Combinatorics, 2015 Fall, USTC

Week 4, October 8, 2016

Random Walk

Consider a real axis with integer points $(0, \pm 1, \pm 2, \pm 3, \cdots)$ marked. A frog leaps among the integer points according to the following rules:

(1). At beginning, it sits at 1.

(2). In each coming step, the frog leaps either by distance 2 to the right (from i to $i+2$), or by distance 1 to the left (from i to $i-1$), each is randomly chosen with probability $\frac{1}{2}$ independently of each other.

Problem. What is the probability that the frog can reach "0"?

In each step, we use "$+$" or "$-$" to indicate the choice of the frog that is either to leap right or leap left. Then the probability space Ω can be viewed as the set of infinite vectors, where each entry is in $\{+, -\}$.

Let A be the event that the frog reaches 0. Let A_i be the event that the frog reaches 0 at the i^{th} step for the first time. So $A = \cup_{i=1}^{+\infty} A_i$ is a disjoint union. So $P(A) = \sum_{i=1}^{+\infty} P(A_i)$.

To compute $P(A_i)$, we can define a_i to be the number of trajectories (or vectors) of the first i steps such that the frog starts at 1 and reaches 0 at the i^{th} step for the first time. So

$$P(A_i) = \frac{a_i}{2^i}.$$

Then,

$$P(A) = \sum_{i=1}^{+\infty} \frac{a_i}{2^i}.$$

Let $f(x) = \sum_{i=0}^{+\infty} a_i x^i$ be the generating function of $\{a_i\}_{i\geq 0}$, where $a_0 := 0$. Thus,

$$P(A) = \sum_{i=1}^{+\infty} \frac{a_i}{2^i} = f\left(\frac{1}{2}\right).$$

We then turn to find the expression of $f(x)$.

Let b_i be the number of trajectories of the first i steps such that the frog starts at "2" and reaches "0" at the i^{th} step for the first time.

Let c_i be the number of trajectories of the first i steps such that the frog starts at "3" and reaches "0" at the i^{th} step for the first time.

We express b_i in terms of $\{a_j\}_{j\geq 1}$. Since the frog only can leap to left by distance 1, if the frog can successfully jump from "i" to "0" in i steps, then this frog must reach "1" first. Let j be the number of steps by which the frog reaches "1" for the first time. So there are a_j trajectories from "2" to "1" at the j^{th} step for the first time.in the remaining $i-j$ steps the frog must jump from "1" to "0" and reach "0" at the coming $(i-j)^{th}$ step for the first time, so there are a_{i-j} trajectories that the frog can finish in exactly $i-j$ steps. In total,

$$b_i = \sum_{j=1}^{i-1} a_j a_{i-j}.$$

As $a_j = 0$,

$$b_i = \sum_{j=0}^{i} a_j a_{i-j}.$$

$$\Rightarrow \sum_{i\geq 0} b_i x^i = (\sum_{i\geq 0} a_i x^i)^2 = f^2(x).$$

Similarly, if we count the number c_i of trajectories from 3 to 0, we can obtain that

$$c_i = \sum_{j=0}^{i} a_j b_{i-j}.$$

$$\Rightarrow \sum_{i\geq 0} c_i x^i = \left(\sum_{i\geq 0} b_i x^i\right)\left(\sum_{i\geq 0} a_i x^i\right) = f^3(x).$$

Let us consider a_i from another point of view. After the first step, either the frog reaches "0" directly (if it leaps to left, so $a_1 = 1$), or it leaps to "3". In the latter case, the frog needs to jump from "3" to "0" using $i-1$ steps. Thus for $i \geq 2$, $a_i = c_{i-1}$.

Thus,

$$f(x) = \sum_{i=0}^{+\infty} a_i x^i = x + \sum_{i\geq 2} a_i x^i = x + \sum_{i\geq 2} c_{i-1} x^i$$

$$= x + x\left(\sum_{j=0}^{+\infty} c_j x^j\right) = x + x \cdot f^3(x).$$

Recall $P(A) = f(1/2) =: a$. Then $a = \frac{1}{2} + \frac{a^3}{2}$, i.e., $(a-1)(a^2+a-1) = 0$.

$$\Rightarrow a = 1, \frac{\sqrt{5}-1}{2} \text{or} \frac{-\sqrt{5}-1}{1}.$$

Since $P(A) \in [0,1]$, we see $P(A) = 1$ or $\frac{\sqrt{5}-1}{2}$.

