatorics, 2016 Fall, USTC

Week 16, Dec 20&22

2-Distance Problems

Theorem 1. *(Frankl-Wilson, 1981) If $\mathscr{F}$ is an L-intersecting family in $2^{[n]}$, then $|\mathscr{F}| \leq \sum_{k=0}^{|L|} \binom{n}{k}$.*

Proof. Let $\mathscr{F} = \{A_1, A_2, ..., A_m\}$ where $|A_1| \leq |A_2| \leq \cdots \leq |A_m|$. For $i \in [m]$, let $f_i(\boldsymbol{x})$ in $\mathbb{R}^n$ by

$$f_i(\boldsymbol{x}) = \prod_{l \in L, l < |A_i|} (\boldsymbol{x} \cdot \mathbf{1}_{\boldsymbol{A_i}} - l).$$

So $f_i(\boldsymbol{x})$ is a polynomial with n variables and with degree$\leq |L|$.

Claim 1: $f_1, f_2, ..., f_m$ are linearly independent.

Pf of Claim 1: Take $\mathbf{1}_{\boldsymbol{A_1}}, \mathbf{1}_{\boldsymbol{A_2}}, ..., \mathbf{1}_{\boldsymbol{A_m}}$, we have

- $f_i(\mathbf{1}_{\boldsymbol{A_i}}) = \prod_{l \in L, l < |A_i|} (|A_i| - l) > 0$
- $f_i(\mathbf{1}_{\boldsymbol{A_j}}) = \prod_{l \in L, l < |A_i|} (|A_i \cap A_j| - l) = 0$

Because $\mathscr{F}$ is L-intersecting $\Rightarrow \exists l \in L$ with $l = |A_j \cap A_i|$ and $l < |A_i|$.

Observation: All vector $\mathbf{1}_{\boldsymbol{A_j}}$ are $0/1-$ vectors. Thus, we can define a new polynomial $\tilde{f}_i(\boldsymbol{x})$ from $f_i(\boldsymbol{x})$ by replacing all term x_i^k by x_i.

So for all $0/1-$ vectors $\boldsymbol{v}$ we still have $\tilde{f}_i(\boldsymbol{v}) = f_i(\boldsymbol{v})$. This also shows that $\tilde{f}_1, \tilde{f}_2, ..., \tilde{f}_m$ are linearly independent. We see each $\tilde{f}_i(\boldsymbol{x})$ is a linear combination of the monomials $\prod_{i \in I} x_i$ where $I \in [n]$ and $|I| \leqslant |L|$. And clearly the number of each monomials is $\sum_{k=0}^{|L|}\binom{n}{k}$ which is also is the dimension of the space containing $\tilde{f}_1, \tilde{f}_2, ..., \tilde{f}_m$. So

$$|\mathscr{F}| = |m| \leq \sum_{k=0}^{|L|}\binom{n}{k}$$

∎

Theorem 2. *Let p be a prime and $L \leqslant Z_p = \{0, 1, ..., p-1\}$. Let $\mathscr{F} \in 2^{[n]}$ be s.t.*

- $|A| \notin L(mod\ P)$
- $|A \cap B| \in L(mod\ p)$ *for* $\forall A \neq B \in \mathscr{F}$

Then $|\mathscr{F}| \leq \sum_{k=0}^{|L|}\binom{n}{k}$

Proof. All operations are mod p. Define $f_i(\boldsymbol{x})$ over Z_p^n for each set in $\mathscr{F} = \{A_1, ..., A_m\}$ by

$$f_i(\boldsymbol{x}) = \prod_{l \in L}(\boldsymbol{x} \cdot \mathbf{1}_{\boldsymbol{A_i}} - l).$$

Then

- $f_i(\mathbf{1}_{\boldsymbol{A_i}}) = \prod_{l \in L}(|A_i| - l) \neq 0(mod\ p)$
- $f_i(\mathbf{1}_{\boldsymbol{A_j}}) = \prod_{l \in L}(|A_i \cap A_j| - l) = 0(mod\ p)$ for $i \neq j$

So $f_1, f_2, ..., f_m$ are linearly independent over Z_p^n.

