Combinatorics, 2016 Fall, USTC

Week 6, October 11/13

Basic of Graphs

Lemma 1 (Sperner's Lemma (a planar version)). *For any assignment of colors described as above, there always exists a small triangle whose three vertices are assigned all three colors* $1, 2, 3$.

Proof. Define an auxiliary graph G:

- Its vertices are the faces of small triangle and the outer face. Let z be the vertex representing the outer face.
- Two vertices of G are adjacent, if the two corresponding faces are neighboring faces and the two endpoints of their common edge are colored by 1 and 2.

We consider the degree of a vertex $v \in V(G) - \{z\}$

(1). If the face of v has NO two vertices with color 1 and 2, then $d_G(v) = 0$.

(2). The face of v has 2 vertices with color 1 and 2. Let the color of the third vertex be k. If $k \in \{1, 2\}$, then $d_G(v) = 2$. Otherwise $k = 3$, then $d_G(v) = 1$

We claim that: $d_G(v)$ is dd iff $d_G(v) = 1$, iff the face of v has colors 1,2,3.

Then we consider $d_G(z)$ and we claim it must be odd. Why? The edge of G incident to z obviously have to go across A_1A_2. Consider the sequence of the colors of the points on A_1A_2, from A_1 to A_2. So $d_G(z) = \#$ of alternations between 1 and 2 in this sequence, which must be odd.

By Corollary, since the graph G has the vertex z with odd degree, there must be another vertex $v \in V(G) - \{z\}$ of odd degree. Then the face of v has colors 1,2,3.

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Theorem 2 (Brouver's Fixed Point Theory in 2-dimension). *Every continuous function $f : \Delta \to R$ has a fixed point x, that is $f(x) = x$.*

Proof. Consider a sequence of refinement of Δ. Define three auxiliary functions $\beta_i : \Delta \to R$ for $i \in \{1, 2, 3\}$ as following: for $\forall a = (x, y) \in \Delta$,

$$\begin{cases} \beta_1(a) & = x \\ \beta_2(a) & = y \\ \beta_3(a) & = 1 - x - y \end{cases}$$

For a given continuous $f : \Delta \to \Delta$, define $M_1 = \{a \in \Delta : \beta_1(a) \geqslant \beta_1(f(a))\}$ for $i \in \{1, 2, 3\}$

Fact 1: $\forall a \in \Delta,\ \exists i \in \{1, 2, 3\}$s.t. $a \in M_i$

Fact 2: if $a \in M_1 \cap M_2 \cap M_3$, then a is a fixed point.

We can define a coloring $C : \Delta \to \{1, 2, 3\}$ such that

(1). Any $a \in \Delta$ colored by i must be $a \in M_i$.

(2). The coloring C satisfies the condition of Sperner's Lemma for each Δ.

We show the following can be done:

- For the point A_i, say $i = 1$, not $A_i = (1, 0) \in M_1$, so we can let $C(A_i) = i$.

- Consider a vertex $a = (x, y) \in A_1A_2$, i.e. $x+y = 1$. Then $a \in M_1 \cap M_2$, otherwise $x + y < 1$ contradiction.

We have proved that for any $f : \Delta \to \Delta$ such $C : \Delta \to \{1, 2, 3\}$ exists. Apply Sperner's Lemma for the C on any Δ_i.

$\Rightarrow \exists$ a small triangle, say $A_1^{(i)}A_2^{(i)}A_3^{(i)}$ in Δ_i, which has 3 colors 1,2,3.

Consider the sequence $A_1^{(1)}, A_1^{(2)}, ..., A_1^{(i)}, ...$ Since everyone's bounded, there is a subsequence $A_1^{(i_1)}, A_1^{(i_2)}, ..., A_1^{(i_j)}, ...$ such that $\lim_{j\to\infty} A_1^{(i_j)} = p \in \Delta$

Since the diameter of $A_1^{(i)}A_2^{(i)}A_3^{(i)}$ is going to be 0 as $j \to \infty$, we see that $\lim_{j\to\infty} A_2^{(i_j)} = p$ and $\lim_{j\to\infty} A_3^{(i_j)} = p$

Note that $\beta_1(A_1^{(i)}) \geqslant \beta_1(f(A_1^{(i)}))$ so $\beta_1(p) \geqslant \beta_1(f(p))$. Similarly,$\beta_2(p) \geqslant \beta_2(f(p))$ and $\beta_3(p) \geqslant \beta_3(f(p))$ $\Rightarrow p \in M_1 \cap M_2 \cap M_3$. By Fact 2, p is a fixed point of f,i.e. $f(p) = p$. ∎

Double Counting

Suppose that we can give two finite sets A and B, and a subset $S \subseteq A\times B$. And if $(a, b) \in S$, we say a and b are incident. Let $N_a = \#$ of elements $b \in B$, $N_b = \#$ of elements $a \in A$. Then $\sum_{a\in A} N_a = |S| = \sum_{b\in B} N_b$. Define a table $X = (x_{ij})$ where

$$x_{ij} = \begin{cases} 1 & i|j \\ 0 & Otherwise \end{cases}$$

Let $T(j) = \#$ of divisions of j=# of i's in j^{th} column.

