Combinatorics, 2016 Fall, USTC

Week 8, October 25 and 27

1 The second proof of Cayley's formula

Definition 1. A *digraph* $D = (V, A)$ consists of a vertex set V and an arc set A where $A \subseteq \{(i, j) : i, j \in V\}$

Let $\mathscr{D} = \{$all digraphs on $[n]$ s.t. each vertex has exactly one arc going out, i.e. the out-degree is 1$\}$, where loops are allowed.

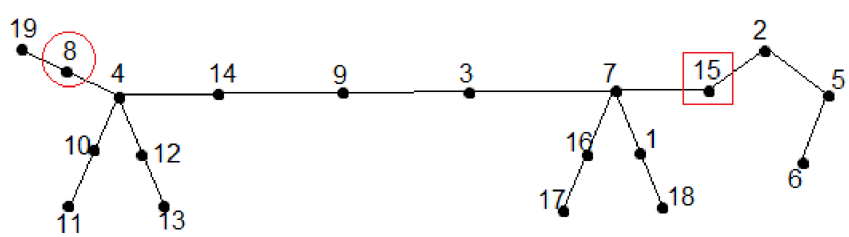
Fact: There exists a bijection between $\mathscr{D}$ and $\mathscr{F}_1 = \{$ all mappings $f : [n] \to [n]\}$.

Proof. For a digraph $D \in \mathscr{D}$, we can define a mapping $f_1 : [n] \to [n]$ s.t. if $i \to j$ is the unique arc going out of i, then $f(i) = j$.

The other direction is also easy to see. ∎

In particular, $|\mathscr{D}| = |\mathscr{F}_1| = n^n$.

Given a spanning tree of K_n, we choose 2 special vertices (one marked by a circle and the other marked by a square). We call such a subject (the spanning tree with 2 special vertices) as a *vertebrate*.



Let $\mathscr{V}$ = { all vertebrate on [n] }. Clearly, $|\mathscr{V}| = ST(K_n)n^2$. So to get the Cayley's formula, it suffices to show $|\mathscr{V}| = n^n$.

Lemma 2. *There exists a bijection between $\mathscr{V}$ and $\mathscr{D}$.*

Consider $w \in \mathscr{V}$, let the unique path P of w between the 2 special vertices $\bigcirc$ and $\square$ be the "chord" of w. So $8 \to 4 \to 14 \to 9 \to 3 \to 7 \to 15$ is the chord of the w in the figure.

We then define a digraph D_1 on $V(P)$ as following:

$$\begin{array}{ccccccc} 8 & 4 & 14 & 9 & 3 & 7 & 15 \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ 3 & 4 & 7 & 8 & 9 & 14 & 15 \end{array}$$

Having the above two rows, the arcs of D_1 are from the vertices in the $2nd$ row to the one above it. Thus, every vertex in D_1 has exactly one edge going out and one edge going in.

Exercise. Then D_1 consists of vertices disjoint cycle. (Possibly containing loops and 2-cycles.)

Next, we extend D_1 to a digraph D on $[n]$, by following:

(1) We go back to the vertebrate W and remain all edges of P.

(2) Then $W - E(P)$ consists of components, each having one vertex from $V(P)$. We direct the edges of the components such that they point to the unique vertex of the component contained in $V(P)$.

(3) These arcs product in (2), together with the arcs of D_1, define a new graph D on $[n]$. This should be easy to see that $D \in \mathscr{D}$.

So we just show that there exists a mapping $\varphi : \mathscr{V} \to \mathscr{D}$, by defining $\varphi(w) = D,\ w \in \mathscr{V}$. We still show that φ is a bijection.

Step 1 Need to define $\varphi^{-1} : \mathscr{D} \to \mathscr{V}$ s.t. $\varphi^{-1} \cdot \varphi = Id$.

How to define φ^{-1}? For each $d \in \mathscr{D}$, for the vertices of D belonging to a directed cycle, there is a national way to define the "chord".

$$\begin{array}{ccccccc} 3 & 4 & 7 & 8 & 9 & 14 & 15 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 8 & 4 & 14 & 9 & 3 & 7 & 15 \end{array}$$

And the remaining vertices give rise to other edges of the corresponding vertebrate w.

Step 2 $\forall D \in \mathscr{D}, \exists w \in \mathscr{V}$ s.t. $\varphi(w) = D$.

