Combinatorics, 2016 Fall, USTC

Week 7, October 18 and 20

Basic of Graphs

Definition 1. A graph G is bipartite if its vertex set can be partitioned into two parts (say A and B) such that each edge joints one vertex in A and another in B. And we say (A, B) is a bipartition of G.

For example, all even cycles C_{2k} are bipartite and all odd cycles C_{2k+1} are not.

Definition 2. Donate $K_{a,b}$ to be the complete bipartite graph with the points of sizes a and b. This is a bipartite graph with edge set $\{(i, j) : i \in A,\ j \in B\}$ where $|A| = a,\ |B| = b$.

Definition 3. For a a graph H, we say a graph G is H-free is G contains NO H as its subgraphs.

For example, $K_{a,b}$ is K_3-free.

Turan Type Problem

For fixed graph H, we want to find the maximum number of edges in an H-free graph G with n vertices. We donate $ex(n, H)$ to the maximum number of edges in an n-vertex H-free graph G.

Theorem 4. $ex(n, C_4) \leqslant \dfrac{n}{4}(1 + \sqrt{4n-3})$

Proof. Let G be a C_4-free graph with n vertices. We want to prove that $e(G) \leqslant \frac{n}{4}(1+\sqrt{4n-3})$.

Consider $S = \{(\{u_1, u_2\}), w : u_1u_2w \textit{ is a path of length 2 in } G\}$. So G is C_4-free, for fixed $\{u_1, u_2\}$, there is at most one w s.t. $(\{u_1, u_2\}, w) \in S$.

So $|S| = \sum\limits_{\{u_1,u_2\}} \#(\{u_1, u_2\}, w) \in S \leqslant \sum\limits_{\{u_1,u_2\}} 1 = \binom{n}{2}$

On the other hand, fixed w, $\#$ $\{u_1, u_2\}$ s.t. $(\{u_1, u_2\}, w) \in S \leqslant \binom{d(w)}{2}$

Putting the above together,

$$\begin{aligned}
\tbinom{n}{2} \geqslant |S| &= \sum_{\{u_1,u_2\}} \#(\{u_1, u_2\}, w) \in S \\
&= \sum_{w \in V(G)} \tbinom{d(w)}{2} \\
&= \frac{n}{2}\Big(\sum_{w\in V(G)} \frac{d^2(w)}{n}\Big) - \frac{1}{2}\sum_{w\in V(G)} d(w) \\
&\geqslant \frac{n}{2}\Big(\sum_{w\in V(G)} \frac{d(w)}{n}\Big)^2 - |E| \\
&= \frac{2|E|^2}{n} - |E|
\end{aligned}$$

That is $|E|^2 - \frac{n}{2}|E| - \frac{n^2(n-1)}{4} \leqslant 0$

$$|E| \leqslant \frac{n}{4}(1+\sqrt{4n-3})$$

∎

Corollary 5. *$ex(n, C_4) \leqslant (\frac{1}{2} + o(n))n^{\frac{3}{2}}$, where $o(n) \to 0$ as $n \to \infty$.*

Theorem 6 (Mantal's Thm)**.** *$ex(n, K_3) = \lfloor \frac{n^2}{4} \rfloor$*

Proof. We first consider the lower bound $ex(n, K_3) \geqslant \lfloor \frac{n^2}{4} \rfloor$ as the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ is K_3-free and has $\lfloor \frac{n}{2} \rfloor$ edges.

So all we need to prove is that $ex(n, K_3) \leqslant \lfloor \frac{n^2}{4} \rfloor$.

We now prove by induction that any n-vertex K_3-free graph G has at most $\frac{n^2}{4}$ edges.

Base case: $n = 1,\ n = 2$ trivial.

Now we assume that any K_3-free graph H with less than n vertices has at most $\dfrac{|V(H)|^2}{4}$ edges. Let G be K_3-free with n vertices. Take any edge of G, say $xy \in E(G)$. Let $N_x = N_G(x) - \{y\}$, $N_y = N_G(y) - \{x\}$

Fact 1: $N_x \cap N_y = \varnothing$ and so $|N_x| + |N_y| \leqslant n - 2$

Let H be a graph obtained from G by deleting vertex x and y. Note that H is also K_3-free and as $n-2$ vertices. By induction, $e(H) \leqslant \frac{(n-2)^2}{4}$. Thus we have that

$$e(G) = e(H) + |N_x| + |N_y| + 1 \leqslant \frac{(n-2)^2}{4} + (n-2) + 1 = \frac{n^2}{4}$$

∎

Theorem 7. *For any $n \geqslant 1$, the n-vertex K_3-free graph G with maximum number of edges is unique and $G = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$*

Proof. By induction on n. Base case $n = 1, 2$ is trivial.