Let $\overline{T(n)} = \frac{1}{n}\sum_{j=1}^{n} T(j)$.

Fact: $|\overline{T(n)} - H(n)| < 1$, where $H(n) = \sum_{i=1}^{n} \frac{1}{i}$ is the n^{th} Hamonic number.

Proof. $\sum_{j=1}^{n} T(j) = \#$ 1's in $n \times n$ table$= \sum_{i=1}^{n} \lfloor \frac{n}{i} \rfloor$

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Sperner Theorem

Def: Let $\mathcal{F} \subseteq 2^{[n]}$ be a family of subsets of [n]. We say $\mathcal{F}$ is independent(or $\mathcal{F}$ is an independent system).If $\forall A, B \in \mathcal{F}$,$A \subsetneq B$ and $B \subsetneq A$,in other words,$\mathcal{F}$ is independent $\Leftrightarrow$ there is no "containment" relationship between any 2 subsets of $\mathcal{F}$.

Fact: For a fixed $k \in [n]$,$\binom{[n]}{k}$ is an independent system.

Theorem 3 (Sperner's Theorem)**.** *For any independent system $\mathcal{F}$ of [n],we have*

$$|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

First proof:(Double-Counting).

Def:(1).A chain of subsets of [n] is a sequence of distinct subsets $A_1 \subseteq A_2 \subseteq A_3 \subseteq ... \subseteq A_k$

(2).A maximal chain is a chain with the property that No other subsets of [n] can be inserted into it.

Fact 1:Any maximal chain looks like:

$$\phi \subseteq \{x_1\} \subseteq \{x_1, x_2\} \subseteq ... \subseteq \{x_1, ..., x_k\} \subseteq ... \subseteq \{x_1, ..., x_n\}.$$

Fact 2:There are exactly $n!$ maximal chains.

Why?Each maximal chain,say $\mathcal{C}:\phi\subseteq\{x_1\}\subseteq\{x_1,x_2\}\subseteq...\subseteq\{x_1,x_2,...,x_n\}$ defines a unique permutation:

$$\pi:[n]\to[n],\pi(i)=x_i$$

We first notice that this bound $|\mathcal{F}|\leq\binom{n}{\lceil\frac{n}{2}\rceil}$.Next,we double count by considering the number of pairs $(\mathcal{C},A)$ such that:

(1). $\mathcal{C}$ is a maximal chain of [n].

(2). $A\in\mathcal{C}\cap\mathcal{F}$.

Recall the double counting.

$$\sum_{\mathcal{C}}N_{\mathcal{C}}=\#pairs(\mathcal{C},A)=\sum_{A}N_A.$$

- $N_{\mathcal{C}}$=#subsets $A\in\mathcal{C}\cap\mathcal{F}$=$|\mathcal{C}\cap\mathcal{F}|\leq 1$.
- N_A = #maximal chains $\mathcal{C}$ s.t. $A\in\mathcal{C}=|A|!(n-|A|)!$

So we have

$$\begin{aligned}n!=\sum_{\mathcal{C}}1\geq\sum_{\mathcal{C}}N_{\mathcal{C}}&=\sum_{A\in\mathcal{F}}N_A\\&=\sum_{A\in\mathcal{F}}|A|!(n-|A|)!=\sum_{A\in\mathcal{F}}\frac{n!}{\binom{n}{|A|}}\\&\geq\sum_{A\in\mathcal{F}}\frac{n!}{\binom{n}{\lfloor\frac{n}{2}\rfloor}}=\frac{n!}{\binom{n}{\lfloor\frac{n}{2}\rfloor}}|\mathcal{F}|\end{aligned}$$

$$\Rightarrow|\mathcal{F}|\leq\binom{n}{\lfloor\frac{n}{2}\rfloor}$$

Second proof:

Def:A chain is symmetric if it consists of subsets of sizes k,k+1,..., $\lfloor\frac{n}{2}\rfloor$,...,n-k-1,n-k for some k.

For example,n=3 $\{\{2\},\{2,3\},[3]\}$ NOT Symmetric.$\{\phi,[3]\}$ NOT Symmetric.