The remaining proof is identical to the proof of Thm 1

$$\Rightarrow |\mathscr{F}| = m \leq \sum_{k=0}^{|L|}\binom{n}{k}$$

∎

Theorem 3. *(Frankl-Wilson) For any prime p, there is a graph G on $n = \binom{p^3}{p^2-1}$ vertices s.t. the size of minimum clique or maximum independent set is $\leqslant \sum_{i=0}^{p-1}\binom{p^3}{i}$*

Proof. Let $G = (V, E)$ be as follows:

- $V = \binom{[p^3]}{p^2-1}$
- for $A, B \in V$, $A \sim_G B$ iff $|A \cap B| = p - 1(mod\ p)$

Consider the max clique with vertices sat $A_1, A_2, ..., A_m \in \binom{[p^3]}{p^2-1}$ ∎

Thus we have

- $|A_i \cap A_j| \neq p - 1(mod\ p)$, for $i \neq j$
- $|A_i| = p^2 - 1 = p - 1(mod\ p)$

By Thm 2 with $L = \{0, 1, 2, ..., p-2\} \subseteq Z_p$ we have $m \leqslant \sum_{i=0}^{p-1}\binom{p^3}{i}$

Consider the maximum independent set, say $B_1, B_2, ..., B_s$, then $|B_i \cap B_j| = p-1(mod\ p)$ for $i \neq j$. So $|B_i \cap B_j| \in \{p-1, 2p-1, ..., p(p-1)-1\} = L^*$ with $|L^*| = p - 1$.

By Thm 1 with L^* we have $s \leqslant \sum_{i=0}^{p-1}\binom{p^3}{i}$

Corollary 4.

$$R(k+1, k+1) \geqslant k^{\Theta(\frac{\log(k)}{\log(\log(k))})}$$

Proof. Let $k = \sum_{i=0}^{p-1}\binom{p^3}{i}$, $n = \binom{p^3}{p^2-1}$.

$$\Rightarrow k \simeq \binom{p^3}{p} \simeq (p^2)^p \simeq p^{2p},\ n \simeq (\frac{p^3}{p^2})^{p^2} \simeq p^{p^2}$$

$$\Rightarrow \log(k) \simeq \Theta(p\log(p))$$

$$\Rightarrow \log(\log(p)) \simeq \log(p)$$

$$\Rightarrow p = \Theta(\frac{\log(k)}{\log(\log(k))}),\ n \simeq (p^{2p})^{p/2} \simeq k^{\Theta(\frac{\log(k)}{\log(\log(k))})}$$

∎

Definition 5. Given a set $S \subseteq R^n$ (bounded), the diameter of S is defined as $Diam(S) = \sup\{d(x,y) : x, y \in S\}$(Euclidean distance between x and y in R^n)

Borswk's Conjecture: Every bounded $S \subseteq R^n$ can be partitioned into $d+1$ sets of strictly smaller diameter.

This was verified for all $S \subseteq R^n$ with $d \leqslant 3$ and for all $S = sphere$. However, using Thm 1 and 2 one show this is false!

Lemma 6. *For prime p, there is a set of $\frac{1}{2}\binom{4p}{2p}$ vectors in $\{-1,1\}^{4p}$ s.t. every subset of size $2\binom{4p}{p-1}$ vectors contains an orthogonal pair of vectors.*

Proof. Let Q=$\{I \in \binom{[4p]}{2p} : 1 \in I\}$, then $|Q| = \frac{1}{2}\binom{4p}{2p}$.

For $\forall I \in Q$, define $\boldsymbol{v}^I \in \{-1,1\}^{4p}$ by

$$\boldsymbol{v_i} = \begin{cases} 1, i \in I \\ -1, i \notin I \end{cases}$$

Claim: $\boldsymbol{v}^I \perp \boldsymbol{v}^J$ iff $|I \cap J| \equiv 0 (mod p)$. Let $\mathscr{F} = \{\boldsymbol{v}^I : I \in Q\}$ with $|\mathscr{F}| = |Q| = \frac{1}{2}\binom{4p}{2p}$.

Proof. $\boldsymbol{v}^I \cdot \boldsymbol{v}^J = |I \cap J| - |I^C \cap J| - |I \cap J^C| + |I^C \cap J^C| = 4p - 2|I \Delta J|$

So $\boldsymbol{v}^I \perp \boldsymbol{v}^J$ iff $|I \Delta J| = 2p = 4p - 2|I \cap J|$ iff $|I \cap J| = p$ ∎

Claim: For any subset $\mathscr{G} \subset \mathscr{F}$ without orthogonal pairs, then $|\mathscr{G}| \leq \sum_{k=0}^{p-1} \binom{4p}{k} < 2\binom{4p}{p-1}$.

