Combinatorics, 2015 Fall, USTC

Week 5, October 8, 2016

Basic of Graphs

Definition 1. A *graph* $G = (V, E)$ consists of a vertex set V and an edge set E, where the elements of V are called *vertices* and the elements of

$$E \subseteq \binom{V}{2} = \{(x, y) : x, y \in V\}$$

are called edges.

If E contains unordered pairs, then G is an undirected graph, otherwise, G is directed graph. In this course, all graphs are undirected.

In this course, all graphs are simple, that is, it contains no loops and multiple edges.

We say vertices i and j are adjacent if $(i, j) \in E$, write as $i \sim_G j$. We say the edge (i, j) is incident to the endpoints i and j. Let $e(G) = \#$ edges in graph $G = (V, E)$, i.e., $e(G) = |E|$. The degree of a vertex v in graph G denote by $d_G(v)$, is the number of edges of G incident to v.

The neighborhood of a vertex v is the set of vertices u s.t. u and v are adjacent, i.e., $N_G(v) = \{u \in V(G) | u \sim_G v\}$.

$$\Rightarrow d_G(v) = |N_G(v)|.$$

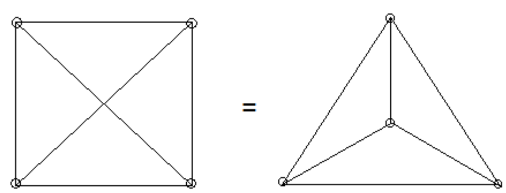
A graph $G' = (V', E')$ is a subgraph of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E \cap \binom{V'}{2}$, and we write as $G' \subseteq G$.

A graph with n vertices is a complete graph or a clique denote by K_n if all pairs of vertices are adjacent. So $e(K_n) = \binom{n}{2}$.

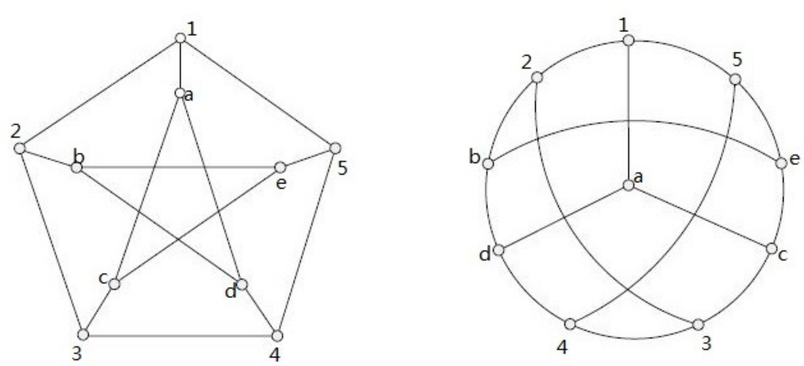
A graph with n vertices is called an independent set, denoted by I_n, if it contains no edge at all.

Given a graph $G = (V, E)$, its complement is a graph $\bar{G} = (V, E^c)$ with the same vertex set V such that $E^c = \binom{V}{2} \setminus E$. So clearly $e(G) + e(\bar{G}) = \binom{n}{2}$ where $n = |V(G)|$.

Definition 2. Two graphs $G = (V, E)$ and $G' = (V', E')$ are *isomorphic* if there exists a bijection $f : V \to V'$ such that $i \sim_G j$ if and only if $f(i) \sim_{G'} f(j)$.



K_4 K_4

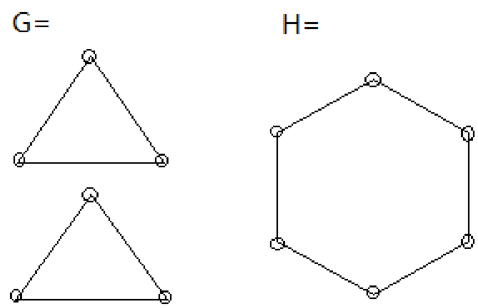


Peterson graph

The degree sequence of a graph $G = (V, E)$ is a sequence of degrees of all vertices listed in a non-decreasing order.

