Combinatorics, 2016 Fall, USTC
Outlines in Week 3

2016.9.20

Generating functions

- **Definition.** The (ordinary) generating function (or GF for short) for an infinity sequence $a_0, a_1, ...$ is a power series
$$f(x) = \sum_{n\geq 0} a_n x^n.$$
We have two ways to view the generating function.
(i). When the power series $\sum_{n\geq 0} a_n x^n$ converges (i.e., there exists a radius $R > 0$ of convergence), we view G.F. as a function of x and we can apply operations of calculus on it, including differentiation and integration. For example, in this case we know that
$$a_n = \frac{f^{(n)}(0)}{n!}.$$
Also recall the following sufficient condition on the radius of convergence that if $|a_n| \leq K^n$ for some constant $K > 0$, then $\sum_{n\geq 0} a_n x^n$ converges in the interval $(-\frac{1}{K}, \frac{1}{K})$.
(ii). When we are not sure of the convergence, we view G.F. as a formal object with additions and multiplications. Let $a(x) = \sum_{n\geq 0} a_n x^n$ and $b(x) = \sum_{n\geq 0} b_n x^n$.
Addition.
$$a(x) + b(x) = \sum_{n\geq 0}(a_n + b_n)x^n.$$
Multiplication. Let $a(x)b(x) = \sum_{n\geq 0} c_n x^n$, where
$$c_n = \sum_{i=0}^{n} a_i b_{n-i}.$$

- $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ holds for any real x with $|x| < 1$. By the point view of (i), we can compute the derivatives of two sides to get more identities, i.e. the first derivative will give
$$\frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} kx^{k-1}.$$

- <u>Problem 1.</u> Let $a_0 = 1$ and $a_n = 2a_{n-1}$ for $n \geq 1$. Find a_n.
We let $f(x) = \sum a_n x^n$ be the generating function. Then we show $f(x) = 1 + 2xf(x)$,so $f(x) = \frac{1}{1-2x}$, which implies that $f(x) = \sum 2^n x^n$ and therefore $a_n = 2^n$.

- From the above problem, we see one of the basic ideas for using GF: in order to find the general expression of a_n, we work on its GF $f(x)$; once we find the formula of $f(x)$, then we can expand $f(x)$ into a power series and find a_n by choosing the coefficient of the right term.

- Recall the following facts:
  Fact 1. If $f(x) = \prod_{i=1}^{k} f_i(x)$ for polynomials $f_1, ..., f_k$, then

$$[x^n]f = \sum_{i_1+i_2+...+i_k=n} \prod_{j=1}^{k} \left([x^{i_j}]f_j\right).$$

  Fact 2. For $j = 1, 2, ..., n$, let

$$f_j(x) := \sum_{i \in I_j} x^i$$

  where I_j is a set containing nonnegative integers. Let $f(x) = f_1 f_2 ... f_n$ be the product. Let b_k be the number of solutions to $i_1 + i_2 + ... + i_n = k$ with each $i_j \in I_j$. Then

$$f(x) = \sum_{k=0}^{\infty} b_k x^k.$$

- Problem 2. Let A_n be the set of strings of length n with entries from the set $\{a, b, c\}$ and with NO "aa" occuring (in the consecutive positions). Find $a_n = |A_n|$ for $n \geq 1$.
  Sol: We first observe that $a_1 = 3, a_2 = 8$ and for any $n \geq 2$

$$a_n = 2a_{n-1} + 2a_{n-2},$$

  therefore $a_0 = 1$. Let $f(x) = \sum_{n \geq 0} a_n x^n$. Then we use the recurrence relation to get

$$f(x) = 1 + 3x + 2x(f(x) - 1) + 2x^2 f(x),$$

  which implies that

$$f(x) = \frac{1+x}{1-2x-2x^2}.$$

  By Partial Fraction Decomposition, we calculate that

$$f(x) = \frac{1-\sqrt{3}}{2\sqrt{3}} \frac{1}{\sqrt{3}+1+2x} + \frac{1+\sqrt{3}}{2\sqrt{3}} \frac{1}{\sqrt{3}-1-2x},$$

  which implies that

$$a_n = \frac{1-\sqrt{3}}{2\sqrt{3}} \frac{1}{\sqrt{3}+1} \left(\frac{-2}{\sqrt{3}+1}\right)^n + \frac{1+\sqrt{3}}{2\sqrt{3}} \frac{1}{\sqrt{3}-1} \left(\frac{2}{\sqrt{3}-1}\right)^n.$$