No we assume this holds for all integers less than n. Let G be an arbitrary K_3-free graph on n vertices and with $n \geqslant 1$, the n-vertex K_3-free graph G with maximum number of edges is unique and $G = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$

$ex(n, K_3) = \lfloor \dfrac{n^2}{4} \rfloor$ edges. We need to show $G = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.

Take an edge $xy \in E(G)$ as before. Then $|N_x| + |N_y| \leqslant n - 2$. Let $H = G - \{xy\}$ and $e(G) \leqslant \dfrac{(n-2)^2}{4}$. Then $\lfloor \dfrac{n^2}{4} \rfloor = e(G) + |N_x| + |N_y| + 1 \leqslant \dfrac{(n-2)^2}{4} + n - 1 = \dfrac{n^2}{4}$

Thus, all inequalities must be equalities!

- $|N_x| + |N_y| = n - 2$
- $e(H) = \lfloor \frac{(n-2)^2}{4} \rfloor$

By induction, $H = K_{\lfloor \frac{n-2}{2} \rfloor, \lceil \frac{n-2}{2} \rceil}$, $N_x \cap N_y = \varnothing$.

Also note that N_x and N_y are independent sets (Otherwise it creats triangles). This implies that $N_x \in A'$ or $N_y \in B'$. Similarly $N_y \in A'$ or $N_x \in B'$.

Since $N_x \cap N_y = \varnothing$ and $|N_x| + |N_y| = n - 2 = |V(H)|$.

We see that $N_x \in A'$ and $N_y \in B'$, or $N_y \in A'$ and $N_x \in B'$. Either case shows that $G = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$. ∎

Trees

Definition 8. A graph G is connected, if for any vertices u and v in G, G contains a path from u to v. Otherwise, we say G is disconnected.

Definition 9. A component of a graph G is a maximal connected subgraph of G.

Fact: G is disconnected if and only if G has ≥ 2 components.

Definition 10. A graph T is called a tree if it is connected but contains no cycles.

Definition 11. A vertex in a tree T with degree one is called a leaf.

Fact 1: Any tree has at least one leaf.

Proof. Suppose not, then any $v \in V(T)$ has degree ≥ 2. By the homework, T contains a cycle of length at least 3, a contradiction. ∎

Theorem 12 (Euler's Formula). *For any tree $T = (V, E)$, $|V| = |E| + 1$.*

Proof. By induction on n.

Base case: $n = 2$. The tree is an edge with two endpoints. The statement holds.

Consider a tree $T = (V, E)$ with n vertices. By Fact 1, T has a leaf v. Then $T - \{v\}$ is still connected and of course it has no cycle. So $T - \{v\}$ is a tree with $n - 1$ vertices. By induction, for $T - \{v\}$,

$$n - 1 = |E(T - \{v\})| + 1 = |E(T)| - 1 + 1,$$

$$\Rightarrow |V(T)| = n = |E(T)| + 1.$$

∎

<u>Fact 2:</u> Any tree T with ≥ 2 vertices has ≥ 2 leaves.

Proof. Suppose not that T has a unique leaf v, so $\forall u \in V(T) \backslash \{v\}$, $d(u) \geq 2$.

$$\sum_{x \in V(T)} d(x) = 2|E| = 2(|V| - 1), \quad \sum_{x \in V(T)} d(x) \geq 2(|V| - 1) + 1,$$

a contradiction. ∎

Theorem 13 (Tree characterization). *Let $T = (V, E)$ be a graph. Then the following are equivalent:*

(i). T is a tree (i.e. connected and no cycle).

(ii). T is connected, but deleting any edge will result in a disconnected graph.

(iii). T has no cycle, but adding any new edge will result in a cycle.

Remark. Here, (ii)$\Leftrightarrow$ a tree is a "minimal" connected graph. (iii)$\Leftrightarrow$ a tree is a "maximal" graph without a cycle.

Proof. (i)$\Rightarrow$(ii): Suppose (ii) fails, then there exists $e = xy \in E(T)$ s.t. $T - \{e\}$ is still connected. Then $T - \{e\}$ has a path P from x to y. But then $P \cup \{e\}$ is a cycle in the tree T, a contradiction.