Problem. If two graphs G and H have the same degree sequence, then they are isomorphic?

The answer is NO.

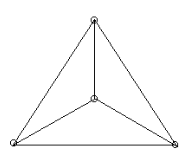


Definition 3. A *path* P_k of length $k-1$ is a graph $v_1 - v_2 - v_3 - \cdots - v_k$, where $v_i \sim v_{i+1}$. Note that the length means the number of edges it contains.

Definition 4. A *cycle* C_k og length k is a graph consisting of $V(C_k) = \{v_i, v_2, ..., v_k\}$, $E(C_k) = \{(v_1, v_2), (v_2, v_3), ..., (v_{k-1}, v_k), (v_k, v_1)\}$.

Definition 5. A graph G is a *planar* graph, if we can draw G on the plane such that its edges intersect only at their endpoints.

Example. K_4 is planar.



Exercise. K_5 is not planar.

Lemma 6 (Hand-shaking Lemma)**.** *In any* $G = (V, E)$*,*

$$\sum_{v \in V} d(v) = 2|E|.$$

Proof. Let $F = \{(e, v) : e \in E(G), v \in V(G) \text{ s.t. } e \text{ is incident to } v\}$, i.e., v is one of the two endpoints of e. Then

$$|F| = \sum_{e \in E(G)} 2 = 2|E|, \ |F| = \sum_{v \in V(G)} d(v) \Rightarrow \textit{Done}.$$

∎

Corollary 7. *In any graph* G*, the number of vertices with odd degree is even.*

Proof. Let $\mathcal{O} = \{v \in V(G) \text{ s.t. } d(v) \text{ is odd }\}$, $\mathcal{E} = \{v \in V(G) \text{ s.t. } d(v) \text{ is even }\}$. Then

$$2|E| = \sum_{v \in V} d(v) = \sum_{v \in \mathcal{O}} d(v) + \sum_{v \in \mathcal{E}} d(v)$$

$$\Rightarrow \sum_{v \in \mathcal{O}} d(v) \text{ is even } \Rightarrow |\mathcal{O}| \text{ is even }.$$

∎

Corollary 8. *In any graph* G*, if there exists a vertex with odd degree, there are at least two vertices with odd degree.*

Let us consider the following application.

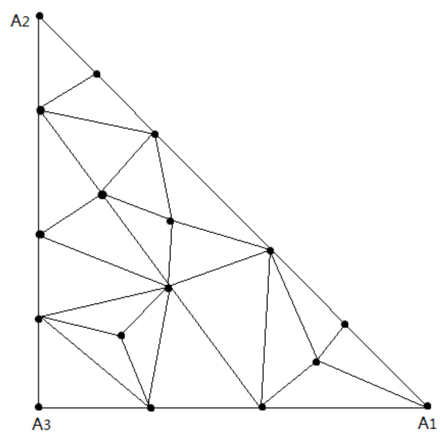
Let us draw a large triangle A_1, A_2, A_3 in the plane, with 3 vertices A_1, A_2, A_3. Then we divided this large triangle arbitrarily into small triangles, s.t. no triangles can have a vertex inside an edge of any other triangle.

Then we assign colors $1, 2, 3$ to all vertices of there triangles, under the following rules:

(1). The vertex A_i is assigned the color i for $\forall i \in 1, 2, 3$.

(2). All vertices lying on the edges A_iA_j of the large triangle must be assigned the color i or j.

(3). All interior vertices can be assigned any color $1, 2$, or 3.



Lemma 9 (Sperner's Lemma (a planar version)). *For any assignment of colors described as above, there always exists a small triangle whose three vertices are assigned all three colors* $1, 2, 3$.

We call such triangle as a rainbow triangle.

$V(G) = \{$ the face of any small triangles, and the outer face$\}$. We define the edges of G as follows:

Two vertices of G, i.e. 2 faces of the drawing, are adjacent in G if they are neighboring faces and the two endpoints of their common edge have colors 1 and 2.