Combinatorics, 2016 Fall, USTC

Week 9, November 1 and 3

Erdős-Ko-Rado's Theorem

Theorem 1 (Erdős-Ko-Rado)**.** *When $n \geqslant 2k$, the largest intersecting family $\mathscr{F} \subseteq \binom{[n]}{k}$ is $\binom{n-1}{k-1}$. If $n > 2k$, then the intersecting family $\mathscr{F}$ with $\mathscr{F} = \binom{n-1}{k-1}$ must be a star.*

Proof. Proof for the extremal case $\mathscr{F} = \binom{n-1}{k-1}$.

We want to show show $\mathscr{F}$ must be a star. From the preview proof, we see that:

- For any cycle permutation π, $|\mathscr{F}_\pi| = k$.
- Moreover, for $\pi = (a_1, a_2, ..., a_n)$, $\mathscr{F}_\pi = \{A_1, A_2, ..., A_k\}$ where $A_j = \{a_j, a_{j+1}, ..., a_{j+k-1}\}$ for $1 \leqslant j \leqslant k$

Fix π, let $\mathscr{F}_\pi = \{A_1, A_2, ..., A_k\}$ and let $A_1 \cap A_2 \cap ... \cap A_k = \{1\}$.

If all subsets of $\mathscr{F}$ contain 1, then $\mathscr{F}$ is a star, we are done.

So we may assume that $\exists A_0 \in \mathscr{F}$ s.t. $1 \notin A_0$.

<u>Claim 1</u>: $\forall B \in \binom{A_1 \cup A_k \setminus \{1\}}{k-1}$ has $B \cup \{1\} \in \mathscr{F}$

Pf of Claim 1: Consider another cycle permutation π' with A_1, A_k unchanged, but the order of the integers insider $A_1 \setminus \{1\}$ and $A_k \setminus \{1\}$ are changed.

Since $A_1, A_k \in \mathscr{F}_{\pi'}$, by (2) all other $k-$sets in $A_1 \cup A_k$ formed by k consective integers on π' are also in $\mathscr{F}_{\pi'} \subseteq \mathscr{F}$. Repeating using the argument, we prove the claim 1.

Claim 2: Note that we have $A_0 \in \mathscr{F}$ with $1 \notin A_0$. Then $A_0 \subseteq A_1 \cup A_k \backslash \{1\}$

Pf of Claim 2: Otherwise, then $|A_1 \cup A_k - A_0| \geqslant k$ (as $|A_1 \cup A_k| = 2k-1$). So, we can pick a $k-$subset $B \subseteq A_1 \cup A_k - A_0$ s.t. $1 \in B$. By Claim 1, $B \in \mathscr{F}$. But $A_0 \cap B = \varnothing$, contraducting that $\mathscr{F}$ is intersecting. This proves Claim 2.

Claim 3: $\binom{A_1 \cup A_k}{k} \subseteq \mathscr{F}$

Pf of Claim 3: Consider any $i \in A_0$, let B_i be s.t.

$$q_{ij} = \begin{cases} A_0 \cup B_i = A_1 \cup A_k \\ A_0 \cap B_i = \{i\} \end{cases}$$

By Claim 1, $B_i \in \mathscr{F}$. By (2) and the same proof of Claim 1, we can obtain that the "new" Claim 1: all $k-$ subsets of $A_1 \cup A_k$ containing i belong to $\mathscr{F}$. This implies that any $k-$subsets B of $A_1 \cup A_k$ with $B \cap A_0 = \varnothing$ belongs to $\mathscr{F}$.

$$\Leftrightarrow \binom{A_1 \cup A_k}{k} \subseteq \mathscr{F}$$

Claim 4: $\binom{A_1 \cup A_k}{k} = \mathscr{F}$

Pf of Claim 4: Suppose that $\exists B \in \mathscr{F}$ s.t. $B \nsubseteq A_1 \cup A_k$, that is $|A_1 \cup A_k - A_0| \geqslant k$. So $\exists B' \subseteq A_1 \cup A_k - B$ with $|B'| = k$. By Claim 3, $B' \in \mathscr{F}$. But $B \cap B' = \varnothing$, a contradiction. This proves Claim 4.

Now, we see $|\mathscr{F}| = \binom{2k-1}{k} = \binom{2k-1}{k-1} < \binom{n-1}{k-1} = |\mathscr{F}|$. This completes the proof.

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Definition 2. A *Kneser graph* $KG(n,k)$ for $n \geqslant 2k$ is a graph with vertex set $\binom{[n]}{k}$ such that for $A, B \in \binom{[n]}{k}$, A is adjacent to B if and only if $A \cap B = \varnothing$.

Now we note that any intersecting family $\mathscr{F}$ of $\binom{[n]}{k}$ is just an independent set in $KG(n,k)$. Therefore, Erdős-Ko-Rado Thm is equivalent to that $\alpha(KG(n,k)) \leqslant \binom{n-1}{k-1}$.