Note that $f(x) = x + xf^3(x)$. Consider the inverse function of $f(x)$, that is, $g(x) := \frac{x}{1+x^3}$. Consider the figure of $g(x)$. We find that $g(x)$ is increasing around $\frac{\sqrt{5}-1}{2}$ but decreasing around 1. Since $f(x) = \sum a_i x^i$ is increasing, $g(x)$ also increases. Thus it doesn't make sense for $g(x)$ being around $x = 1$.

This explains that $P(A) = \frac{\sqrt{5}-1}{2}$. $\blacksquare$

Exponential Generating Function

Let $\mathbb{N}, \mathbb{N}_e$ and $\mathbb{N}_o$ be the sets of nonnegative integers, nonnegative even integers and nonnegative odd integers, respectively.

Recall:

- The ordinary generating function of the sequence $\{a_n\}_{n\geq 0}$ is the power series $f(x) = \sum_{n\geq 0} a_n x^n$.
- Let $f_j(x) := \sum_{i\in I_j} x^i$ for $j = 1, 2, ..., n$ and a_k be the number of integer solutions to $i_1 + i_2 + ... + i_n = k$, where $i_j \in I_j$, that is

$$a_k := \sum_{i_1+i_2+...+i_n=k \text{ for } i_j\in I_j} 1.$$

Then $\prod_{j=1}^{n} f_j(x)$ is the ordinary generating function of $\{a_k\}$.

Problem (1)**.** Let S_n be the number of selections of n letters chosen from an unlimited supply of a's, b's and c's such that both of the numbers of a's and b's are even.

We can write S_n as

$$S_n = \sum_{e_1+e_2+e_3=n,\ e_1,e_2\in\mathbb{N}_e,\ e_3\in\mathbb{N}} 1.$$

Using the previous fact, we see that $S_n = [x^n]f$, where

$$f(x) = \left(\sum_{i\in\mathbb{N}_e} x^i\right)^2 \left(\sum_{j\in\mathbb{N}} x^j\right) = \left(\frac{1}{1-x^2}\right)^2 \cdot \frac{1}{1-x}.$$

Problem (2)**.** Let T_n be the number of arrangements (or words) of n letters chosen from an unlimited supply of a's, b's and c's such that both of the numbers of a's and b's are even. What is the value of T_n?

To solve this, we define a new kind of generating functions.

Definition 1. The *exponential generating function* for the sequence $\{a_n\}_{n\geq 0}$ is the power series

$$f(x) = \sum_{n=0}^{\infty} a_n \cdot \frac{x^n}{n!}.$$

Fact: If we have n letters including x a's, y b's and z c's (i.e. $x+y+z=n$), then we can form $\frac{n!}{x!y!z!}$ distinct words using them.

Therefore, a selection (say x a's, y b's and z c's) can contribute $\frac{n!}{x!y!z!}$ arrangements to T_n. This implies that

$$T_n = \sum_{e_1+e_2+e_3=n,\ e_1,e_2\in\mathbb{N}_e,\ e_3\in\mathbb{N}} \frac{n!}{e_1!e_2!e_3!}.$$

Similar to define the above $f(x)$ for S_n, we define the following for T_n. Let

$$g(x) := \left(\sum_{i\in\mathbb{N}_e} \frac{x^i}{i!}\right)^2 \left(\sum_{j\in\mathbb{N}} \frac{x^j}{j!}\right).$$

Claim:

$$[x^n]g = \frac{T_n}{n!}.$$

Proof. To see this, we expand $g(x)$. Then the term x^n in $g(x)$ becomes

$$\sum_{\substack{e_1+e_2+e_3=n,\\ e_1,e_2\in\mathbb{N}_e,\ e_3\in\mathbb{N}}} \frac{x^{e_1}}{e_1!}\cdot\frac{x^{e_2}}{e_2!}\cdot\frac{x^{e_3}}{e_3!} = \left(\sum_{\substack{e_1+e_2+e_3=n,\\ e_1,e_2\in\mathbb{N}_e,\ e_3\in\mathbb{N}}} \frac{n!}{e_1!e_2!e_3!}\right)\frac{x^n}{n!} = T_n\cdot\frac{x^n}{n!}.$$

So $[x^n]g = \frac{T_n}{n!}$, i.e., $g(x)$ is the exponential generating function of $\{T_n\}$. $\blacksquare$