Combining step 1 and 2, we see φ is a bijection.

2 The third proof of Cayley's formula (using Linear Algebra)

Definition 3. For a graph G in $[n]$, define the Laplace matrix $Q = (q_{ij})_{n\times n}$ of G as follows:

$$q_{ii} = d_G(i),\ i \in [n]$$

$$q_{ij} = \begin{cases} -1,\ if\ ij \in E(G) \\ 0, otherwise\ for\ i \neq j. \end{cases}$$

Note that the sum of each row/column is 0.

Also, K_n has the Laplace matrix

$$A = \begin{pmatrix} n-1 & -1 & ... & -1 & -1 \\ -1 & n-1 & ... & -1 & -1 \\ ... & ... & ... & ... & ... \\ -1 & -1 & -1 & n-1 & -1 \\ -1 & -1 & -1 & -1 & n-1 \end{pmatrix}_{n \times n}.$$

Exercise. $det(A_{11}) =?$

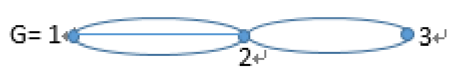
For an $n \times n$ matrix Q, let Q_{ij} be the $(n-1) \times (n-1)$ matrix obtained from Q by deleting the i^{th} row and j^{th} column.

Theorem 4. *$\forall$ graph G, $ST(G) = det(Q_{11})$.*

In fact, we will show that the statement also holds for multigraphs.

Definition 5. A *multigraph* is a graph where we allow multiple edges between two vertices (but no loops).

Example.



Then $ST(G) = 6$.

For multigraph G, we can define laplace matrix similarly:

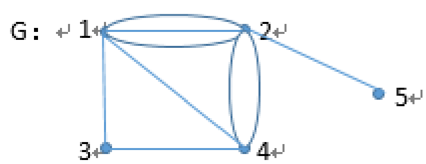
$$\begin{cases} q_{ii} = d_G(i), \\ q_{ij} = -m, \text{ if } i \neq j \text{ and } \exists\ m \text{ edges between } i \text{ and } j. \end{cases}$$

Theorem 6. *For any multigraph G (which has no loops), $ST(G) = det(Q_{11})$, where Q_{ij} is the $(n-1) \times (n-1)$ matrix obtained from the laplace matrix Q of G by deleting the i^{th} row and j^{th} column.*

Proof. By induction on the number of edges of G.

Base case, say $e(G) = 1$, which is trivial.

Now consider a multigraph G and assume this holds for any multigraph with less than $e(G) - 1$ edges.



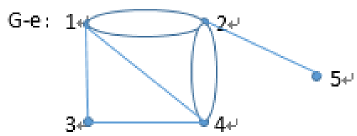
$$\Longrightarrow Q = \begin{pmatrix} 5 & -3 & -1 & -1 & 0 \\ -3 & 5 & 0 & -2 & -1 \\ -1 & 0 & 2 & -1 & 0 \\ -1 & -2 & -1 & 4 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{pmatrix}$$

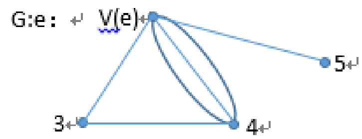
Definition 7. Let e be a fixed edge of G.

$G - e =$ the multigraph obtained from G by deleting the edge e.

$G : e =$ the multigraph obtained from G by constructing the edge e, i.e. merging the two endpoints of e into a new vertex.

By doing this,we may introduce new multiple edges. For example, fix $e = \overline{12}$, then for the G above,





Let Q' and Q'' be the laplace matrixes of $G-e$ and $G:e$ respectively. So

$$Q' = \begin{pmatrix} 4 & -2 & -1 & -1 & 0 \\ -2 & 5 & 0 & -2 & -1 \\ -1 & 0 & 2 & -1 & 0 \\ -1 & -2 & -1 & 4 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{pmatrix}$$

implying that

$$Q'_{11} = Q_{11} - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Let $Q_{11,22}$ be the matrix obtained from Q by deleting the first 2 rows and the first 2 columns. Then $Q''_{11} = Q_{11,22}$.

Claim 1. $det(Q'_{11}) + det(Q''_{11}) = det(Q_{11})$.

Proof. Obviously. ∎

Claim 2. $ST(G) = ST(G-e) + ST(G:e)$.

Proof. We divide the spanning trees of G into two classes:

- The 1^{st} class contains those spanning trees of G NOT containing e, which are exactly $ST(G-e)$.