  Note that a_n must be an integer but its expression is of a combination of irrational terms! Observe that $\left|\frac{-2}{\sqrt{3}+1}\right| < 1$, so $\left(\frac{-2}{\sqrt{3}+1}\right)^n \to 0$ as $n \to \infty$. Thus, when n is sufficiently large, a_n is about the value of the second term $\frac{1+\sqrt{3}}{2\sqrt{3}} \frac{1}{\sqrt{3}-1} \left(\frac{2}{\sqrt{3}-1}\right)^n$; equivalently a_n will be the nearest integer to that.

- **Definition.** For any real r and integer $k \geq 0$, let

$$\binom{r}{k} = \frac{r(r-1)...(r-k+1)}{k!}.$$

- **Newton's Binomial Theorem.** For any real r,

$$(1+x)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^k$$

  holds for any $x \in (-1, 1)$.

  Pf: Taylor series.

- **Corollary.** Let $r = -n$ for integer $n \geq 0$. Then $\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}$. Therefore

$$(1+x)^{-n} = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} x^k.$$

- Problem 3. Let a_n be the number of ways to pay n Chinese Yuan using 1-Yuan bills, 2-Yuan bills and 5-Yuan bills (assume there exist such bills). What is the generating function of this sequence $\{a_n\}$?

  Sol: Observe that a_n corresponds to the number of integer solutions (i_1, i_2, i_3) to

  $i_1+i_2+i_3 = n$, where $i_1 \in I_1 := \{0, 1, 2, ...\}, i_2 \in I_2 := \{0, 2, 4, ...\}$ and $i_3 \in I_3 := \{0, 5, 10, ...\}$.

  Let $f_j(x) := \sum_{m \in I_j} x^m$ for $j = 1, 2, 3$. Then $f(x) := \prod_{1 \leq j \leq 3} f_j(x)$ is such that $[x^n]f = a_n$. That is, the generating function of $\{a_n\}$ is $f(x) = \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^5}$.

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Integer partition

- How many ways are there to write a natural number n as a sum of several natural numbers?

- The answer is not too difficult if we count *ordered partitions* of n. Here "ordered partition" means that we will view $1+1+2, 1+2+1$ as two different partitions of 4.

  For $1 \leq k \leq n$, let a_k be the number of ordered partitions of n such that n is partitioned into k natural numbers. Then this counts the number of integer solutions to

$$i_1 + i_2 + ... + i_k = n, \quad \text{where each} \quad i_j \geq 1.$$

  So $a_k = \binom{n-1}{k-1}$.
  Therefore the total number of ordered partitions of n is $\sum_{1 \leq k \leq n} \binom{n-1}{k-1} = 2^{n-1}$.

- We then consider the *unordered partitions*. For instance, we will view $1+2+3$ and $3+2+1$ as the same one.

  Let p_n be the number of partitions of n in this sense.

  Let n_j be the number of the j's in such a partition of n. Then it holds that

$$\sum_{j\geq 1} j \cdot n_j = n.$$

  If we use i_j to express the contribution of the addends equal to j in a partition of n (i.e., $i_j = j \cdot n_j$), then

$$\sum_{j\geq 1} i_j = n, \quad \text{where} \quad i_j \in \{0, j, 2j, 3j, ...\}.$$

  Note that in the above summation, j can run from 1 to infinity, or run from 1 to n.

  So p_n is the coefficient of x_n in the product

$$P_n(x) := (1 + x + x^2 + ...)(1 + x^2 + x^4 + ...)...(1 + x^n + x^{2n} + ...) = \prod_{k=1}^{n} \frac{1}{1 - x^k}.$$

- What is the generating function $P(x)$ of $\{p_n\}$ then?