(ii)$\Rightarrow$(i): Suppose (i) fails, then T contains a cycle C. If we delete any edge from C, $T - \{e\}$ remains connected, a contradiction.

(i)$\Rightarrow$(iii): For any new edge $f = xy$, as T is connected, T has a path P from x to y. Thus, $P \cup \{f\}$ gives a cycle.

(iii)$\Rightarrow$(i): Suppose (i) fails, so T is disconnected. Then T has two components, say D_1 and D_2. Pick $x \in D_1$ and $y \in D_2$. If we add the new edge $f = xy$, then it is easy to see that $T + \{f\}$ still has NO cycles, a contradiction.

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Definition 14. Given a graph $G = (V, E)$, a graph $H = (V', E')$ is a spanning subgraph of G if H is a subgraph of G and $V = V'$.

Fact 3: Any connected graph G contains a spanning tree.

Proof. Deleting edges of G until it satisfies the property (ii) in the above. ∎

Definition 15. Given a connected graph G with n vertices, say $v_1, ..., v_n$. Let $ST(G) = \#$ of (labelled) spanning trees in G.

Theorem 16 (Cayley's Formula)**.** *$\forall n \geq 2$, $ST(K_n) = n^{n-2}$.*

Proof 1. We first count the number of spanning trees in K_n with degrees, say, $d(v_i) = d_i$, where $\sum_{i=1}^n d_i = 2(n-1)$.

Lemma: *Let $d_1, d_2, ..., d_n$ be positive integers with $\sum_{i=1}^n d_i = 2(n-1)$. Then the number of spanning trees on vertex set $\{v_1, ..., v_n\}$ and satisfying $d(v_i) = d_i$ is equal to*

$$\frac{(n-2)!}{(d_1-1)!(d_2-1)!\cdots(d_n-1)!}.$$

Proof of Lemma: We prove by induction on n.

Base case: $n = 2 \Rightarrow d_1 = d_2 = 2$. It holds.

We assume that this holds for any sequence of $n-1$ integers. Consider $d_1, ..., d_n$. As $(\sum d_i)/n < 2$, there exists some $d_i = 1$. (We may assume that $d_n = 1$.) So V_n is a leaf. Let $\mathcal{F} = \{\text{spanning trees with } d(v_i) = d_i\}$. $\forall i \in [n-1]$, $\mathcal{F}_i = \{T - \{v_n\} : T \in \mathcal{F}, \text{the unique neighbor of } v_n \text{ in } T \text{ is } v_i\}$. So $|\mathcal{F}| = \sum_{i=1}^{n-1} |\mathcal{F}_i|$. And any tree in $\mathcal{F}_i$ satisfies that

$$\begin{cases} d(v_j) = d_j & j \neq i, j \in [n-1] \\ d(v_i) = d_i. \end{cases}$$

By induction on $n-1$,

$$|\mathcal{F}_i| = \frac{(n-3)!}{(d_1-1)!\cdots(d_i-2)!\cdots(d_{n-1}-1)!} = \frac{(n-3)!(d_i-1)}{\prod_{j=1}^{n-1}(d_j-1)!}, \ \forall i \in [n-1].$$

$$
\begin{aligned}
|\mathcal{F}| &= \frac{(n-3)!}{\prod_{j=1}^{n-1}(d_j-1)!}\left(\sum_{i=1}^{n-1}(d_i-1)\right) \\
&= \frac{(n-3)!}{\sum_{j=1}^{n-1}(d_j-1)!}\left(2n-2-(n-1)-1\right) \\
&= \frac{(n-2)!}{\prod_{j=1}^{n}(d_j-1)!}.
\end{aligned}
$$

∎

<u>Binomial Theorem:</u>

$$
(x+y)^n = \sum_{\substack{i+j=n \\ i,j\geq 0}} \frac{n!}{i!j!}x^i y^j
$$

$$
\Rightarrow (x_1+x_2+\cdots x_k)^n = \sum_{i_1+\cdots i_k=n} \frac{n!}{i_1!\cdots i_k!}x_1^{i_1}\cdots x_k^{i_k}
$$

$$
\Rightarrow k^n = \sum_{i_1+\cdots i_k=n} \frac{n!}{i_1!\cdots i_k!}.
$$

Proof 1:

$$
ST(K_n) = \sum_{\substack{\sum_{i=1}^{n} d_i=2(n-2) \\ d_i\geq 1}} \frac{(n-2)!}{\prod_{j=1}^{n}(d_j-1)!} = n^{n-2}.
$$

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