- The 2^{nd} class contains those spanning trees of G containing e. And we see that the trees in the 2^{nd} class are in a one-to-one correspondence with the spanning trees of $G:e$.

∎

By induction, $ST(G-e) = det(Q'_{11}), ST(G:e) = det(Q''_{11})$.

By Claim 1 and 2, $ST(G) = det(Q_{11})$.

∎

Proof of Cayley's Formula.

Proof. Recall that the laplace matrix of K_n:

$$Q = \begin{pmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-1 \end{pmatrix}$$

Therefore $ST(G) = det(Q_{11}) = n^{n-2}$. ∎

3 Intersecting Family

Definition 8. A family $\mathcal{F} \subset 2^{[n]}$ is *intersecting* if for any $A, B \in \mathcal{F}, |A \cap B| \geq 1$.

Fact: For any intersecting family $\mathcal{F} \subset 2^{[n]}$, we have $|\mathcal{F}| \leq 2^{n-1}$.

Proof. Consider all pairs $\{A, A^c\}, \forall A \subset [n]$. Note that there are exactly 2^{n-1} such pairs, and $\mathcal{F}$ can have at most 1 subset from every pairs. This proves $\mathcal{F} \leq 2^{n-1}$. ∎

Tight:

- $\mathcal{F} = \{A \subset [n] : 1 \in A\}$,
- For n odd,$\mathcal{F} = \{A \in [n] : |A| > \frac{n}{2}\}$.

A harder problem:

What is the largest intersecting family $\mathcal{F} \subset \binom{[n]}{k}$?

e.g.: $\mathcal{F} = \{A \in \binom{[n]}{k} : 1 \in A\}$ is such an example.

Theorem 9 (Erdős-Ko-Rado's Theorem)**.** *For $n \geq 2k$, the largest intersecting family $\mathcal{F} \subset \binom{[n]}{k}$ has size $\binom{n-1}{k-1}$.*

Moreover, if n>2k, then the largest intersecting family $\mathcal{F} \subset \binom{[n]}{k}$ must be: $\mathcal{F} = \{A \in \binom{[n]}{k} : i \in A\}$ for some $i \in [n]$.

Proof. Take a cyclic permutation $\pi = (a_1, a_2, ..., a_n)$ of $[n]$. Note that there are $(n-1)!$ cyclic permutations of $[n]$ in total.

Let $\mathcal{F}_\pi = \{A \in \mathcal{F}$, A appears as k consecutive numbers in the circuit of $\pi.\}$

Claim: For each cyclic permutation π, assume $n \geq 2k$, then $|\mathcal{F}_\pi| \leq k$.

Proof of Claim. Pick $A \in \mathcal{F}_\pi$, say $A = \{a_1, a_2, ..., a_k\}$. We call the edges $a_n a_1, a_k a_{k+1}$ as the boundary edges of A, and the edges $a_1 a_2, a_2 a_3, ..., a_{k-1} a_k$ as the inner-edges of A. We observe that for any distinct $A, B \in \mathcal{F}_\pi$, the boundary-edges of A and B are distinct. For any $B \in \mathcal{F}_\pi - \{A\}$, as $A \cap B \neq \emptyset$, we see that one of the boundary-edges of B must be an inner-edge of A. But A has $k-1$ inner-edges, so we see that there are at most $k-1$ many subsets in $\mathcal{F}_\pi - \{A\}$. so $|\mathcal{F}_\pi| \leq k$. ∎

Next we do a double-counting.

Let $N = \#\text{pairs } (\pi, A)$, where π is a cyclic permutation of $[n]$, and $A \in \mathcal{F}_\pi$.

By Claim, $N = \sum_\pi |\mathcal{F}_\pi| \leq k(n-1)!$.

Fix A, how many cyclic π s.t. $A \in \mathcal{F}_\pi$?

The answer is $k!(n-k)!$.

So #cyclic permutations π s.t. π contains the elements of A as k consecutive numbers is $k!(n-k)!$.

So $k(n-1)! \geq N = \sum_{A \in \mathcal{F}} k!(n-k)! = |\mathcal{F}| k!(n-k)!$.

$$\implies |\mathcal{F}| \leq \frac{k \cdot (n-1)!}{k!(n-k)!} = \binom{n-1}{k-1}.$$

∎