Combinatorics, 2016 Fall, USTC
Outlines in Week 2

2016.9.13

Inclusion-Exclusion Theorem

Let $A_1, ..., A_n$ be n subsets of the groud set Ω.

- Definition. Let $A_\emptyset = \Omega$; and for any nonempty subset $I \subseteq [n]$, let

$$A_I = \cap_{i \in I} A_i.$$

  If $|I| = k$,then we call A_I as a k-fold inclusion. For any integer $k \geq 0$, write

$$S_k = \sum_{I \in \binom{[n]}{k}} |A_I|$$

  to be the sum of the sizes of all k-fold intersections.

- **Inclusion-Exclusion formula.**

$$|A_1 \cup A_2 \cup ... \cup A_n| = \sum_{k=1}^{n} (-1)^{k+1} S_k.$$

  Sometime we also use the following version of Inclusion-Exclusion formula,

$$|A_1^c \cap A_2^c \cap ... \cap A_n^c| = |\Omega \backslash \left(\cup_{i=1}^n A_i \right)| = \sum_{k=0}^{n} (-1)^k S_k,$$

  where $A_i^c = \Omega \backslash A_i$ means the complement of subset A_i. We point out that $S_0 = |A_\emptyset| = |\Omega|$. It also holds that

$$|A_1^c \cap A_2^c \cap ... \cap A_n^c| = \sum_{I \subset [n]} (-1)^{|I|} |A_I|.$$

- **Proof 1:**uses the characterization funtions.For any subset $X \subseteq \Omega$, we define its characterization function $\mathbf{1}_X : \Omega \to \{0, 1\}$ by

$$\mathbf{1}_X(x) = \begin{cases} 1, x \in X \\ 0, x \in \Omega \backslash X \end{cases} \quad (0.1)$$

  So we have $\sum_{x \in \Omega} \mathbf{1}_X(x) = |X|$. Let $A = A_1 \cup A_2 \cup ... \cup A_n$,consider $f(x) = \prod_{i=1}^n (\mathbf{1}_A - \mathbf{1}_{A_i})$.
  Fact 1:$f(x) \equiv 0$ for any $x \in \Omega$.
  Fact 2:$\prod_{i=1}^n \mathbf{1}_{A_i} = \mathbf{1}_A$

- **Proof 2:**considers the contributions of each element $a \in \Omega$ to both sides.We show: for each $a \in \Omega$, the contributions of a to both sides always equal.

Applications

- Definition. Let $\varphi(n)$ be the number of integers $m \in [n]$ which are relatively prime to n. Here, m is relatively prime to n means that the greatest common divisor of m and n is 1.

- **Fact:** If n can be written as $n = p_1^{a_1} p_2^{a_2} ... p_t^{a_t}$, where $p_1, ..., p_t$ are distinct primes in $[n]$, then

$$\varphi(n) = n \prod_{i=1}^{t} \left(1 - \frac{1}{p_i}\right).$$

We proved this by considering $\Omega = [n]$ and the sets $A_i = \{m \in [n] : p_i | m\}$ for $i = 1, ..., t$. Note that $\varphi(n) = |\Omega \backslash (\cup_{i=1}^{t} A_i)|$

- Definition. A permutation $\sigma : X \to X$ is called a **derangement** of X if $\sigma(i) \neq i$ for any $i \in X$. We use D_n to denote the set of all derangements of $[n]$.

- **Fact:** For any integer $n \geq 1$,

$$|D_n| = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

We apply inclusion-exclusion by considering $A_i = \{\sigma | \sigma(i) = i\}$ for $i = 1, ..., n$.

- **Observation:** $\frac{|D_n|}{n!} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \to e^{-1} asn \to \infty$,so $|D_n| \sim \frac{n!}{e}$.

- **Recall.** (i) $S(n, k)$ is the number of partitions of a set of size n into k nonempty parts.

  (ii) $S(n, k)k!$ is the number of surjective functions from Y to X, where $|Y| = n$ and $|X| = k$.

- **Fact:**

$$S(n, k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k - i)^n.$$

To prove this, we use inclusion-exclusion (again!) by considering $\Omega = X^Y$ and its subsets $A_i := \{f : Y \to X \backslash \{i\}\}$.