<u>Recall:</u> Two Taylor series: $e^x = \sum_{j\geq 0} \frac{x^j}{j!}$ and $e^{-x} = \sum_{j\geq 0} \frac{(-1)^j}{j!} x^j$ imply that

$$\frac{e^x + e^{-x}}{2} = \sum_{j\in\mathbb{N}_e} \frac{x^j}{j!} \quad \text{and} \quad \frac{e^x - e^{-x}}{2} = \sum_{j\in\mathbb{N}_o} \frac{x^j}{j!}.$$

Using the previous fact, we get

$$g(x) = \left(\frac{e^x + e^{-x}}{2}\right)^2 \cdot e^x = \frac{e^{3x} + 2e^x + e^{-x}}{4} = \sum_{n\geq 0} \left(\frac{3^n + 2 + (-1)^n}{4}\right) \cdot \frac{x^n}{n!}.$$

Therefore, we get that

$$T_n = \frac{3^n + 2 + (-1)^n}{4}.$$

$\blacksquare$

As we see, ordinary generation functions can be used to find the number of selections; while exponential generation functions can be used to find the number of arrangements or some combinatorial objects **involving ordering**.

Exercise (1)**.** Find the number a_n of ways to send n students to 4 different classrooms (say R_1, R_2, R_3, R_4) such that each room has at least 1 students.

Solution.

$$a_n = \sum i_1 + i_2 + i_3 + i_4 = n,\ i_j \geq 1 \frac{n!}{i_1! i_2! i_3! i_4!}$$

$$\Rightarrow f(x) = \sum_{n=0}^{+\infty} \frac{a_n}{n!} x^n = \left(\sum_{i=1}^{+\infty} \frac{x^i}{i!}\right)^4 = (e^x - 1)^4 \Rightarrow a_n = \cdots .$$

$\blacksquare$

<u>Fact 1:</u> Given k subsets $I_1, ..., I_k$ of non-negative integers, let $f_j(x) = \sum_{i\in I_j} \frac{x^i}{i!}$

and $b_n = \sum_{i_1+\cdots i_k=n,\ i_j\in I_j} \frac{n!}{i_1!\cdots i_k!}$. Then $\prod_{j=1}^k f_j$ is the exponential generation function for $\{b_n\}_{n\geq 0}$.

Recall the other fact that ordinary generation function. Given $I_1, ..., I_k$, let $f_j = \sum_{i\in I_j} x^i$ and $a_n = \sum_{i_1+\cdots i_k=n,\ i_j\in I_j} 1$. Then $\prod_{j=1}^k f_j$ is the ordinary generation function for $\{a_n\}_{n\geq 0}$.

Next, we can extend Fact 1.

Fact 2: Given polynomial $f_1, ..., f_k$. Say $f_j(x) = \sum_{n=0}^{\infty} \frac{a_n^{(j)}}{n!} x^n$, let $f(x) = \prod_{j=1}^k f_j(x)$. Then $f(x) = \sum_{n=0}^{+\infty} \frac{A_n}{n!} x^n$ if and only if

$$A_n = \sum_{i_1+\cdots+i_k=n,\ i_j\geq 0} \frac{n!}{i_1!\cdots i_k!} \left(\prod_{j=1}^k a_i^{(j)}\right).$$

Proof.

$$[x^n]f = \sum_{i_1+\cdots+i_k=n,\ i_j\geq 0} \left(\prod_{j=1}^n [x]^{i_j} f_j\right).$$

∎

Exercise (2)**.** Let a_n be the number of arrangements of some type A for a group of n people, and let b_n be the number of arrangements of some type B for a group of n people.

Define a new arrangement of n people called type C, as follows:

- Divide the n person into 2 groups (say 1^{st} and 2^{nd}).
- Then arrange the first group by an arrangement of type A, and arrange the second group by arrangement of type B.

Let $c_n :=$ # arrangements of type C. Let $A(x), B(x), C(x)$ be the exponential generation function for $\{a_n\}, \{b_n\}, \{c_n\}$ respectively. Prove that

$C(x) = A(x)B(x)$.

Proof.

$$C_n = \sum_{i+j=n} \frac{n!}{i!j!} a_i b_j$$

$\Rightarrow$ By Fact 2, $C(x) = A(x)B(x)$. ∎

Basic of Graphs

Definition 2. A graph $G = (V, E)$ consists of a finite set V of vertices (V is called the vertex set) and a set E of edges (E is called the edge set) such that

$$E \subseteq V \times V = \{(u, v) : u \in V, v \in V\},$$

where such (u, v)s are unordered pairs.