Proof. Consider the corresponding subset $Q' \subset Q$ of $\mathscr{G}$, i.e. $Q' = \{I \in Q : \boldsymbol{v}^I \in \mathscr{G}$. By claim 1, Q' is a subfamily of $\binom{[4p]}{2p}$ such that

- $|A| = 2p \equiv 0(modp), \forall A \in Q'$
- $|A \cap B| \neq 0(modp), \forall A \neq B \in Q'$

By thm 2, $|\mathscr{G}| = |Q'| \leq \sum_{k=0}^{p-1} \binom{4p}{k}$. ∎

$\implies$ maximal subset without orthogonal pairs$\leq \sum_{k=0}^{p-1} \binom{4p}{k} < 2\binom{4p}{p-1}$. ∎

Theorem 7. *For sufficiently large d, there exists a bounded set $S \subset \mathbb{R}^d$(a finite set) such that any partition of S into $|\cdot|^{\sqrt{d}}$ subsets contains a subset of the same diameter.*

Remark. As $|\cdot|^{\sqrt{d}} >> d+1$ for large d, this disproves Borsuk's conj.

Definition 8. A tensor product of vectors $\boldsymbol{v} \in \mathbb{R}^n$ is $\boldsymbol{w} = \boldsymbol{v} \otimes \boldsymbol{v} \in \mathbb{R}^{n^2}$ by $w_{ij} = v_i \cdot v_j$ for all $1 \leq i, j \leq n$

Proof. Take the family $\mathscr{F}$ from the lemma, so $\mathscr{F} \subset \{-1,1\}^n$(where n=4p)$\subset \mathbb{R}^n$. Let $X = \{\boldsymbol{v} \otimes \boldsymbol{v} : \boldsymbol{v} \in \mathscr{F}\}$ s.t. $X \subset \mathbb{R}^{n^2}$. For any $\boldsymbol{w} = \boldsymbol{v} \otimes \boldsymbol{v} \in X$,

$$||\boldsymbol{w}||^2 = \sum_{1\leq i,j\leq n} w_{ij}^2 = \sum_{1\leq i,j\leq n} v_i^2 v_j^2 = (\sum_{i=1}^{n} v_i^2)(\sum_{j=1} v_j^2) = n^2$$

$$\implies ||\boldsymbol{w}|| = n$$

For $\boldsymbol{w} = \boldsymbol{v} \otimes \boldsymbol{v}, \boldsymbol{w}' = \boldsymbol{v}' \otimes \boldsymbol{v}' \in X$, we have

$$\boldsymbol{w} \cdot \boldsymbol{w}' = \sum_{1 \le i,j \le n} w_{ij} w'_{ij} = \sum_{1 \le i,j \le n} (v_i v'_i)(v_j v'_j) = (\sum v_i v'_i)^2 = (\boldsymbol{v} \cdot \boldsymbol{v}')^2.$$

This implies that

$$\boldsymbol{w} \perp \boldsymbol{w}' \Longleftrightarrow \boldsymbol{v} \perp \boldsymbol{v}'$$

Also, $||\boldsymbol{w} - \boldsymbol{w}'||^2 = ||\boldsymbol{w}||^2 + ||\boldsymbol{w}'||^2 - 2\boldsymbol{w} \cdot \boldsymbol{w}' = 2n^2 - 2(\boldsymbol{v} \cdot \boldsymbol{v}')^2 \le 2n^2$

$$\Longrightarrow \begin{cases} Diam(X) = \sqrt{2}n \\ |X| = |\mathscr{F}| = \dfrac{1}{2}\dbinom{[4p]}{2p} \end{cases}$$

By the lemma, any subset of $2\binom{4p}{p-1}$ vectors in $\mathscr{F}$ contains an orthogonal pair of vector $\boldsymbol{v}\&\boldsymbol{v}'$. Thus, any subset of $2\binom{4p}{p-1}$ vectors in X must contain a pair $\boldsymbol{w} = \boldsymbol{v} \otimes \boldsymbol{v}, \boldsymbol{w}' = \boldsymbol{v}' \otimes \boldsymbol{v}'$ with $\boldsymbol{v} \perp \boldsymbol{v}'$ and thus of the maximum distance $||\boldsymbol{w} - \boldsymbol{w}'|| = \sqrt{2}n$. Thus, if we want to decrease the diameter, we must partition X into subsets, each of which has less than $2\binom{4p}{p-1}$ vectors, so the number of subsets is at least

$$\frac{|X|}{2\binom{4p}{p-1}} = \frac{\frac{1}{2}\binom{4p}{2p}}{2\binom{4p}{p-1}} = \frac{1}{4}\frac{(3p+1)\cdots 92p+1)}{(2p)\cdots(p)} \ge \frac{1}{4}\cdot(\frac{3}{2})^{p+1} \ge C\cdot(\frac{3}{2})^{\frac{\sqrt{d}}{4}} \ge 1\cdot 1^{\sqrt{d}}.$$

where $d = n^2 = 16p^2$ is the dimension of X. ∎

<u>$Bollobás'$ **Thm**</u>

Recall: (Sperner's Thm)