Def:A partition of $2^{[n]}$ into symmetric chains is a way of expressing $2^{[n]}$ as a disjoint union of symmetric chains.

Theorem 4. *The family $2^{[n]}$ has a partition into symmetric chains.*

Proof of Sperner's Thm(Assuming Thm 2)

Note that any symmetric chain contains exactly one subset of size $\lfloor\frac{n}{2}\rfloor$.Since there are $\binom{n}{\lfloor\frac{n}{2}\rfloor}$ many subsets of size $\lfloor\frac{n}{2}\rfloor$,we see that any partition of $2^{[n]}$ into symmetric chains has to consists of exactly $\binom{n}{\lfloor\frac{n}{2}\rfloor}$ symmetric chains.

For each $A \in 2^{[n]}$,we define a sequence "$a_1a_2...a_n$" consisting of left and right parentheses by defining

$$a_i = \begin{cases} "(", if \quad i \in A \\ ")", otherwise \end{cases}$$

e.g. n=7,$A=\{2,5,6\}$,the sequence is))()()(()))(() $\rightarrow$)))(.

We then define the "partial pairing of parentheses" as following:

(1). First,we pair up all pairs "()" of adjoint parentheses.

(2). Then,we delete these already paired parentheses.

(3). Repeat the above process until nothing can be done.

Note that when this process stops,the remaining unpaired parentheses must look like this:

$$))))((((($$

We say two subsets $A, B \in 2^{[n]}$ have the same partial pairing,if the paired parentheses are the same(even in the same positions).

n=11:$A_1 = \{5, 6, 8\}$))))(()()))

$A_2 = \{5, 6, 8, 11\}$))))(()())(

$A_3 = \{4, 5, 6, 8, 11\}$)))((()())(

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$A_6 = \{1, 2, 3, 4, 5, 6, 8, 11\}$ ((((((()())(

$$\Rightarrow \{A_1, A_2, ..., A_6\} \quad is \quad a \quad symmetric \quad chain.$$

we can define an equivalence " $\sim$ " on $2^{[n]}$ by letting $A \sim B$ iff A, B have the same partial pairing.

Exercise:Each equivalence class indeed forms a symmetric chain. This proves Thm2.

Littlewood-Offord Problem Fix a vector $\boldsymbol{a} = (a_1, a_2, ..., a_n)$ with each $|a_i| \geq 1$. Let $S = \{\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, ..., \epsilon_n) : \epsilon_i \in \{1, -1\}, \boldsymbol{\epsilon} \cdot \boldsymbol{a} \in (-1, 1)\}$,Then $|S| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Proof. For $\forall \boldsymbol{\epsilon} \in S$,define $A_{\boldsymbol{\epsilon}} = \{i \in [n] : a_i \epsilon_i > 0\}$.Let $\mathcal{F} = \{A_{\boldsymbol{\epsilon}} : \boldsymbol{\epsilon} \in S\}$.

$$\Rightarrow |S| = |\mathcal{F}|$$

We want $\mathcal{F}$ is independent system.

Suppose NOT,say $A_{\boldsymbol{\epsilon}_1}, A_{\boldsymbol{\epsilon}_2} \in \mathcal{F}$,$A_{\boldsymbol{\epsilon}_1} \subseteq A_{\boldsymbol{\epsilon}_2}$

$$\begin{cases} \boldsymbol{\epsilon}_1 \centerdot \boldsymbol{a} \in (-1,1) \\ \boldsymbol{\epsilon}_2 \centerdot \boldsymbol{a} \in (-1,1) \end{cases}$$

$$\boldsymbol{\epsilon}_1 \centerdot \boldsymbol{a} = \sum_{i \in A_{\boldsymbol{\epsilon}_1}} |a_i| - \sum_{i \notin A_{\boldsymbol{\epsilon}_1}} |a_i| = 2 \sum_{i \in A_{\boldsymbol{\epsilon}_1}} |a_i| - \sum_{i=1}^{n} |a_i|$$

$$\boldsymbol{\epsilon}_2 \centerdot \boldsymbol{a} - \boldsymbol{\epsilon}_1 \centerdot \boldsymbol{a} = 2(\sum_{i \in A_{\boldsymbol{\epsilon}_2}} |a_i| - \sum_{j \in A_{\boldsymbol{\epsilon}_1}} |a_j|) \geq 2|a_j| \geq 2. for \quad some \quad j \in A_{\boldsymbol{\epsilon}_2} \backslash A_{\boldsymbol{\epsilon}_1}$$

But this is a contradiction as $|\boldsymbol{\epsilon}_2 \centerdot \boldsymbol{a} - \boldsymbol{\epsilon}_1 \centerdot \boldsymbol{a}| < 2$.

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