Definition 3. The *adjacency matrix* $A_G = (a_{ij})_{n\times n}$ of an $n-$ vertex graph G is defined by

$$a_{ii} = 0$$

$$a_{ij} = \begin{cases} 1, \ if \ ij \in E(G) \\ 0, \ otherwise \ for \ i \neq j \end{cases}$$

Definition 4. The eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant ... \geqslant \lambda_n$ of A_G is also called *the eigenvalues of the graph* G. The eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n$ of A_G s.t.

$$\begin{cases} A_G\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i \\ ||\boldsymbol{v}_i|| = 1 \\ \boldsymbol{v}_i \perp \boldsymbol{v}_j \end{cases}$$

are called the orthonormal *eigenvectors of* G.

Definition 5. A graph G is *regular* if all vertices have the same degree.

Theorem 6 (Hoffman's Theorem)**.** *If an n-vertex graph G is regular with eigenvalues* $\lambda_1 \geqslant \lambda_2 \geqslant ... \geqslant \lambda_n$, *then* $\alpha(G) \leqslant n \cdot \dfrac{-\lambda_n}{\lambda_1 - \lambda_n}$

Proof. Let $\boldsymbol{v}_1, ..., \boldsymbol{v}_n$ be the corresponding eigenvectors of $\lambda_1, ..., \lambda_n$ s.t.

$$\begin{cases} A_G\boldsymbol{v}_1 = \lambda_i \boldsymbol{v}_1, \\ ||\boldsymbol{v}_i|| = 1, \\ < \boldsymbol{v}_i, \boldsymbol{v}_j >= 0, \ \forall i \neq j. \end{cases}$$

Let I be an independent set of G with $|I| = \alpha(G)$. Let $\mathbf{1}_I \in R^n$ s.t. its i^{th} coordinate is 1 if $i \in I$, and is 0 if $i \notin I$. Then we can write

$$\mathbf{1}_I = \sum_{i=1}^{n} \alpha_i \boldsymbol{v}_i.$$

Then

$$|I| =< \mathbf{1}_I, \mathbf{1}_I >= \sum_{i=1}^{n} \alpha_i^2 \tag{1}$$

and $\alpha_i =< \mathbf{1}_I, \boldsymbol{v}_i >$.

Since G is regular, (say every vertex has degree d,) We have that $\lambda_1 = d$ and $\boldsymbol{v}_1 = (1/\sqrt{n}, ..., 1/\sqrt{n})^T$. (Think why $\lambda_1 = d$ is maximum?) So

$$\alpha_1 =< \mathbf{1}_I, \boldsymbol{v}_1 >= \frac{|I|}{\sqrt{n}} \tag{2}$$

Since I is an independent set of G,

$$\mathbf{1}_I^T A_G \mathbf{1}_I = \sum_{i,j} x_i a_{ij} x_j = 0,$$

where

$$\mathbf{1}_I = (x_i), \;\; x_i = \begin{cases} 1, & i \in I \\ 0, & i \notin I. \end{cases}$$

Also,

$$\begin{aligned}
0 = \mathbf{1}_I^T A_G \mathbf{1}_I &= \sum_{i=1}^{n} \alpha_i^2 \lambda_i \\
&\geq \alpha_1^2 \lambda_1 + (\alpha_2^2 + \cdots + \alpha_n^2)\lambda_n \\
&\overset{\text{by (1) (2)}}{=} \frac{|I|^2}{n}\lambda_1 + \left(|I| - \frac{|I|^2}{n}\right)\lambda_n
\end{aligned}$$

$$\Rightarrow 0 \geq \frac{|I|^2}{n}\lambda_1 + \left(|I| - \frac{|I|^2}{n}\right)\lambda_n$$

$$\Rightarrow |I|\left(\frac{|I|}{n}\lambda_1 + \lambda_n - \frac{|I|}{n}\lambda_n\right) \leq 0$$

$$\Rightarrow \frac{|I|}{n}(\lambda_1 - \lambda_n) \leq -\lambda_n$$

$$\Rightarrow \alpha(G) = |I| \leq n \cdot \frac{-\lambda}{\lambda_1 - \lambda_n}.$$

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Lemma 7. *The eigenvalues of Kneser graph $KG(n,k)$ are:*

$$u_j := (-1)^j \binom{n-k-j}{k-j} \text{ of multiplicity } \binom{n}{j} - \binom{n}{j-1}$$

for every $0 \leq j \leq k$.

Remark. For more information, see GTM 207, 9.3 and 9.4.

Recall: Any intersecting family $\mathscr{F}$ is an independent set of $KG(n,k)$. Let $\alpha(G) = \max_I |I|$ over all independent sets I of G. Thus, Erdős-Ko-Rado's Theorem $\Leftrightarrow \alpha(KG(n,k)) \leq \binom{n-1}{k-1}$.

The second proof of Erdős-Ko-Rado's Theorem. Consider the eigenvalues of

$KG(n,k)$, say $\lambda_1 \geq \lambda_2 \cdots \lambda_{\binom{n}{k}}$, where $\lambda_1 = \binom{n-k}{k} = u_0$, $\lambda_{\binom{n}{k}} = -\binom{n-k-1}{k-1} = u_1$.

By Hoffman's bound,

$$\alpha(KG(n,k)) \leq \binom{n}{k} \frac{-\lambda_{\binom{n}{k}}}{\lambda_1 - \lambda_{\binom{n}{k}}} = \binom{n}{k} \frac{\binom{n-k-1}{k-1}}{\binom{n-k}{k} + \binom{n-k-1}{k-1}} = \binom{n-1}{k-1}$$

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