Let $\mathscr{F} \subset 2^{[n]}$ be: $\forall A \neq B \in \mathscr{F}, A \subsetneq B, B \subsetneq A$, then $|\mathscr{F}| \le \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

LYM-Inequality: For such $\mathscr{F}$, $\sum_{A \in \mathscr{F}} \frac{1}{\binom{n}{|A|}} \le 1$.

Theorem 9. *(Bollobás' Thm) Let* $A_1, A_2, ..., A_m$ *and* $B_1, B_2, ..., B_m$ *be the sequences of sets in [n] s.t.*

- $A_i \cap B_j \neq \phi, \forall i \neq j$
- $A_i \cap B_i = \phi, \forall i.$

Then,

$$\sum_{i=1}^{m} \frac{1}{\binom{a_i+b_i}{a_i}} \leq 1$$

where $a_i = |A_i|, b_i = |B_i|$.

Remark. Condition: $A_i \cap B_j \neq \phi, \forall i \neq j$ can't be weakened to i<j, or the base case doesn't hold any more. Counter example:

- $A_1 = \{1\} = B_2, A_2 = B_1 = \phi.$
- $A_1 = \{1\} = B_2, A_2 = \{3\} = B_1, A_3 = \{3\}, B_3 = \{1, 2\}$

Remark. $Bollobás \implies LYM \implies Sperner's$

Proof. Let $X = \cup_{i=1}^{m}(A_i \cup B_i)$. We prove by induction on $n = |X|$.

Base case: $n = 1 \leftrightarrow A_1 = \{1\}; B_1 = \phi$, OK.

Assume this holds for $|X| \leq n-1$. For $\forall x \in X$, define $I_x = \{1 \leq i \leq m : x \notin A_i\}$.

Define $\mathscr{F}_x = \{A_i : i \in I_x\} \cup \{B_i - \{x\} : i \in I_x\}$. Note that any set of _$x$ doesn't contain x, so $\mathscr{F}_x$ has less than n elements. Hence we apply induction hypothesis for each $\mathscr{F}_x$ to get:

$$\sum_{i \in I_x} \frac{1}{\binom{|A_i|+|B_i-\{x\}|}{|A_i|}} \leq 1 \tag{1}$$

We summing up the above inequalities for all $x \in X$ to get:

$$\sum_{x\in X}\sum_{i\in I_x}\frac{1}{\binom{|A_i|+|B_i-\{x\}|}{|A_i|}} \leq n \tag{2}$$

For each i, it contributes either 0, or $\frac{1}{\binom{a_i+b_i}{a_i}}$ or $\frac{1}{\binom{a_i+b_i-1}{a_i}}$ to each x. The term $\frac{1}{\binom{a_i+b_i}{a_i}}$ corresponds to points $x \notin A_i \cup B_i$, thus this term appears exactly $(n - a_i - b_i)$ times.

While, the term $\frac{1}{\binom{a_i+b_i-1}{a_i}}$ corresponds to points $x \notin A_i \& x \in B_i$, thus this term appears exactly b_i times.

$$(2) \Longrightarrow \sum_{i=1}^{m}[(n-a_i-b_i)\frac{1}{\binom{a_i+b_i}{a_i}} + b_i\frac{1}{\binom{a_i+b_i-1}{a_i}}] \leq n$$

Since $\frac{\binom{k-1}{l}}{\binom{k}{l}} = \frac{k-l}{k}$, we get $\frac{1}{\binom{a_i+b_i-1}{a_i}} = \frac{1}{\binom{a_i+b_i}{a_i}} \cdot \frac{a_i+b_i}{b_i}$, plugging in,

$$\sum_{i=1}^{m}[(n-a_i-b_i)\frac{1}{\binom{a_i+b_i}{a_i}} + \frac{a_i+b_i}{\binom{a_i+b_i}{a_i}}] \leq n$$

$$\Longleftrightarrow \sum_{i-1}^{m} n \cdot \frac{1}{\binom{a_i+b_i}{a_i}} \leq n$$

$$\Longleftrightarrow \sum_{i=1}^{m} \frac{1}{\binom{a_i+b_i}{a_i}} \leq n$$

∎

Definition 10. Let $\mathbb{F}$ be a field, a set $A \subset \mathbb{F}^n$ is general position if any n vectors in A are linearly independant over $\mathbb{F}$.