  As the index j in the summation can be viewed from 1 to $+\infty$, the generating function $P(x)$ is an infinite product of polynomials

$$P(x) = \prod_{k=1}^{+\infty} \frac{1}{1 - x^k}.$$

The Catalan number

- First let us recall the definition of $\binom{r}{k}$ for real number r and positive integer k, and the Newton's binomial Theorem. We obtained that

$$\binom{\frac{1}{2}}{k} = \frac{(-1)^{k-1}2}{4^k} \cdot \frac{(2k-2)!}{k!(k-1)!}.$$

- Let n-gon be a polygon with n corners, labelled as corner 1, corner 2,..., corner n.

- **Definition.** A triangulation of the n-gon is a way to add lines between corners to make triangles such that these lines do not cross inside of the polygon.

- Let b_{n-1} be the number of triangulations of the n-gon, for $n \geq 3$. It is not hard to see that $b_2 = 1, b_3 = 2, b_4 = 5$. We want to find the general formula of b_n.

  Consider the triangle T in a triangulation of n-gon which contains corners 1 and 2. The triangle T should contain a third corner, say i. Since $3 \leq i \leq n$, we can divide the set of triangulations of n-gon into cases.

Case 1. If $i = 3$ or n, the triangle T divides the n-gon into triangle T itself plus a $(n-1)$-gon, which results in b_{n-2} triangulations of n-gon.

Case 2. For $4 \leq i \leq n-1$, the triangle T divides the n-gon into three regions: a $(n-i+2)$-gon, triangle T and a $(i-1)$-gon, therefore it results in $b_{i-2} \times b_{n-i+1}$ many triangulations of n-gon. Therefore, combining Case 1 and 2, we get that

$$b_{n-1} = b_{n-2} + \sum_{i=4}^{n-1} b_{i-2}b_{n-i+1} + b_{n-2} = b_{n-2} + \sum_{j=2}^{n-3} b_j b_{n-j-1} + b_{n-2}$$

By letting $b_0 = 0$ and $b_1 = 1$, we get

$$b_{n-1} = \sum_{j=0}^{n-1} b_j b_{n-1-j} \quad \text{or} \quad b_k = \sum_{j=0}^{k} b_j b_{k-j} \quad \text{for} \quad k \geq 2.$$

Let $f(x) = \sum_{k\geq 0} b_k x^k$. Note that $f^2(x) = \sum_{k\geq 0} \left(\sum_{j=0}^{k} b_j b_{k-j}\right) x^k$. Therefore

$$f(x) = x + \sum_{k\geq 2} b_k x^k = x + \sum_{k\geq 2} \left(\sum_{j=0}^{k} b_j b_{k-j}\right) x^k = x + \sum_{k\geq 0} \left(\sum_{j=0}^{k} b_j b_{k-j}\right) x^k = x + f^2(x).$$

Solving $f^2(x) - f(x) + x = 0$, we get that $f(x) = \frac{1+\sqrt{1-4x}}{2}$ or $\frac{1-\sqrt{1-4x}}{2}$. But notice that $f(0) = 0$, so it has to be the case that

$$f(x) = \frac{1-\sqrt{1-4x}}{2}.$$

Next, we apply the Newton's binomial theorem to get that

$$f(x) = \frac{1}{2} - \frac{1}{2}\sum_{k\geq 0} \binom{\frac{1}{2}}{k}(-4x)^k = \sum_{k\geq 1} \frac{(-1)^{k+1}4^k}{2}\binom{\frac{1}{2}}{k} x^k.$$

After plugging the obtained expression of $\binom{\frac{1}{2}}{k} = \frac{(-1)^{k-1}2}{4^k} \cdot \frac{(2k-2)!}{k!(k-1)!}$, we get that

$$f(x) = \sum_{k\geq 1} \frac{(2k-2)!}{k!(k-1)!} x^k = \sum_{k\geq 1} \frac{1}{k}\binom{2k-2}{k-1} x^k.$$

Note that $f(x)$ is the generating function of $\{b_k\}$, therefore

$$b_k = \frac{1}{k}\binom{2k-2}{k-1}.$$

- **Theorem.** The total number of triangulations of the $(k+2)$-gon is $\frac{1}{k+1}\binom{2k}{k}$, which is also called **the k^{th} Catalan number**.