Examples. For $a \in \mathbb{F}$, define $\boldsymbol{m}(a) = (1, a, a^2, ..., a^{n-1}) \in \mathbb{F}^n$(moment curve). Then $\{\boldsymbol{m}(a) : a \in \mathbb{F}$ is a general position.

Next, we use the so-called "general position" argument to prove a version of $Bollobás's$ Thm, which is weaker than the previous one. But, on the other hand, the condition can be generalized to $A_i \cap B_j \neq \phi$ for $\forall i < j$.

Theorem 11. *(Bollobás' Thm(the skew version)) Let $A_1, ..., A_m$ be sets of size r and $B_1, ..., B_m$ be the sets of size s, such that :*

- $A_i \cap B_j \neq \phi, \forall i \neq j$
- $A_i \cap B_i = \phi, \forall i.$

Then,

$$m \leq \binom{r+s}{s}$$

.

Proof. (By $Lovász$): Let X=$\cup_i(A_i \cup B_i)$.

Take a set $V \subset \mathbb{R}^{r+1}$ of vectors $\boldsymbol{v} = (v_0, v_1, ..., v_r)$ such that

- V is in general position
- $|V| = |X|$

Identify the elements of X with vectors in V. Hence, we will view A_i as a subset in V containing r vectors and B_j as a subset in V containing s vectors.

For each B_j, define $f_j(\boldsymbol{x}) = \prod_{\boldsymbol{v} \in B_j} < \boldsymbol{v}, \boldsymbol{x} >= \prod_{\boldsymbol{v} \in B_j}(v_0x_0 + \cdots + v_rx_r)$. For $x \in \mathbb{R}^{r+1}$, note that

$$f_j(\boldsymbol{x}) = 0 \quad iff \quad < \boldsymbol{v}, \boldsymbol{x} >= 0 \quad for \quad some \quad \boldsymbol{v} \in B_j. \tag{3}$$

Consider the subspace span A_i, which is spanned by the r vector in A_i, since $A_i \subset V \subset \mathbb{R}^{r+1}$ and V is in general position, we see that all r vectors in

A_i are linearly independent and thus $dim(spanA_i) = r$. So, $(spanA_i)^{\perp}$ has dimension 1. Choose $\boldsymbol{a}_i \in (spanA_i)^{\perp}$ for i=1,...,m. Then for each $\boldsymbol{v} \in V$,

$$< \boldsymbol{v}, \boldsymbol{a}_i >= 0 \quad iff \quad \boldsymbol{v} \in spanA_i \quad iff \quad \boldsymbol{v} \in A_i. \tag{4}$$

(O.W. $\boldsymbol{v} \notin A_i$, $\{\boldsymbol{v}\} \cup A_i$ has r+1 vectors in V, which must be linearly independent, contradicting to $\boldsymbol{v} \in spanA_i$)

Combing (3)&(4), $f_j(\boldsymbol{a}_i) = \prod_{\boldsymbol{v} \in B_j} < \boldsymbol{v}, \boldsymbol{a}_i >= 0$ iff $A_i \cap B_j \neq \phi$

$$\Longrightarrow \begin{cases} f_j(\boldsymbol{a}_i) = 0, \forall i < j \\ f_j(\boldsymbol{a}_i) \neq 0, \forall j \end{cases}$$

This shows that $f_1, ..., f_m$ are linearly independent.

Next, we give an upper bound on the dimension of the space containing $f_1, ..., f_m$.

Recall: $f_j(\boldsymbol{x}) = \prod_{\boldsymbol{v} \in B_j}(v_0 x_0 + \cdots + v_r x_r)$, it is homogeneous with degree $s = |B_j|$ and r+1 variables $(x_0, x_1, ..., x_r)$. So this polynomial space can be generated by all monomials of follows:

$$x_0^{i_0} x_1^{i_1} \cdots x_r^{i_r}, \quad where \quad i_0 + i_1 + \cdots + i_r = s, i_j \geq 0$$

There are $\binom{r+s}{r}$ many solutions! So $m \leq$ the dimension= $\binom{r+s}{s}$.

∎